Chapter 9

RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

In this chapter, we will introduce perhaps the most important idea associated with vectors, the solution of systems of equations. In previous chapters, the solution of systems of equations was introduced in situations dealing with two equations in two unknowns. Geometrically, the solution of two equations in two unknowns is the point of intersection between two lines on the *xy*-plane. In this chapter, we are going to extend these ideas and consider systems of equations in R^3 and interpret their meaning. We will be working with systems of up to three equations in three unknowns, and we will demonstrate techniques for solving these systems.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the intersection between a line and a plane and between two lines in three-dimensional space, **Section 9.1**
- algebraically solve systems of equations involving up to three equations in three unknowns, **Section 9.2**
- determine the intersection of two or three planes, **Sections 9.3, 9.4**
- determine the distance from a point to a line in two- and three-dimensional space, **Section 9.5**
- determine the distance from a point to a plane, **Section 9.6**
- solve distance problems relating to lines and planes in three-dimensional space and interpret the results geometrically, **Sections 9.5, 9.6**

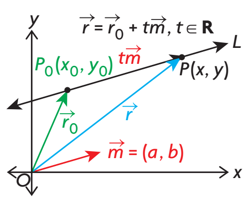
Review of Prerequisite Skills

In this chapter, you will examine how lines can intersect with other lines and planes, and how planes can intersect with other planes. Intersection problems are geometric models of linear systems. Before beginning, you may wish to review some equations of lines and planes.

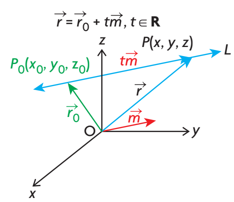
Type of Equation	Lines	Planes
Vector equation	$\vec{r} = \vec{r}_0 + t\vec{m}$	$\vec{r} = \vec{r}_0 + s\vec{a} + t\vec{b}$
Parametric equation	$x = x_0 + ta$ $y = y_0 + tb$ $z = z_0 + tc$	$x = x_0 + sa_1 + tb_1$ $y = y_0 + sa_2 + tb_2$ $z = z_0 + sa_3 + tb_3$
Cartesian equation	$Ax + By + C = 0$	$Ax + By + Cz + D = 0$ in three-dimensional space

In the table above,

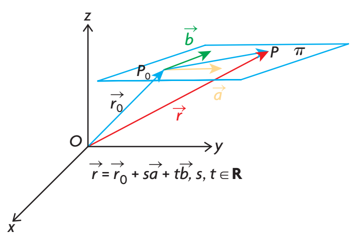
- $\vec{r}_0$ is the position vector whose tail is located at the origin and whose head is located at the point (x_0, y_0) in R^2 and (x_0, y_0, z_0) in R^3
- $\vec{m}$ is a direction vector whose components are (a, b) in R^2 and (a, b, c) in R^3
- $\vec{a}$ and $\vec{b}$ are noncollinear direction vectors whose components are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively in R^3
- s and t are parameters where $s \in \mathbf{R}$ and $t \in \mathbf{R}$
- (A, B) is a normal to the line defined by $Ax + By + C = 0$ in R^2
- (A, B, C) is a normal to the plane defined by $Ax + By + Cz + D = 0$ in R^3



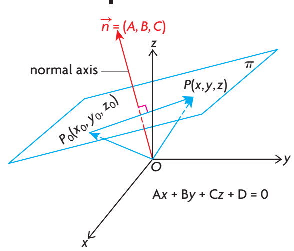
Vector Equation of a Line in R^2



Vector Equation of a Line in R^2



Vector Equation of a Plane in R^3



Scalar Equation of a Plane in R^3

Exercise

1. Determine if the point P_0 is on the given line.

a. $P_0(2, -5)$, $\vec{r} = (10, -12) + t(8, -7)$, $t \in \mathbf{R}$

b. $P_0(1, 2)$, $12x + 5y - 13 = 0$

c. $P_0(7, -3, 8)$, $\vec{r} = (1, 0, -4) + t(2, -1, 4)$, $t \in \mathbf{R}$

d. $P_0(1, 0, 5)$, $\vec{r} = (2, 1, -2) + t(4, -1, 2)$, $t \in \mathbf{R}$

2. Determine the vector and parametric equations of the line that passes through each of the following pairs of points:

a. $P_1(2, 5)$, $P_2(7, 3)$

b. $P_1(-3, 7)$, $P_2(4, -7)$

c. $P_1(-1, 0)$, $P_2(-3, -11)$

d. $P_1(1, 3, 5)$, $P_2(6, -7, 0)$

e. $P_1(2, 0, -1)$, $P_2(-1, 5, 2)$

f. $P_1(2, 5, -1)$, $P_2(12, -5, -7)$

3. Determine the Cartesian equation of the plane passing through point P_0 and perpendicular to $\vec{n}$.

a. $P_0(4, 1, -3)$, $\vec{n} = (2, 6, -1)$

b. $P_0(-2, 0, 5)$, $\vec{n} = (0, 7, 0)$

c. $P_0(3, -1, -2)$, $\vec{n} = (4, -3, 0)$

d. $P_0(0, 0, 0)$, $\vec{n} = (6, -5, 3)$

e. $P_0(4, 1, 8)$, $\vec{n} = (11, -6, 0)$

f. $P_0(2, 5, 1)$, $\vec{n} = (1, 1, -1)$

4. Determine the Cartesian equation of the plane that has the vector equation $\vec{r} = (2, 1, 0) + s(1, -1, 3) + t(2, 0, -5)$, $s, t \in \mathbf{R}$.

5. Which of the following lines is parallel to the plane $4x + y - z = 10$? Do any of the lines lie on this plane?

L_1: $\vec{r} = (3, 0, 2) + t(1, -2, 2)$, $t \in \mathbf{R}$

L_2: $x = -3t$, $y = -5 + 2t$, $z = -10t$, $t \in \mathbf{R}$

L_3: $\dfrac{x - 1}{4} = \dfrac{y + 6}{-1} = \dfrac{z}{1}$

6. Determine the Cartesian equations of the planes that contain the following sets of points:

a. $A(1, 0, -1)$, $B(2, 0, 0)$, $C(6, -1, 5)$

b. $P(4, 1, -2)$, $Q(6, 4, 0)$, $R(0, 0, -3)$

7. Determine the vector and Cartesian equations of the plane containing $P(1, -4, 3)$ and $Q(2, -1, 6)$ and parallel to the y-axis.

8. Determine the Cartesian equation of the plane that passes through $A(-1, 3, 4)$ and is perpendicular to $2x - y + 3z - 1 = 0$ and $5x + y - 3z + 6 = 0$.

CAREER LINK Investigate

CHAPTER 9: RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

Much of the world's reserves of fossil fuels are found in places that are not accessible to water for shipment. Due to the enormous volumes of oil that are currently being extracted from the ground in places such as northern Alberta, Alaska, and Russia, shipment by trucks would be very costly. Instead, pipelines are built to move the fuel to a place where it can be processed or loaded onto a large sea tanker for shipment. The construction of the pipelines is a costly undertaking, but, once completed, pipelines save vast amounts of time, energy, and money.

A team of pipeline construction engineers is needed to design a pipeline. The engineers have to study surveys of the land that the pipeline will cross and choose the best path. Often the least difficult path is above ground, but engineers will choose to have the pipeline go below ground. In plotting the course for the pipeline, vectors can be used to determine if the intended path of the pipeline will cross an obstruction or to determine where two different pipelines will meet.

Case Study—Pipeline Construction Engineer

New pipelines must be a certain distance away from existing pipelines and buildings, depending on the type of product that the pipeline is carrying. To calculate the distance between the two closest points on two pipelines, the lines are treated as skew lines on two different planes. (Skew lines are lines that never intersect because they lie on parallel planes.)

Suppose that an engineer wants to lay a pipeline according to the line L_1: $r = (0, 2, 1) + s(2, -1, 1)$, $s \in \mathbf{R}$. There is an existing pipeline that has a pathway determined by L_2: $r = (1, 0, 1) + t(1, -2, 0)$, $t \in \mathbf{R}$. Determine whether the proposed pathway for the new pipeline is less than 2 units away from the existing pipeline.

DISCUSSION QUESTIONS

1. Construct two parallel planes, π_1 and π_2. The first plane contains L_1 and a second intersecting line that has a direction vector of $\vec{a} = (1, -2, 0)$, the same direction vector as L_2. The second plane contains L_2 and a second intersecting line that has a direction vector of $\vec{b} = (2, -1, 1)$, the same direction vector as L_1.
2. Find the distance between π_1 and π_2.
3. Write the equation of L_1 and L_2 in parametric form.
4. Determine the point on each of the two lines in problem 3 that produces the minimal distance.

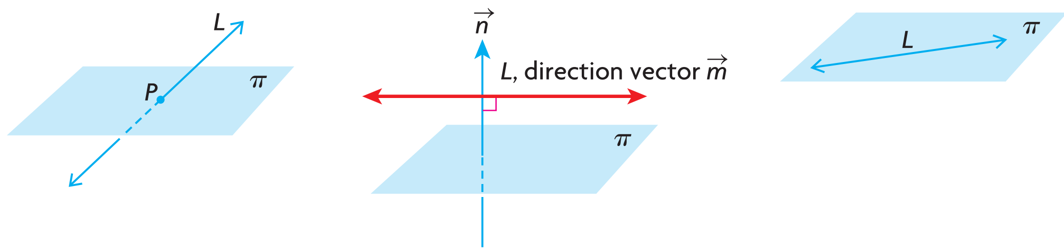
Section 9.1—The Intersection of a Line with a Plane and the Intersection of Two Lines

We start by considering the intersection of a line with a plane.

Intersection between a Line and a Plane

Before considering mathematical techniques for the solution to this problem, we consider the three cases for the intersection of a line with a plane.

Case 1: L intersects π at a point. Case 2: L is parallel to, but not on, π. Case 3: L is on π.



Case 1: The line L intersects the plane π at exactly one point, P.
Case 2: The line L does *not* intersect the plane so it is parallel to the plane. There are no points of intersection.
Case 3: The line L lies on the plane π. Every point on L intersects the plane. There are an infinite number of points of intersection.

For the intersection of a line with a plane, there are three different possibilities, which correspond to zero, one, or an infinite number of intersection points. It is not possible to have a finite number of intersection points, other than zero or one. These three possible intersections are considered in the following examples.

EXAMPLE 1 **Selecting a strategy to determine the point of intersection between a line and a plane**

Determine points of intersection between the line $L: \vec{r} = (3, 1, 2) + s(1, -4, -8)$, $s \in \mathbf{R}$, and the plane $\pi: 4x + 2y - z - 8 = 0$, if any exist.

Solution

To determine the required point of intersection, first convert the line from its vector form to its corresponding parametric form. The parametric form is $x = 3 + s$, $y = 1 - 4s$, $z = 2 - 8s$. Using parametric equations allows for direct substitution into π.

$$4(3 + s) + 2(1 - 4s) - (2 - 8s) - 8 = 0 \quad \text{(Use substitution)}$$
$$12 + 4s + 2 - 8s - 2 + 8s - 8 = 0 \quad \text{(Isolate } s\text{)}$$
$$4s = -4$$
$$s = -1$$

This means that the point where L meets π corresponds to a single point on the line with a parameter value of $s = -1$. To obtain the coordinates of the required point, $s = -1$ is substituted into the parametric equations of L. The point of intersection is

$$x = 3 + (-1) = 2$$
$$y = 1 - 4(-1) = 5$$
$$z = 2 - 8(-1) = 10$$

Check (by substitution):
The point lies on the plane because
$4(2) + 2(5) - 10 - 8 = 8 + 10 - 10 - 8 = 0.$

The point that satisfies the equation of the plane and the line is $(2, 5, 10)$. Now we consider the situation in which the line does not intersect the plane.

EXAMPLE 2 Connecting the algebraic representation to the situation with no points of intersection

Determine points of intersection between the line
$L: x = 2 + t, y = 2 + 2t, z = 9 + 8t, t \in \mathbf{R}$,
and the plane $\pi: 2x - 5y + z - 6 = 0$, if any exist.

Solution

Method 1:
Because the line L is already in parametric form, we substitute the parametric equations into the equation for π.

$$2(2 + t) - 5(2 + 2t) + (9 + 8t) - 6 = 0 \quad \text{(Use substitution)}$$
$$4 + 2t - 10 - 10t + 9 + 8t - 6 = 0 \quad \text{(Isolate } t\text{)}$$
$$0t = -3$$

Since there is no value of t that, when multiplied by zero, gives -3, there is no solution to this equation. Because there is no solution to this equation, there is no point of intersection. Thus, L and π do not intersect. L is a line that lies on a plane that is parallel to π.

Method 2:
It is also possible to show that the given line and plane do not intersect by first considering $\vec{n} = (2, -5, 1)$, which is the normal for the plane, and $\vec{m} = (1, 2, 8)$, which is the direction vector for the line, and calculating their dot product. If the dot product is zero, this implies that the line is either on the plane or parallel to the plane.

$$\begin{aligned} \vec{n} \times \vec{m} &= (2, -5, 1) \cdot (1, 2, 8) \\ &= 2(1) - 5(2) + 1(8) \\ &= 0 \end{aligned}$$ (Definition of dot product)

We can prove that the line does not lie on the plane by showing that the point $(2, 2, 9)$, which we know is on the line, is not on the plane.

Substituting $(2, 2, 9)$ into the equation of the plane, we get $2(2) - 5(2) + 9 - 6 = -3 \neq 0$.

Since the point does not satisfy the equation of the plane, the point is not on the plane. The line and the plane are parallel and do not intersect.

Next, we examine the intersection of a line and a plane where the line lies on the plane.

EXAMPLE 3 Connecting the algebraic representation to the situation with infinite points of intersection

Determine points of intersection of the line $L: \vec{r} = (3, -2, 1) + s(14, -5, -3)$, $s \in \mathbf{R}$, and the plane $x + y + 3z - 4 = 0$, if any exist.

Solution

Method 1:
As before, we convert the equation of the line to its parametric form. Doing so, we obtain the equations $x = 3 + 14s$, $y = -2 - 5s$, and $z = 1 - 3s$.

$$(3 + 14s) + (-2 - 5s) + 3(1 - 3s) - 4 = 0$$ (Use substitution)

$$3 + 14s - 2 - 5s + 3 - 9s - 4 = 0$$ (Isolate s)

$$0s = 0$$

Since any real value of s will satisfy this equation, there are an infinite number of solutions to this equation, each corresponding to a real value of s. Since any value will work for s, every point on L will be a point on the plane. Therefore, the given line lies on the plane.

Method 2:
Again, this result can be achieved by following the same procedure as in the previous example. If $\vec{n} = (1, 1, 3)$ and $\vec{m} = (14, -5, -3)$, then $\vec{n} \times \vec{m} = 1(14) + 1(-5) + 3(-3) = 0$, implying that the line and plane are parallel. We substitute the coordinates $(3, -2, 1)$, which is a point on the line, into the equation of the plane and find that $3 + (-2) + 3(1) - 4 = 0$. So this point lies on the plane as well. Since the line and plane are parallel, and $(3, -2, 1)$ lies on the plane, the entire line lies on the plane.

Next, we consider the intersection of a line with a plane parallel to a coordinate plane.

EXAMPLE 4 **Reasoning about the intersection between a line and the *yz*-plane**

Determine points where $L: x = 2 - s,\ y = -1 + 3s,\ z = 4 - 2s,\ s \in \mathbf{R}$, and $\pi: x = -3$ intersect, if any exist.

Solution

At the point of intersection, the x-values for the line and the plane will be equal.

Equating the two gives $2 - s = -3$, or $s = 5$. The y- and z-values for the point of intersection can now be found by substituting $s = 5$ into the other two parametric equations. Thus, $y = -1 + 3s = -1 + 3(5) = 14$ and $z = 4 - 2s = 4 - 2(5) = -6$. The point of intersection between L and π is $(-3, 14, -6)$.

Intersection between Two Lines

Thus far, we have discussed the possible intersections between a line and a plane. Next, we consider the possible intersection between two lines.

There are four cases to consider for the intersection of two lines in R^3.

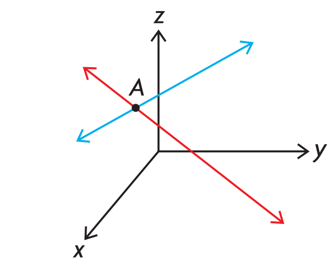
Intersecting Lines
Case 1: The lines are not parallel and intersect at a single point.
Case 2: The lines are coincident, meaning that the two given lines are identical. There are an infinite number of points of intersection.

Non-intersecting Lines
Case 3: The two lines are parallel, and there is no intersection.
Case 4: The two lines are not parallel, and there is no intersection. The lines in this case are called **skew lines**. (Skew lines do not exist in R^2, only in R^3.)

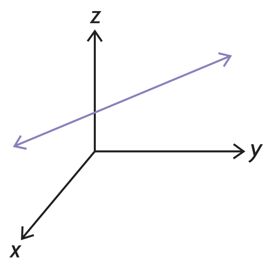
These four cases are shown in the diagram below.

Intersecting Lines

Case 1: Intersecting Lines at a Point

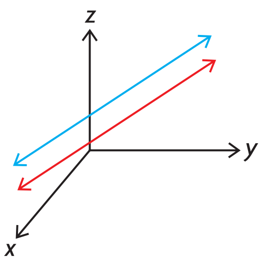


Case 2: Coincident Lines

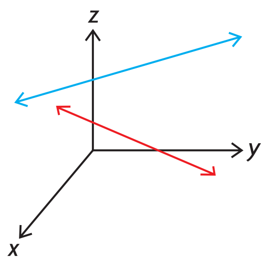


Non-intersecting Lines

Case 3: Parallel Lines



Case 4: Skew Lines



EXAMPLE 5 Selecting a strategy to determine the intersection of two lines in R^3

For $L_1: \vec{r} = (-3, 1, 4) + s(-1, 1, 4), s \in \mathbf{R}$, and $L_2: \vec{r} = (1, 4, 6) + t(-6, -1, 6), t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

Before calculating the coordinates of points of intersection between the two lines, we note that these lines are not parallel to each other because their direction vectors are not scalar multiples of each other—that is, $(-1, 1, 4) \neq k(-6, -1, 6)$. This indicates that these lines either intersect each other exactly once or are skew lines. If these lines intersect, there must be a single point that is on both lines. To use this idea, the vector equations for L_1 and L_2 must be converted to parametric form.

L_1	L_2
$x = -3 - s$	$x = 1 - 6t$
$y = 1 + s$	$y = 4 - t$
$z = 4 + 4s$	$z = 6 + 6t$

We can now select any two of the three equations from each line and equate them. Comparing the x and y components gives $-3 - s = 1 - 6t$ and $1 + s = 4 - t$. Rearranging and simplifying gives

① $s - 6t = -4$

② $s + t = 3$

Subtracting ② from ① yields the following:

$$-7t = -7$$
$$t = 1$$

Substituting $t = 1$ into equation ①,

$$s - 6(1) = -4$$
$$s = 2$$

We find $s = 2$ and $t = 1$. These two values can now be substituted into the parametric equations to find the corresponding values of x, y, and z.

L_1	L_2
$x = -3 - 2 = -5$	$x = 1 - 6(1) = -5$
$y = 1 + 2 = 3$	$y = 4 - 1 = 3$
$z = 4 + 4(2) = 12$	$z = 6 + 6(1) = 12$

Since we found that substituting $s = 2$ and $t = 1$ into the corresponding parametric equations gives the same values of x, y, and z, the point of intersection is $(-5, 3, 12)$.

It is important to understand that when finding the points of intersection between any pair of lines, the parametric values must be substituted back into the original equations to check that a consistent result is obtained. In other words, $s = 2$ and $t = 1$ must give the same point for each line. In this case, there were consistent values, and so we can be certain that the point of intersection is $(-5, 3, 12)$.

In the next example, we will demonstrate the importance of checking for consistency to find the possible point of intersection.

EXAMPLE 6 Connecting the solution to a system of equations to the case of skew lines

For $L_1: x = -1 + s, y = 3 + 4s, z = 6 + 5s, s \in \mathbf{R}$, and $L_2: x = 4 - t, y = 17 + 2t, z = 30 - 5t, t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

We use the same approach as in the previous example. In this example, we'll start by equating corresponding y- and z-coordinates.

L_1	L_2
$x = -1 + s$	$x = 4 - t$
$y = 3 + 4s$	$y = 17 + 2t$
$z = 6 + 5s$	$z = 30 - 5t$

Comparing y- and z-values, we get $3 + 4s = 17 + 2t$ and $6 + 5s = 30 - 5t$. Rearranging and simplifying gives

① $4s - 2t = 14$

② $5s + 5t = 24$

③ $10s - 5t = 35 \qquad \frac{5}{2} \times$ ①

$$15s = 59 \qquad ② + ③$$

$$s = \frac{59}{15}$$

If $s = \frac{59}{15}$ is substituted into either equation ① or equation ②, we obtain the value of t.

Substituting into equation ①,

$$4\left(\frac{59}{15}\right) - 2t = 14$$

$$\frac{236}{15} - \frac{210}{15} = 2t$$

$$t = \frac{13}{15}$$

We found that $s = \frac{59}{15}$ and $t = \frac{13}{15}$. These two values can now be substituted back into the parametric equations to find the values of x, y, and z.

L_1	L_2
$x = -1 + \frac{59}{15} = \frac{44}{15}$	$x = 4 - \frac{13}{15} = \frac{47}{15}$
$y = 3 + 4\left(\frac{59}{15}\right) = \frac{281}{15}$	$y = 17 + 2\left(\frac{13}{15}\right) = \frac{281}{15}$
$z = 6 + 5\left(\frac{59}{15}\right) = \frac{77}{3}$	$z = 30 - 5\left(\frac{13}{15}\right) = \frac{77}{3}$

For these lines to intersect at a point, we must obtain equal values for each coordinate. From observation, we can see that the x-coordinates are different, which implies that these lines do not intersect. Since the two given lines do not intersect and have different direction vectors, they must be skew lines.

IN SUMMARY

Key Ideas

- Line and plane intersections can occur in three different ways.

  Case 1: The line L intersects the plane π at exactly one point, P.

  Case 2: The line L does *not* intersect the plane and is parallel to the plane π. In this case, there are no points of intersection and solving the system of equations results in an equation that has no solution ($0 \times$ variable $=$ a nonzero number).

  Case 3: The line L lies on the plane π. In this case, there are an infinite number of points of intersection between the line and the plane, and solving the system of equations results in an equation with an infinite number of solutions ($0 \times$ variable $= 0$).

- Line and line intersections can occur in four different ways.

  Case 1: The lines intersect at a single point.

  Case 2: The two lines are parallel, and there is no intersection.

  Case 3: The two lines are not parallel and do not intersect. The lines in this case are called *skew lines*.

  Case 4: The two lines are parallel and coincident. They are the same line.

Exercise 9.1

PART A

1. Tiffany is given the parametric equations for a line L and the Cartesian equation for a plane π and is trying to determine their point of intersection. She makes a substitution and gets $(1 + 5s) - 2(2 + s) - 3(-3 + s) - 6 = 0$.

   a. Give a possible equation for both the line and the plane.

   b. Finish the calculation, and describe the nature of the intersection between the line and the plane.

2. a. If a line and a plane intersect, in how many different ways can this occur? Describe each case.

   b. It is only possible to have zero, one, or an infinite number of intersections between a line and a plane. Explain why it is not possible to have a finite number of intersections, other than zero or one, between a line and a plane.

C 3. A line has the equation $\vec{r} = s(1, 0, 0)$, $s \in \mathbf{R}$, and a plane has the equation $y = 1$.

   a. Describe the line.

   b. Describe the plane.

c. Sketch the line and the plane.

d. Describe the nature of the intersection between the line and the plane.

PART B

4. For each of the following, show that the line lies on the plane with the given equation. Explain how the equation that results implies this conclusion.

   a. $L: x = -2 + t, y = 1 - t, z = 2 + 3t, t \in \mathbf{R}; \pi : x + 4y + z - 4 = 0$

   b. $L: \vec{r} = (1, 5, 6) + t(1, -2, -2), t \in \mathbf{R}; \pi : 2x - 3y + 4z - 11 = 0$

5. For each of the following, show that the given line and plane do not intersect. Explain how the equation that results implies there is no intersection.

   a. $L: \vec{r} = (-1, 1, 0) + s(-1, 2, 2), s \in \mathbf{R}; \pi : 2x - 2y + 3z - 1 = 0$

   b. $L: x = 1 + 2t, y = -2 + 5t, z = 1 + 4t, t \in \mathbf{R};$
   $\pi : 2x - 4y + 4z - 13 = 0$

6. Verify your results for question 5 by showing that the direction vector of the line and the normal for the plane meet at right angles, and the given point on the line does not lie on the plane.

7. For the following, determine points of intersection between the given line and plane, if any exist:

   a. $L: \vec{r} = (-1, 3, 4) + p(6, 1, -2), p \in \mathbf{R}; \pi : x + 2y - z + 29 = 0$

   b. $L: \dfrac{x - 1}{4} = \dfrac{y + 2}{-1} = z - 3; \pi : 2x + 7y + z + 15 = 0$

8. Determine points of intersection between the following pairs of lines, if any exist:

   a. $L_1: \vec{r} = (3, 1, 5) + s(4, -1, 2), s \in \mathbf{R};$
   $L_2: x = 4 + 13t, y = 1 - 5t, z = 5t, t \in \mathbf{R}$

   b. $L_3: \vec{r} = (3, 7, 2) + m(1, -6, 0), m \in \mathbf{R};$
   $L_4: \vec{r} = (-3, 2, 8) + s(7, -1, -6), s \in \mathbf{R}$

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9. Determine which of the following pairs of lines are skew lines:

   a. $\vec{r} = (-2, 3, 4) + p(6, -2, 3), p \in \mathbf{R};$
   $\vec{r} = (-2, 3, -4) + q(6, -2, 11), q \in \mathbf{R}$

   b. $\vec{r} = (4, 1, 6) + t(1, 0, 4), t \in \mathbf{R}; \vec{r} = (2, 1, -8) + s(1, 0, 5), s \in \mathbf{R}$

   c. $\vec{r} = (2, 2, 1) + m(1, 1, 1), m \in \mathbf{R};$
   $\vec{r} = (-2, 2, 1) + p(3, -1, -1), p \in \mathbf{R}$

   d. $\vec{r} = (9, 1, 2) + m(5, 0, 4), m \in \mathbf{R}; \vec{r} = (8, 2, 3) + s(4, 1, -2), s \in \mathbf{R}$

10. The line with the equation $\vec{r} = (-3, 2, 1) + s(3, -2, 7), s \in \mathbf{R}$, intersects the z-axis at the point $Q(0, 0, q)$. Determine the value of q.

11. a. Show that the lines $L_1: \vec{r} = (-2, 3, 4) + s(7, -2, 2)$, $s \in \mathbf{R}$, and $L_2: \vec{r} = (-30, 11, -4) + t(7, -2, 2)$, $t \in \mathbf{R}$, are coincident by writing each line in parametric form and comparing components

b. Show that the point $(-2, 3, 4)$ lies on L_2. How does this show that the lines are coincident?

12. The lines $\vec{r} = (-3, 8, 1) + s(1, -1, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (1, 4, 2) + t(-3, k, 8)$, $t \in \mathbf{R}$, intersect at a point.

a. Determine the value of k.

b. What are the coordinates of the point of intersection?

A 13. The line $\vec{r} = (-8, -6, -1) + s(2, 2, 1)$, $s \in \mathbf{R}$, intersects the xz- and yz-coordinate planes at the points A and B, respectively. Determine the length of line segment AB.

14. The lines $\vec{r} = (2, 1, 1) + p(4, 0, -1)$, $p \in \mathbf{R}$, and $\vec{r} = (3, -1, 1) + q(9, -2, -2)$, $q \in \mathbf{R}$, intersect at the point A.

a. Determine the coordinates of point A.

b. What is the distance from point A to the xy-plane?

T 15. The lines $\vec{r} = (-1, 3, 2) + s(5, -2, 10)$, $s \in \mathbf{R}$, and $\vec{r} = (4, -1, 1) + t(0, 2, 11)$, $t \in \mathbf{R}$, intersect at point A.

a. Determine the coordinates of point A.

b. Determine the vector equation for the line that is perpendicular to the two given lines and passes through point A.

16. a. Sketch the lines $L_1: \vec{r} = p(0, 1, 0)$, $p \in \mathbf{R}$, and $L_2: \vec{r} = q(0, 1, 1)$, $q \in \mathbf{R}$.

b. At what point do these lines intersect?

c. Verify your conclusion for part b. algebraically.

PART C

17. a. Show that the lines $\frac{x}{1} = \frac{y-7}{-8} = \frac{z-1}{2}$ and $\frac{x-4}{3} = \frac{z-1}{-2}$, $y = -1$, lie on the plane with equation $2x + y + 3z - 10 = 0$.

b. Determine the point of intersection of these two lines.

18. A line passing through point $P(-4, 0, -3)$ intersects the two lines with equations $L_1: \vec{r} = (1, 1, -1) + s(1, 1, 0)$, $s \in \mathbf{R}$, and $L_2: \vec{r} = (0, 1, 3) + t(-2, 1, 3)$, $t \in \mathbf{R}$. Determine a vector equation for this line.

Section 9.2—Systems of Equations

To solve problems in real-life situations, we often need to solve systems of linear equations. Thus far, we have seen systems of linear equations in a variety of different contexts dealing with lines and planes. The following is a typical example of a system of two equations in two unknowns:

① $2x + y = -9$

② $x + 2y = -6$

Each of the equations in this system is a linear equation. A linear equation is an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$, where $a_1, a_2, a_3, \dots, a_n$ and b are real numbers with the variables $x_1, x_2, x_3, \dots, x_n$ being the unknowns. Typical examples of linear equations are $y = 2x - 3$, $x + 4y = 9$, and $x + 3y - 2z - 2 = 0$. All the variables in each of these equations are raised to the first power only (degree of one). Linear equations do not include any products or powers of variables, and there are no trigonometric, logarithmic, or exponential functions making up part of the equation. Typical examples of nonlinear equations are $x - 3y^2 = 3$, $2x - xyz = 4$, and $y = \sin 2x$.

A system of linear equations is a set of one or more linear equations. When we solve a system of linear equations, we are trying to find values that will simultaneously satisfy the unknowns in each of the equations. In the following example, we consider a system of two equations in two unknowns and possible solutions for this system.

EXAMPLE 1 **Reasoning about the solutions to a system of two equations in two unknowns**

The number of solutions to the following system of equations depends on the value(s) of a and b. Determine values of a and b for which this system has no solutions, an infinite number of solutions, and one solution.

① $x + 4y = a$

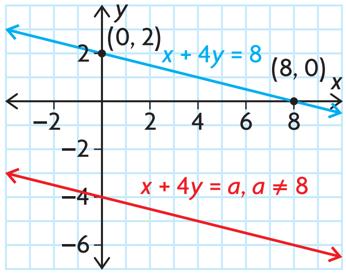
② $x + by = 8$

Solution

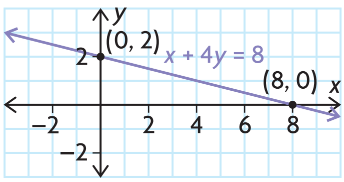
Each of the equations in this system represents a line in R^2. For these two lines, there are three cases to consider, each depending on the values of a and b.

Case 1: These equations represent two parallel and non-coincident lines. If these lines are parallel, they must have the same slope, implying that $b = 4$.

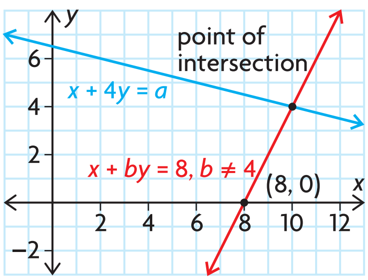
This means that the second equation is $x + 4y = 8$, and the slope of each line is $-\frac{1}{4}$. If $a \neq 8$, this implies that the two lines are parallel and have different equations. Since the lines would be parallel and not intersect, there is no solution to this system when $b = 4$ and $a \neq 8$.



Case 2: These equations represent two parallel and coincident lines. This means that the two equations must be equivalent. If $a = 8$ and $b = 4$, then both equations are identical and this system would be reduced to finding values of x and y that satisfy the equation $x + 4y = 8$. Since there are an infinite number of points that satisfy this equation, the original system will have an infinite number of solutions.



Case 3: These two equations represent two intersecting, non-coincident lines. The third possibility for these two lines is that they intersect at a single point in R^2. These lines will intersect at a single point if they are not parallel—that is, if $b \neq 4$. In this case, the solution is the point of intersection of these lines.



This system of linear equations is typical in that it can only have zero, one, or an infinite number of solutions. In general, it is not possible for any system of linear equations to have a finite number of solutions greater than one.

Number of Solutions to a Linear System of Equations

A linear system of equations can have zero, one, or an infinite number of solutions.

In Example 2, the idea of equivalent systems is introduced as a way of understanding how to solve a system of equations. Equivalent systems of equations are defined in the following way:

Definition of Equivalent Systems

Two systems of equations are defined as equivalent if every solution to one system is also a solution to the second system of equations, and vice versa.

The idea of equivalent systems is important because, when solving a system of equations, what we are attempting to do is create a system of equations that is easier to solve than the previous system. To construct an equivalent system of equations, the new system is obtained in a series of steps using a set of well-defined operations. These operations are referred to as **elementary operations**.

Elementary Operations Used to Create Equivalent Systems

1. Multiply an equation by a nonzero constant.
2. Interchange any pair of equations.
3. Add a nonzero multiple of one equation to a second equation to replace the second equation.

In previous courses, when we solved systems of equations, we often multiplied two equations by different constants and then added or subtracted to eliminate variables. Although these kinds of operations can be used algebraically to solve systems, elementary operations are used because of their applicability in higher-level mathematics.

The use of elementary operations to create equivalent systems is illustrated in the following example.

EXAMPLE 2 Using elementary operations to solve a system of two equations in two unknowns

Solve the following system of equations:

① $2x + y = -9$

② $x + 2y = -6$

Solution

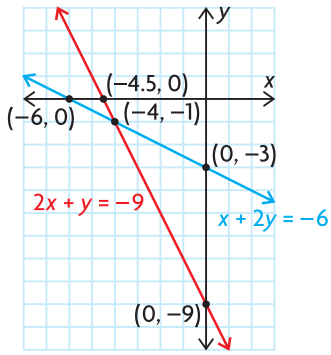
1: Interchange equations ① and ②.

① $x + 2y = -6$

② $2x + y = -9$

The equations have been interchanged to make the coefficient of x in the first equation equal to 1. This is always a good strategy when solving systems of linear equations.

This original system of equations is illustrated in the following diagram.



After Step 1

2: Multiply equation ① by -2, and then add equation ② to eliminate the variable x from the second equation to create equation ③. Note that the coefficient of the x-term in the new equation is 0.

① $x + 2y = -6$

③ $0x - 3y = 3$ $\quad -2 \times$ ① + ②

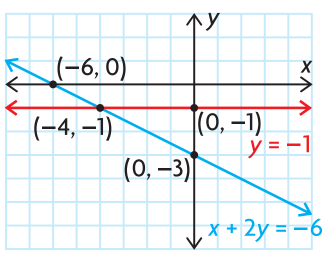
When solving a system of equations, the elementary operations that are used are specified beside the newly created equation.

3: Multiply each side of equation ③ by $-\frac{1}{3}$ to obtain a new equation that is labelled equation ④.

① $x + 2y = -6$

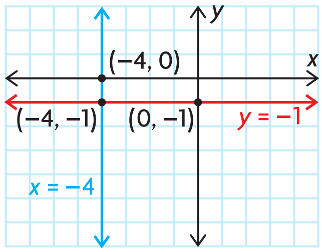
④ $0x + y = -1$ $\quad -\dfrac{1}{3} \times$ ③

This new system of equations is illustrated in the diagram below.



After Step 3

The new system of equations that we produced is easier to solve than the original system. If we substitute $y = -1$ into equation ①, we obtain $x + 2(-1) = -6$ or $x = -4$.
The solution to this system of equations is $x = -4$, $y = -1$, which is shown in the graph below.



Final system

The equivalence of the systems of equations was illustrated geometrically at different points in the calculations. As each elementary operation is applied, we create an equivalent system such that the two lines always have the point $(-4, -1)$ in common. When we create the various equivalent systems, the solution to each set of equations remains the same. This is what we mean when we use elementary operations to create equivalent systems.

The solution to the original system of equations is $x = -4$ and $y = -1$. This means that these two values of x and y must satisfy each of the given equations. It is easy to verify that these values satisfy each of the given equations. For the first equation, $2(-4) + (-1) = -9$. For the second equation, $-4 + 2(-1) = -6$.

A solution to a system of equations must satisfy each equation in the system for it to be a solution to the overall system. This is demonstrated in the following example.

EXAMPLE 3 Reasoning about the solution to a system of two equations in three unknowns

Determine whether $x = -3$, $y = 5$, and $z = 6$ is a solution to the following system:

① $2x + 3y - 5z = -21$

② $x - 6y + 6z = 8$

Solution

For the given values to be a solution to this system of equations, they must satisfy both equations.

Substituting into the first equation,

$2(-3) + 3(5) - 5(6) = -6 + 15 - 30 = -21$

Substituting into the second equation,

$-3 - 6(5) + 6(6) = -3 - 30 + 36 = 3 \neq 8$

Since the values of x, y, and z do not satisfy both equations, they are not a solution to this system.

If a system of equations has no solutions, it is said to be **inconsistent**. If a system has at least one solution, it is said to be **consistent**.

Consistent and Inconsistent Systems of Equations

A system of equations is consistent if it has either one solution or an infinite number of solutions. A system is inconsistent if it has no solutions.

In the next example, we show how to use elementary operations to solve a system of three equations in three unknowns.

EXAMPLE 4 Using elementary operations to solve a system of three equations in three unknowns

Solve the following system of equations for x, y, and z using elementary operations:

① $x - y + z = 1$

② $2x + y - z = 11$

③ $3x + y + 2z = 12$

Solution

1: Use equation ① to eliminate x from equations ② and ③.

① $x - y + z = 1$

④ $0x + 3y - 3z = 9 \qquad -2 \times ① + ②$

⑤ $0x + 4y - z = 9 \qquad -3 \times ① + ③$

2: Use equations ④ and ⑤ to eliminate y from equation ⑤, and then scale equation ④.

① $x - y + z = 1$

⑥ $0x + y - z = 3 \qquad \frac{1}{3} \times ④$

⑦ $0x + 0y + 3z = -3 \qquad -\frac{4}{3} \times ④ + ⑤$

We can now solve this system by using a method known as **back substitution**. Start by solving for z in equation ⑦, and use this value to solve for y in equation ⑥. From there, we use the values for y and z to solve for x in equation ①.

From equation ⑦, $3z = -3$
$$z = -1$$

If we then substitute into equation ⑥,

$$y - (-1) = 3$$
$$y = 2$$

If $y = 2$ and $z = -1$, these values can now be substituted into equation ① to obtain $x - 2 + (-1) = 1$, or $x = 4$.

Therefore, the solution to this system is $(4, 2, -1)$.

Check:
These values should be substituted into each of the original equations and checked to see that they satisfy each equation.

To solve this system of equations, we used elementary operations and ended up with a triangle of zeros in the lower left part:

$$\begin{aligned} ax + by + cz &= d \\ 0x + ey + fz &= g \\ 0x + 0y + hz &= i \end{aligned}$$

The use of elementary operations to create the lower triangle of zeros is our objective when solving systems of equations. Large systems of equations are solved using computers and elementary operations to eliminate unknowns. This is by far the most efficient and cost-effective method for their solution.

In the following example, we consider a system of equations with different possibilities for its solution.

EXAMPLE 5 Connecting the value of a parameter to the nature of the intersection between two lines in R^2

Consider the following system of equations:

① $x + ky = 4$
② $kx + 4y = 8$

Determine the value(s) of k for which this system of equations has

a. no solutions
b. one solution
c. an infinite number of solutions

Solution

Original System of Equations:

① $x + ky = 4$
② $kx + 4y = 8$

1: Multiply equation ① by $-k$, and add it to equation ② to eliminate x from equation ②.

① $x + ky = 4$

③ $0x - k^2y + 4y = -4k + 8$, $-k \times$ ① + ②

Actual Solution to Problem:
To solve the problem, it is only necessary to deal with the equation $0x - k^2y + 4y = -4k + 8$ to determine the necessary conditions on k.

$$-k^2y + 4y = -4k + 8 \qquad \text{(Factor)}$$

$$y(-k^2 + 4) = -4(k - 2) \qquad \text{(Multiply by } -1\text{)}$$

$$y(k^2 - 4) = 4(k - 2)$$

$$(k - 2)(k + 2)y = 4(k - 2)$$

There are three different cases to consider.

Case 1: $k = 2$
If $k = 2$, this results in the equation $(2 - 2)(2 + 2)y = 4(2 - 2)$, or $0y = 0$.

Since this equation is true for all real values of y, we will have an infinite number of solutions. Substituting $k = 2$ into the original system of equations gives

① $x + 2y = 4$

② $2(x + 2y) = 2(4)$

This system can then be reduced to just a single equation, $x + 2y = 4$, which, as we have seen, has an infinite number of solutions.

Case 2: $k = -2$
If $k = -2$, this equation becomes $(-2 - 2)(-2 + 2)y = 4(-2 - 2)$, or $0y = -16$.

There are no solutions to this equation. Substituting $k = -2$ into the original system of equations gives

① $x - 2y = 4$

② $-2(x - 2y) = -2(-4)$

This system can be reduced to the two equations, $x - 2y = 4$ and $x - 2y = -4$, which are two parallel lines that do not intersect. Thus, there are no solutions.

Case 3: $k \neq \pm 2$
If $k \neq \pm 2$, we get an equation of the form $ay = b$, $a \neq 0$. This equation will always have a unique solution for y, which implies that the original system of equations will have exactly one solution, provided that $k \neq \pm 2$.

IN SUMMARY

Key Idea

- A system of two (linear) equations in two unknowns geometrically represents two lines in R^2. These lines may intersect at zero, one, or an infinite number of points, depending on how the lines are related to each other.

Need to Know

- Elementary operations can be used to solve a system of equations. The operations are defined as follows:
  1. Multiply an equation by a nonzero constant.
  2. Interchange any pair of equations.
  3. Add a multiple of one equation to a second equation to replace the second equation.

  As each elementary operation is applied, we create an equivalent system, which gets progressively easier to solve.
- The solution to a system of equations consists of the values of the variables that satisfy all the equations in the system simultaneously.
- A system of equations is consistent if it has either one solution or an infinite number of solutions. The system is inconsistent if it has no solutions.

Exercise 9.2

PART A

1. Given that k is a nonzero constant, which of the following are linear equations?

   a. $kx - \frac{1}{k}y = 3$

   b. $2 \sin x = kx$

   c. $2^k x + 3y - z = 0$

   d. $\frac{1}{x} - y = 3$

2. a. Create a system of three equations in three unknowns that has $x = -3$, $y = 4$, and $z = -8$ as its solution.

   b. Solve this system of equations using elementary operations.

3. Determine whether $x = -7$, $y = 5$, and $z = \frac{3}{4}$ is a solution to the following systems:

   a. ① $x - 3y + 4z = -19$
   ② $x - 8z = -13$
   ③ $x + 2y = 3$

   b. ① $3x - 2y + 16z = -19$
   ② $3x - 2y = -23$
   ③ $8x - y + 4z = -58$

PART B

4. Solve each system of equations, and state whether the systems given in parts a. and b. are equivalent or not. Explain.

   a. ① $x = -2$
      ② $3y = -9$

   b. ① $3x + 5y = -21$
      ② $\frac{1}{6}x - \frac{1}{2}y = \frac{7}{6}$

K 5. Solve each of the following systems using elementary operations:

   a. ① $2x - y = 11$
      ② $x + 5y = 11$

   b. ① $2x + 5y = 19$
      ② $3x + 4y = 11$

   c. ① $-x + 2y = 10$
      ② $3x + 5y = 3$

C 6. Solve the following systems of equations, and explain the nature of each intersection:

   a. ① $2x + y = 3$
      ② $2x + y = 4$

   b. ① $7x - 3y = 9$
      ② $35x - 15y = 45$

7. Write a solution to each equation using parameters.

   a. $2x - y = 3$

   b. $x - 2y + z = 0$

8. a. Determine a linear equation that has $x = t$, $y = -2t - 11$, $t \in \mathbf{R}$, as its general solution.

   b. Show that $x = 3t + 3$, $y = -6t - 17$, $t \in \mathbf{R}$, is also a general solution to the linear equation found in part a.

9. Determine the value(s) of the constant k for which the following system of equations has

   a. no solutions
   b. one solution
   c. infinitely many solutions

   ① $x + y = 6$
   ② $2x + 2y = k$

10. For the equation $2x + 4y = 11$, determine

    a. the number of solutions
    b. a generalized parametric solution
    c. an explanation as to why it will *not* have any integer solutions

11. a. Solve the following system of equations for x and y:

    ① $x + 3y = a$
    ② $2x + 3y = b$

    b. Explain why this system of equations will always be consistent, irrespective of the values of a and b.

12. Solve each system of equations using elementary operations.

a. ① $x + y + z = 0$
② $x - y = 1$
③ $y - z = -5$

b. ① $2x - 3y + z = 6$
② $x + y + 2z = 31$
③ $x - 2y - z = -17$

c. ① $x + y = 10$
② $y + z = -2$
③ $x + z = -4$

d. ① $\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 14$
② $\frac{x}{4} + \frac{y}{5} + \frac{z}{3} = -21$
③ $\frac{x}{5} + \frac{y}{3} + \frac{z}{4} = 7$

e. ① $2x - y = 0$
② $2y - z = 7$
③ $2z - x = 0$

f. ① $x + y + 2z = 13$
② $2y - 3z = -12$
③ $x - y + 4z = 19$

A 13. A system of equations is given by the lines L_1: $ax + by = p$, L_2: $dx + ey = q$, and L_3: $gx + hy = r$. Sketch the lines under the following conditions:

a. when the system of equations represented by these lines has no solutions

b. when the system of equations represented by these lines has exactly one solution

c. when the system of equations represented by these lines has an infinite number of solutions

T 14. Determine the solution to the following system of equations:

① $x + y + z = a$
② $x + y = b$
③ $y + z = c$

PART C

15. Consider the following system of equations:

① $x + 2y = -1$
② $2x + k^2y = k$

Determine the values of k for which this system of equations has

a. no solutions

b. an infinite number of solutions

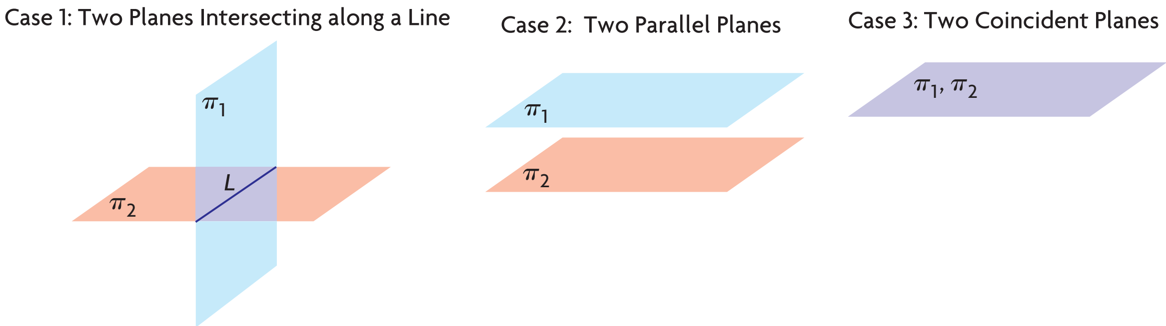
c. a unique solution

Section 9.3—The Intersection of Two Planes

In the previous section, we introduced elementary operations and their use in the solution of systems of equations. In this section, we will again examine systems of equations but will focus specifically on dealing with the intersections of two planes. Algebraically, these are typically represented by a system of two equations in three unknowns.

In our discussion on the intersections of two planes, there are three different cases to be considered, each of which is illustrated below.

Possible Intersections for Two Planes



Case 1: Two planes can intersect along a line. The corresponding system of equations will therefore have an infinite number of solutions.

Case 2: Two planes can be parallel and non-coincident. The corresponding system of equations will have no solutions.

Case 3: Two planes can be coincident and will have an infinite number of solutions.

Solutions for a System of Equations Representing Two Planes

The system of equations corresponding to the intersection of two planes will have either zero solutions or an infinite number of solutions.

It is not possible for two planes to intersect at a single point.

EXAMPLE 1

Reasoning about the nature of the intersection between two planes (Case 2)

Determine the solution to the system of equations $x - y + z = 4$ and $x - y + z = 5$. Discuss how these planes are related to each other.

Solution

Since the two planes have the same normals, $\vec{n_1} = \vec{n_2} = (1, -1, 1)$, this implies that the planes are parallel. Since the equations have different constants on the right side, the equations represent parallel and non-coincident planes. This indicates that there are no solutions to this system because the planes do not intersect.

The corresponding system of equations is

① $x - y + z = 4$

② $x - y + z = 5$

Using elementary operations, the following equivalent system of equations is obtained:

① $\quad x - y + z = 4$

③ $0x + 0y + 0z = 1 \qquad -1 \times$ ① + ②

Since there are no values that satisfy equation ③, there are no solutions to this system, confirming our earlier conclusion.

EXAMPLE 2

Reasoning about the nature of the intersection between two planes (Case 3)

Determine the solution to the following system of equations:

① $\quad x + 2y - 3z = -1$

② $4x + 8y - 12z = -4$

Solution

Since equation ② can be written as $4(x + 2y - 3z) = 4(-1)$, the two equations represent coincident planes. This means that there are an infinite number of values that satisfy the system of equations. The solution to the system of equations can be written using parameters in equation ①. If we let $y = s$ and $z = t$, then $x = -2s + 3t - 1$.

The solution to the system is $x = -2s + 3t - 1$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. This is the equation of a plane, expressed in parametric form. Every point that lies on the plane is a solution to the given system of equations.

If we had solved the system using elementary operations, we would have arrived at the following equivalent system:

① $x + 2y - 3z = -1$

③ $0x + 0y + 0z = 0 \qquad -4 \times ① + ②$

There are an infinite number of ordered triples (x, y, z) that satisfy both equations ① and ③, confirming our earlier conclusion.

The normals of two planes give us important information about their intersection.

Intersection of Two Planes and their Normals

If the planes π_1 and π_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals, we know the following:

1. If $\vec{n_1} = k\vec{n_2}$ for some scalar, k, the planes are coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection. If they are parallel and non-coincident, there are no points of intersection.
2. If $\vec{n_1} \neq k\vec{n_2}$, the two planes intersect in a line. This results in an infinite number of points of intersection.

EXAMPLE 3 Reasoning about the nature of the intersection between two planes (Case 1)

Determine the solution to the following system of equations:

① $x - y + z = 3$

② $2x + 2y - 2z = 3$

Solution

When solving a system involving two planes, it is useful to start by determining the normals for the two planes. The first plane has normal $\vec{n_1} = (1, -1, 1)$, and the second plane has $\vec{n_2} = (2, 2, -2)$. Since these vectors are not scalar multiples of each other, the normals are not parallel, which implies that the two planes intersect. Since the planes intersect and do not coincide, they intersect along a line.

We will use elementary operations to solve the system.

① $x - y + z = 3$

③ $0x + 4y - 4z = -3 \qquad -2 \times ① + ②$

To determine the equation of the line of intersection, a parameter must be introduced. From equation ③, which is written as $4y - 4z = -3$, we start by letting $z = s$. Substituting $z = s$ gives

$$\begin{aligned} 4y - 4s &= -3 \\ 4y &= 4s - 3 \\ y &= s - \frac{3}{4} \end{aligned}$$

Substituting $z = s$ and $y = s - \frac{3}{4}$ into equation ①, we obtain

$$\begin{aligned} x - \left(s - \frac{3}{4}\right) + s &= 3 \\ x &= \frac{9}{4} \end{aligned}$$

Therefore, the line of intersection expressed in parametric form is $x = \frac{9}{4}$, $y = s - \frac{3}{4}$, $z = s$, $s \in \mathbf{R}$.

Check:
To check, we'll substitute into each of the two original equations.

Substituting into equation ①,

$$x - y + z = \frac{9}{4} - \left(s - \frac{3}{4}\right) + s = \frac{9}{4} + \frac{3}{4} - s + s = 3$$

Substituting into equation ②,

$$2x + 2y - 2z = 2\left(\frac{9}{4}\right) + 2\left(s - \frac{3}{4}\right) - 2s = \frac{9}{2} - \frac{3}{2} = 3$$

This confirms our conclusion.

EXAMPLE 4 **Selecting the most efficient strategy to determine the intersection between two planes**

Determine the solution to the following system of equations:

① $2x - y + 3z = -2$

② $x - 3z = 1$

Solution

As in the first example, we note that the first plane has normal $\overrightarrow{n_1} = (2, -1, 3)$ and the second $\overrightarrow{n_2} = (1, 0, -3)$. These normals are not scalar multiples of each other, implying that the two planes have a line of intersection.

To find the line of intersection, it is not necessary to use elementary operations to reduce one of the equations. Since the second equation is missing a y-term, the best approach is to write the second equation using a parameter for z. If $z = s$, then $x = 3s + 1$. Now it is a matter of substituting these parametric values into the first equation and determining y in terms of s. Substituting gives

$$\begin{aligned} 2(3s + 1) - y + 3(s) &= -2 \\ 6s + 2 - y + 3s &= -2 \\ 9s + 4 &= y \end{aligned}$$

The line of intersection is given by the parametric equations $x = 3s + 1$, $y = 9s + 4$, and $z = s$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①,
$2(3s + 1) - (9s + 4) + 3s = 6s + 2 - 9s - 4 + 3s = -2$
Substituting into equation ②, $(3s + 1) - 3s = 1$

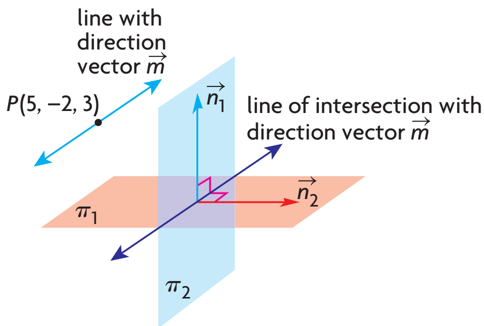
In the next example, we will demonstrate how a problem involving the intersection of two planes can be solved in more than one way.

EXAMPLE 5 Selecting a strategy to solve a problem involving two planes

Determine an equation of a line that passes through the point $P(5, -2, 3)$ and is parallel to the line of intersection of the planes π_1: $x + 2y - z = 6$ and π_2: $y + 2z = 1$.

Solution

Method 1:
Since the required line is parallel to the line of intersection of the planes, then the direction vectors for both of these lines must be parallel. Since the line of intersection is contained in both planes, its direction vector must then be perpendicular to the normals of each plane.



If $\vec{m}$ represents the direction vector of the required line, and it is perpendicular to $\vec{n_1} = (1, 2, -1)$ and $\vec{n_2} = (0, 1, 2)$, then we can choose $\vec{m} = \vec{n_1} \times \vec{n_2}$.

$$\begin{aligned} \text{Thus, } \vec{m} &= (1, 2, -1) \times (0, 1, 2) \\ &= (2(2) - (-1)(1), -1(0) - 1(2), 1(1) - 2(0)) \\ &= (5, -2, 1) \end{aligned}$$

Thus, the required line that passes through $P(5, -2, 3)$ and has direction vector $\vec{m} = (5, -2, 1)$ has parametric equations $x = 5 + 5t$, $y = -2 - 2t$, and $z = 3 + t$, $t \in \mathbf{R}$.

Method 2:
We start by finding the equation of the line of intersection between the two planes. In equation ②, if $z = t$, then $y = -2t + 1$ by substitution. Substituting these values into equation ① gives

$$x + 2(-2t + 1) - t = 6$$
$$x - 4t + 2 - t = 6$$
$$x = 5t + 4$$

The line of intersection has $x = 5t + 4$, $y = -2t + 1$, and $z = t$ as its parametric equations $t \in \mathbf{R}$. Since the direction vector for this line is $(5, -2, 1)$, we can choose the direction vector for the required line to also be $(5, -2, 1)$.

The equation for the required line is $x = 5 + 5t$, $y = -2 - 2t$, $z = 3 + t$, $t \in \mathbf{R}$.

IN SUMMARY

Key Ideas

- A system of two (linear) equations in three unknowns geometrically represents two planes in R^3. These planes may intersect at zero points or an infinite number of points, depending on how the planes are related to each other.

  Case 1: Two planes can intersect along a line and will therefore have an infinite number of points of intersection.

  Case 2: Two planes can be parallel and non-coincident. In this case, there are no points of intersection.

  Case 3: Two planes can be coincident and will have an infinite number of points of intersection.

Need to Know

- If the normals of two planes are known, examining how these are related to each other provides information about how the two planes are related.

  If planes π_1 and π_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals, we know the following:

  1. If $\vec{n_1} = k\vec{n_2}$ for some scalar k, the planes are either coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection, and if they are parallel and non-coincident, there are no points of intersection.
  2. If $\vec{n_1} \neq k\vec{n_2}$ for some scalar k, the two planes intersect in a line. This results in an infinite number of points of intersection.

Exercise 9.3

PART A

1. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = 1$
   ③ $0x + 0y + 0z = 3$

   a. Explain what this equivalent system means.
   b. Give an example of a system of equations that might lead to this solution.

2. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $2x - y + 2z = 1$
   ③ $0x + 0y + 0z = 0$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

C 3. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = -1$
   ③ $0x + 0y + 2z = -4$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

PART B

K 4. Consider the following system of equations:

   ① $2x + y + 6z = p$
   ② $x + my + 3z = q$

   a. Determine values of m, p, and q such that the two planes are coincident. Are these values unique? Explain.
   b. Determine values of m, p, and q such that the two planes are parallel and not coincident. Are these values unique? Explain.
   c. A value of m such that the two planes intersect at right angles. Is this value unique? Explain.
   d. Determine values of m, p, and q such that the two planes intersect at right angles. Are these values unique? Explain.

5. Consider the following system of equations:

① $x + 2y - 3z = 0$

② $y + 3z = 0$

a. Solve this system of equations by letting $z = s$.

b. Solve this system of equations by letting $y = t$.

c. Show that the solution you found in part a. is the same as the solution you found in part b.

6. The following systems of equations involve two planes. State whether the planes intersect, and, if they do intersect, specify if their intersection is a line or a plane.

a. ① $x + y + z = 1$
② $2x + 2y + 2z = 2$

b. ① $2x - y + z + 1 = 0$
② $2x - y + z + 2 = 0$

c. ① $x - y + 2z = 2$
② $x + y + 2z = -2$

d. ① $x + y + 2z = 4$
② $x - y = 6$

e. ① $2x - y + 2z = 2$
② $-x + 2y + z = 1$

f. ① $x - y + 2z = 0$
② $z = 4$

7. Determine the solution to each system of equations in question 6.

8. A system of equations is given as follows:

① $x + y + 2z = 1$

② $kx + 2y + 4z = k$

a. For what value of k does the system have an infinite number of solutions? Determine the solution to the system for this value of k.

b. Is there any value of k for which the system does not have a solution? Explain.

9. Determine the vector equation of the line that passes through $A(-2, 3, 6)$ and is parallel to the line of intersection of the planes π_1: $2x - y + z = 0$ and π_2: $y + 4z = 0$.

A 10. For the planes $2x - y + 2z = 0$ and $2x + y + 6z = 4$, show that their line of intersection lies on the plane with equation $5x + 3y + 16z - 11 = 0$.

T 11. The line of intersection of the planes π_1: $2x + y - 3z = 3$ and π_2: $x - 2y + z = -1$ is L.

a. Determine parametric equations for L.

b. If L meets the xy-plane at point A and the z-axis at point B, determine the length of line segment AB.

PART C

12. Determine the Cartesian equation of the plane that is parallel to the line with equation $x = -2y = 3z$ and that contains the line of intersection of the planes with equations $x - y + z = 1$ and $2y - z = 0$.

Mid-Chapter Review

1. Determine the point of intersection between the line $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$, and each of the following planes:
   a. the xy-plane
   b. the xz-plane
   c. the yz-plane
2. $A(2, 1, 3)$, $B(3, -2, 5)$, and $C(-8, -5, 7)$ are three points in R^3 that form a triangle.
   a. Determine the parametric equations for any two of the three medians. (A median is a line drawn from one vertex to the midpoint of the opposite side.)
   b. Determine the point of intersection of the two medians you found in part a.
   c. Determine the equation of the third median for this triangle.
   d. Verify that the point of intersection you found in part b. is a point on the line you found in part c.
   e. State the coordinates of the point of intersection of the three medians.
3. a. Determine an equation for the line of intersection of the planes $5x + y + 2z + 15 = 0$ and $4x + y + 2z + 8 = 0$.
   b. Determine an equation for the line of intersection of the planes $4x + 3y + 3z - 2 = 0$ and $5x + 2y + 3z + 5 = 0$.
   c. Determine the point of intersection between the line you found in part a. and the line you found in part b.
4. a. Determine the line of intersection of the planes π_1: $3x + y + 7z + 3 = 0$ and π_2: $x - 13y - 3z - 38 = 0$.
   b. Determine the line of intersection of the planes π_3: $x - 3y + z + 11 = 0$ and π_4: $6x - 13y + 8z - 28 = 0$.
   c. Show that the lines you found in parts a. and b. do not intersect.
5. Consider the following system of equations:

   ① $x + ay = 9$

   ② $ax + 9y = -27$

   Determine the value(s) of a for which the system of equations has
   a. no solution
   b. an infinite number of solutions
   c. one solution

6. Show that $\frac{x - 11}{2} = \frac{y - 4}{-4} = \frac{z - 27}{5}$ and $x = 0, y = 1 - 3t, z = 3 + 2t$, $t \in \mathbf{R}$, are skew lines.

7. a. Determine the intersection of the lines
   $(x - 3, y - 20, z - 7) = t(2, -4, 5), t \in \mathbf{R}$, and $\frac{x - 5}{2} = y - 2 = \frac{z + 4}{-3}$.
   b. What conclusion can you make about these lines?

8. Determine the point of intersection between the lines
   $x = 1 + 2s, y = 4 - s, z = -3s, s \in \mathbf{R}$, and
   $x = -3, y = t + 3, z = 2t, t \in \mathbf{R}$.

9. Determine the point of intersection for each pair of lines.
   a. $\vec{r} = (5, 1, 7) + s(2, 0, 5), s \in \mathbf{R}$, and
      $\vec{r} = (-1, -1, 3) + t(4, 2, -1), t \in \mathbf{R}$
   b. $\vec{r} = (2, -1, 3) + s(5, -1, 6), s \in \mathbf{R}$, and
      $\vec{r} = (-8, 1, -9) + t(5, -1, 6), t \in \mathbf{R}$

10. You are given a pair of vector equations that both represent lines in R^3.
    a. Explain all the possible ways that these lines could be related to each other. Support your explanation with diagrams.
    b. Explain how you could use the equations you are given to help you identify which of the situations you described in part a. you are dealing with.

11. a. Explain when a line and a plane can have an infinite number of points of intersection.
    b. Give an example of a pair of vector equations (one for a line and one for a plane) that have an infinite number of points of intersection.

12. Use elementary operations to solve each system of equations.
    a. ① $2x + 3y = 30$
       ② $x - 2y = -13$
    b. ① $x + 4y - 3z + 6 = 0$
       ② $2x + 8y - 6z + 11 = 0$
    c. ① $x - 3y - 2z = -9$
       ② $2x - 5y + z = 3$
       ③ $-3x + 6y + 2z = 8$

13. For the system of equations given in parts a. and b. of question 12, describe the corresponding geometrical representation.

14. L is the line of intersection of planes $x - y = 1$ and $y + z = -3$, and L_1 is the line of intersection of the planes $y - z = 0$ and $x = -\frac{1}{2}$.
    a. Determine the point of intersection of L and L_1.
    b. Determine the angle between the lines of intersection.
    c. Determine the Cartesian equation of the plane that contains the point you found in part a. and the two lines of intersection.

Section 9.4—The Intersection of Three Planes

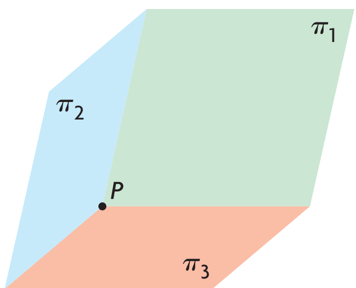
In the previous section, we discussed the intersection of two planes. In this section, we will extend these ideas and consider the intersection of three planes. Algebraically, the three planes are typically represented by a system of three linear equations in three unknowns.

First we will consider consistent systems. Later in the section, we will consider inconsistent systems.

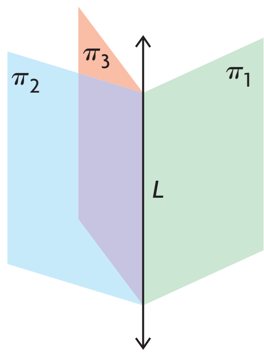
Consistent Systems for Three Equations Representing Three Planes

There are four cases that should be considered for the intersection of three planes. These four cases, which all result in one or more points of intersection between all three planes, are shown below.

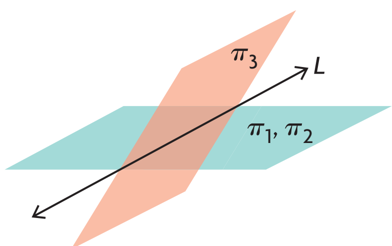
Case 1: P is the point of intersection of three planes.



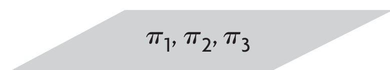
Case 2a: L is the line of intersection of three planes.



Case 2b: L is the line of intersection of two coincident planes and a third plane not parallel to the coincident planes.



Case 3: The plane of intersection of three coincident planes is the plane itself.



Possible Intersections for Three Planes

A description is given below for each situation represented in the diagram on the previous page.

Case 1: There is just one solution to the corresponding system of equations. This is a single point. The coordinates of the point of intersection will satisfy each of the three equations.

This case can be visualized by looking at the ceiling in a rectangular room. The point where the plane of the ceiling meets two walls represents the point of intersection of three planes. Although planes are not usually at right angles to each other and they extend infinitely far in all directions, this gives some idea of how planes can intersect at a point.

Case 2: There are an infinite number of solutions to the related system of equations. Geometrically, this corresponds to a line, and the solution is given in terms of one parameter. There are two sub-cases to consider.

Case 2a: The three planes intersect along a line and are mutually non-coincident.

Case 2b: Two planes are coincident, and the third plane cuts through these two planes intersecting along a line.

Case 3: Three planes are coincident, and there are an infinite number of solutions to the related system of equations. The number of solutions corresponds to the infinite number of points on a plane, and the solution is given in terms of two parameters. In this case, there are three coincident planes that have identical equations or can be reduced to three equivalent equations.

In the following examples, we use elementary operations to determine the solution of three equations in three unknowns.

EXAMPLE 1

Using elementary operations to solve a system of three equations in three unknowns

Determine the intersection of the three planes with the equations $x - y + z = -2$, $2x - y - 2z = -9$, and $3x + y - z = -2$.

Solution

For the intersection of the three planes, we must find the solution to the following system of equations:

① $x - y + z = -2$

② $2x - y - 2z = -9$

③ $3x + y - z = -2$

1: Create two new equations, ④ and ⑤, each containing an x-term with a coefficient of 0.

To determine the required intersection, we use elementary operations and solve the system of equations as shown earlier.

$$\begin{array}{lll} ① & x - y + z = -2 & \\ ④ & 0x + y - 4z = -5 & -2 \times ① + ② \\ ⑤ & 0x + 4y - 4z = 4 & -3 \times ① + ③ \end{array}$$

2: We create equation ⑥ by eliminating y from equation ⑤.

$$\begin{array}{lll} ① & x - y + z = -2 & \\ ④ & 0x + y - 4z = -5 & \\ ⑥ & 0x + 0y + 12z = 24 & -4 \times ④ + ⑤ \end{array}$$

This equivalent system can now be solved by first solving equation ⑥ for z.

Thus, $12z = 24$

$z = 2$

If we use the method of back substitution, we can substitute into equation ④ and solve for y.

Substituting into equation ④,

$y - 4(2) = -5$

$y = 3$

If we now substitute $y = 3$ and $z = 2$ into equation ①, we obtain the value of x.

$x - 3 + 2 = -2$

$x = -1$

Thus, the three planes intersect at the point with coordinates $(-1, 3, 2)$.

Check:
Substituting into equation ①, $x - y + z = -1 - 3 + 2 = -2$.

Substituting into equation ②, $2x - y - 2z = 2(-1) - 3 - 2(2) = -9$.

Substituting into equation ③, $3x + y - z = 3(-1) + 3 - 2 = -2$.

Checking each of the equations confirms the solution.

Two of the other possibilities involving consistent systems are demonstrated in the next two examples.

EXAMPLE 2 Selecting a strategy to determine the intersection of three planes

Determine the solution to the following system of equations:

$$\text{①}\quad 2x - y + z = 1$$
$$\text{②}\quad 3x - 5y + 4z = 3$$
$$\text{③}\quad 3x + 2y - z = 0$$

Solution

In this situation, there is not a best way to solve the system. Because the coefficients of x for two of the equations are equal, however, the computation might be easier if we arrange them as follows (although, in situations like this, it is often a matter of individual preference).

$$\text{①}\quad 3x + 2y - z = 0 \qquad \textbf{(Interchange equations ① and ③)}$$
$$\text{②}\quad 3x - 5y + 4z = 3$$
$$\text{③}\quad 2x - y + z = 1$$

Again, we try to create a zero for the coefficient of x in two of the equations.

Applying elementary operations gives the following system of equations:

$$\text{①}\quad 3x + 2y - z = 0$$
$$\text{④}\quad 0x - 7y + 5z = 3 \qquad -1 \times \text{①} + \text{②}$$
$$\text{⑤}\quad 0x - \frac{7}{3}y + \frac{5}{3}z = 1 \qquad -\frac{2}{3} \times \text{①} + \text{③}$$

Before proceeding with further computations, we should observe that equations ④ and ⑤ are scalar multiples of each other and that, if equation ⑤ is multiplied by 3, there will be two identical equations.

$$\text{①}\quad 3x + 2y - z = 0$$
$$\text{④}\quad 0x - 7y + 5z = 3$$
$$\text{⑥}\quad 0x - 7y + 5z = 3 \qquad 3 \times \text{⑤}$$

By using elementary operations again, we create the following equivalent system:

$$\text{①}\quad 3x + 2y - z = 0$$
$$\text{④}\quad 0x - 7y + 5z = 3$$
$$\text{⑦}\quad 0x + 0y + 0z = 0 \qquad -1 \times \text{④} + \text{⑥}$$

Equation ⑦, in conjunction with equations ① and ④, indicates that this system has an infinite number of solutions. To solve this system, we let $z = t$ and solve for y in equation ④.

Thus, $-7y = -5t + 3$. Dividing by -7, we get $y = \frac{5}{7}t - \frac{3}{7}$.

We determine the parametric equation for x by substituting in equation ①.
Substituting $z = t$ and $y = \frac{5}{7}t - \frac{3}{7}$ into equation ① gives

$$3x + 2\left(\frac{5}{7}t - \frac{3}{7}\right) - t = 0$$

$$3x + \frac{3}{7}t - \frac{6}{7} = 0$$

$$x = -\frac{1}{7}t + \frac{2}{7}$$

Therefore, the solution to this system is $x = -\frac{1}{7}t + \frac{2}{7}$, $y = \frac{5}{7}t - \frac{3}{7}$, and $z = t$.

To help simplify the verification, we will remove the fractions from the *direction numbers* of this line by multiplying them by 7. (Recall that we cannot multiply the points by 7.)

In simplified form, the solution to the system of equations is
$x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$.

Check:
Substituting into equation ①,

$$3\left(-t + \frac{2}{7}\right) + 2\left(5t - \frac{3}{7}\right) - 7t = -3t + \frac{6}{7} + 10t - \frac{6}{7} - 7t = 0$$

Substituting into equation ②,

$$3\left(-t + \frac{2}{7}\right) - 5\left(5t - \frac{3}{7}\right) + 4(7t) = -3t + \frac{6}{7} - 25t + \frac{15}{7} + 28t = 3$$

Substituting into equation ③,

$$2\left(-t + \frac{2}{7}\right) - \left(5t - \frac{3}{7}\right) + 7t = -2t + \frac{4}{7} - 5t + \frac{3}{7} + 7t = 1$$

The solution to the system of equations is a line with parametric equations $x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$. This is a line that has direction vector $\vec{m} = (-1, 5, 7)$ and passes through the point $\left(\frac{2}{7}, -\frac{3}{7}, 0\right)$.

It is useful to note that the normals for these three planes are $\vec{n_1} = (3, 2, -1)$, $\vec{n_2} = (3, -5, 4)$, and $\vec{n_3} = (2, -1, 1)$. Because none of these normals are collinear, this situation corresponds to Case 2a.

EXAMPLE 3 More on solving a consistent system of equations

Determine the solution to the following system of equations:

① $2x + y + z = 1$

② $4x - y - z = 5$

③ $8x - 2y - 2z = 10$

Solution

Again, using elementary operations,

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3$ $\quad -2 \times ① + ②$

⑤ $0x - 6y - 6z = 6$ $\quad -4 \times ① + ③$

Continuing, we obtain

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3$

⑥ $0x + 0y + 0z = 0$ $\quad -2 \times ④ + ⑤$

Equation ⑥ indicates that this system has an infinite number of solutions.

We can solve this system by using a parameter for either y or z.

Substituting $y = s$ in equation ④ gives $-3s - 3z = 3$ or $z = -s - 1$.

Substituting into equation ①, $2x + s + (-s - 1) = 1$ or $x = 1$.

Therefore, the solution to this system is $x = 1$, $y = s$, and $z = -s - 1$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①, $2(1) + s + (-s - 1) = 2 + s - s - 1 = 1$.

Substituting into equation ②, $4(1) - s - (-s - 1) = 4 - s + s + 1 = 5$.

There is no need to check in our third equation since $2 \times$ ② = ③. Equation ③ represents the same plane as equation ②.

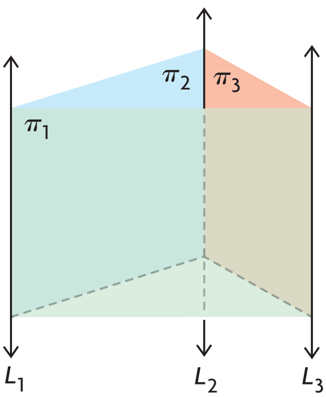
It is worth noting that the normals of the second and third planes, $\overrightarrow{n_2} = (4, -1, -1)$ and $\overrightarrow{n_3} = (8, -2, -2)$, are scalar multiples of each other, and that the constants on the right-hand side are related by the same factor. This indicates that the two equations represent the same plane. Since neither of these normals and the first plane's normal $\overrightarrow{n_1} = (2, 1, 1)$ are scalar multiples of each other, the first plane must intersect the two coincident planes along a line passing through the point $(1, 0, -1)$ with direction vector $\overrightarrow{m} = (0, 1, -1)$. This corresponds to Case 2b.

Inconsistent Systems for Three Equations Representing Three Planes

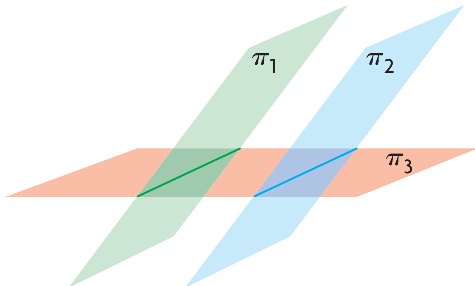
There are four cases to consider for inconsistent systems of equations that represent three planes.

The four cases, which all result in no points of intersection between all three planes, are shown below.

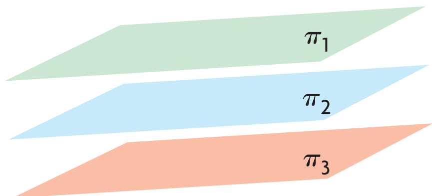
Case 1: A triangular prism is formed by three parallel lines.



Case 2: Two parallel planes intersect a third plane.



Case 3: Three planes are parallel; none are coincident.



Case 4: Two planes are coincident; a third plane is parallel to and non-coincident with the first two planes.

Non-intersections for Three Planes

A description of each case above is given below.

Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism as shown. This means that, if you consider any two of the three planes, they intersect in a line and each of these three lines is parallel to the others. In the diagram, the lines L_1, L_2, and L_3 represent the lines of intersection between the three pairs of planes, and these lines have direction vectors that are identical to, or scalar multiples of, each other.

Even though the planes intersect in a pair-wise fashion, there is no common intersection between all three of the planes.

As well, the normals of the three planes are not scalar multiples of each other, and the system is inconsistent. The only geometric possibility is that the planes form a triangular prism. This idea is discussed in Example 4.

Case 2: We consider two parallel planes, each intersecting a third plane. Each of the parallel planes has a line of intersection with the third plane, but there is no intersection between all three planes.

Cases 3 and 4: In these two cases, which again implies that all three planes do not have any points of intersection.

EXAMPLE 4 **Selecting a strategy to solve an inconsistent system of equations**

Determine the solution to the following system of equations:

① $x - y + z = 1$

② $x + y + 2z = 2$

③ $x - 5y - z = 1$

Solution

Applying elementary operations to this system, the following system is obtained:

① $x - y + z = 1$

④ $0x + 2y + z = 1$ $\quad -1 \times$ ① + ②

⑤ $0x - 4y - 2z = 0$ $\quad -1 \times$ ① + ③

At this point, it can be observed that there is an inconsistency between equations ④ and ⑤. If equation ④ is multiplied by -2, it becomes $0x - 4y - 2z = -2$, which is inconsistent with equation ⑤ $(0x - 4y - 2z = 0)$. This implies that there is no solution to the system of equations. It is instructive, however, to continue using elementary operations and observe the results.

① $x - y + z = 1$

④ $0x + 2y + z = 1$

⑥ $0x + 0y + 0z = 2$ $\quad 2 \times$ ④ + ⑤

Equation ⑥, $0x + 0y + 0z = 2$, tells us there is no solution to the system, because there are no values of x, y, and z that satisfy this equation. The system is inconsistent.

If we use the normals for these three equations, we can calculate direction vectors for each pair of intersections. The normals for the three planes are $\overrightarrow{n_1} = (1, -1, 1)$, $\overrightarrow{n_2} = (1, 1, 2)$, and $\overrightarrow{n_3} = (1, -5, -1)$.
Let $\overrightarrow{m_1}$ be a direction vector for the line of intersection between π_1 and π_2.
Let $\overrightarrow{m_2}$ be a direction vector for the line of intersection between π_1 and π_3.
Let $\overrightarrow{m_3}$ be a direction vector for the line of intersection between π_2 and π_3.

Therefore, we can choose

$$\begin{aligned}\vec{m_1} &= \vec{n_1} \times \vec{n_2}\\ &= (1, -1, 1) \times (1, 1, 2)\\ &= (-1(2) - 1(1), 1(1) - 1(2), 1(1) - (-1)(1))\\ &= (-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\vec{m_2} &= \vec{n_1} \times \vec{n_3}\\ &= (1, -1, 1) \times (1, -5, -1)\\ &= (-1(-1) - 1(-5), 1(1) - 1(-1), 1(-5) - (-1)(1))\\ &= -2(-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\vec{m_3} &= \vec{n_2} \times \vec{n_3}\\ &= (1, 1, 2) \times (1, -5, -1)\\ &= (1(-1) - 2(-5), 2(1) - 1(-1), 1(-5) - 1(1))\\ &= -3(-3, -1, 2)\end{aligned}$$

We can see from our calculations that the system of equations corresponds to Case 1 for systems of inconsistent equations (triangular prism).

This conclusion could have been anticipated without doing any calculations. We have shown that the system of equations is inconsistent, and, because the normals are not scalar multiples of each other, we can reach the same conclusion.

In the following example, we deal with another inconsistent system.

EXAMPLE 5

Reasoning about an inconsistent system of equations

Determine the solution to the following system of equations:

① $x - y + 2z = -1$

② $x - y + 2z = 3$

③ $x - 3y + z = 0$

Solution

Using elementary operations,

① $x - y + 2z = -1$

④ $0x + 0y + 0z = 4$ $\quad -1 \times$ ① + ②

③ $x - 3y + z = 0$

It is only necessary to use elementary operations once, and we obtain equation ④. As before, we create an equivalent system of equations that does not have a solution, implying that the original system has no solution.

It should be noted that it is not necessary to use elementary operations in this example. Because equations ① and ② are the equations of non-coincident parallel planes, no intersection is possible. This corresponds to Case 2 for systems of inconsistent equations, since the third plane is not parallel to the first two.

EXAMPLE 6 Identifying coincident and parallel planes in an inconsistent system

Solve the following system of equations:

① $x + y + z = 5$

② $x + y + z = 4$

③ $x + y + z = 5$

Solution

It is clear, from observation, that this system of equations is inconsistent. Equations ① and ③ represent the same plane, and equation ② represents a plane that is parallel to, but different from, the other plane. This corresponds to Case 4 for systems of inconsistent equations, so there are no solutions.

IN SUMMARY

Key Idea

- A system of three (linear) equations in three unknowns geometrically represents three planes in R^3. These planes may intersect at zero, one, or an infinite number of points, depending on how the planes are related to each other.

Need to Know

- Consistent Systems for Three Equations Representing Three Planes

  Case 1 (one solution): There is a single point.

  Case 2 (infinite number of solutions): The solution uses one parameter.

  Case 2a: The three planes intersect along a line.

  Case 2b: Two planes are coincident, and the third plane cuts through these two planes.

  Case 3 (infinite number of solutions): The solution uses two parameters. There are three planes that have identical equations (after reducing the equations) that coincide with one another.

- Inconsistent Systems for Three Equations Representing Three Planes (No Intersection)

  Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism.

  Case 2: Two non-coincident parallel planes each intersect a third plane.

  Case 3: The three planes are parallel and non-coincident.

  Case 4: Two planes are coincident and parallel to the third plane.

Exercise 9.4

PART A

1. A student is manipulating a system of equations and obtains the following equivalent system:

① $x - 3y + z = 2$

② $0x + y - z = -1$

③ $0x + 0y + 3z = -12$

a. Determine the solution to this system of equations.

b. How would your solution be interpreted geometrically?

2. When manipulating a system of equations, a student obtains the following equivalent system:

① $x - y + z = 4$

② $0x + 0y + 0z = 0$

③ $0x + 0y + 0z = 0$

a. Give a system of equations that would produce this equivalent system.

b. How would you interpret the solution to this system geometrically?

c. Write the solution to this system using parameters for x and y.

d. Write the solution to this system using parameters for y and z.

3. When manipulating a system of equations, a student obtains the following equivalent system:

① $2x - y + 3z = -2$

② $x - y + 4z = 3$

③ $0x + 0y + 0z = 1$

a. Give *two* systems of equations that could have produced this result.

b. What does this equivalent system tell you about possible solutions for the original system of equations?

4. When manipulating a system of equations, a student obtains the following equivalent system:

① $x + 2y - z = 4$

② $x + 0y - 2z = 0$

③ $2x + 0y + 0z = -6$

a. Without using any further elementary operations, determine the solution to this system.

b. How can the solution to this system be interpreted geometrically?

PART B

5. a. Without solving the following system, how can you deduce that these three planes must intersect in a line?

   ① $2x - y + z = 1$

   ② $x + y - z = -1$

   ③ $-3x - 3y + 3z = 3$

   b. Find the solution to the given system using elementary operations.

C 6. Explain why there is no solution to the following system of equations:

   ① $2x + 3y - 4z = -5$

   ② $x - y + 3z = -201$

   ③ $5x - 5y + 15z = -1004$

7. Avery is solving a system of equations using elementary operations and derives, as one of the equations, $0x + 0y + 0z = 0$.

   a. Is it true that this equation will always have a solution? Explain.

   b. Construct your own system of equations in which the equation $0x + 0y + 0z = 0$ appears, but for which there is no solution to the constructed system of equations.

K 8. Solve the following systems of equations using elementary operations. Interpret your results geometrically.

   a. ① $2x + y - z = -3$

   ② $x - y + 2z = 0$

   ③ $3x + 2y - z = -5$

   b. ① $\frac{x}{3} - \frac{y}{4} + z = \frac{7}{8}$

   ② $2x + 2y - 3z = -20$

   ③ $x - 2y + 3z = 2$

   c. ① $x - y = -199$

   ② $x + z = -200$

   ③ $y - z = 201$

   d. ① $x - y - z = -1$

   ② $y - 2 = 0$

   ③ $x + 1 = 5$

9. Solve each system of equations using elementary operations. Interpret your results geometrically.

   a. ① $x - 2y + z = 3$
      ② $2x + 3y - z = -9$
      ③ $5x - 3y + 2z = 0$

   b. ① $x - 2y + z = 3$
      ② $x + y + z = 2$
      ③ $x - 3y + z = -6$

   c. ① $x - y + z = -2$
      ② $x + y + z = 2$
      ③ $x - 3y + z = -6$

10. Determine the solution to each system.

   a. ① $x - y + z = 2$
      ② $2x - 2y + 2z = 4$
      ③ $x + y - z = -2$

   b. ① $2x - y + 3z = 0$
      ② $4x - 2y + 6z = 0$
      ③ $-2x + y - 3z = 0$

A

11. a. Use elementary operations to show that the following system does not have a solution:
      ① $x + y + z = 1$
      ② $x - 2y + z = 0$
      ③ $x - y + z = 0$

   b. Calculate the direction vectors for the lines of intersection between each pair of planes, as shown in Example 4.

   c. Explain, in your own words, why the planes represented in this system of equations must correspond to a triangular prism.

   d. Explain how the same conclusion could have been reached without doing the calculations in part b.

12. Each of the following systems does not have a solution. Explain why.

   a. ① $x - y + 3z = 3$
      ② $x - y + 3z = 6$
      ③ $3x - 5z = 0$

   b. ① $5x - 2y + 3z = 1$
      ② $5x - 2y + 3z = -1$
      ③ $5x - 2y + 3z = 13$

   c. ① $x - y + z = 9$
      ② $2x - 2y + 2z = 18$
      ③ $2x - 2y + 2z = 17$

   d. ① $3x - 2y + z = 4$
      ② $9x - 6y + 3z = 12$
      ③ $6x - 4y + 2z = 5$

13. Determine the solution to each system of equations, if a solution exists.

a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

PART C

14. The following system of equations represents three planes that intersect in a line:

① $2x + y + z = 4$
② $x - y + z = p$
③ $4x + qy + z = 2$

a. Determine p and q.
b. Determine an equation in parametric form for the line of intersection.

T 15. Consider the following system of equations:

① $4x + 3y + 3z = -8$
② $2x + y + z = -4$
③ $3x - 2y + (m^2 - 6)z = m - 4$

Determine the value(s) of m for which this system of equations will have

a. no solution
b. one solution
c. an infinite number of solutions

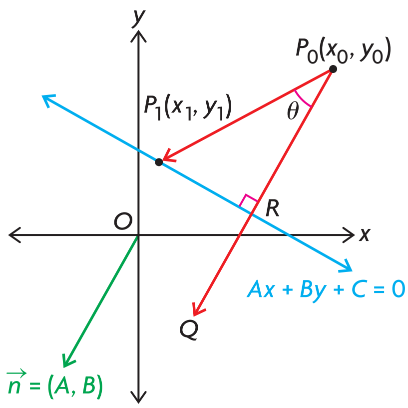
16. Determine the solution to the following system of equations:

① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$
② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$
③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Section 9.5—The Distance from a Point to a Line in R^2 and R^3

In this section, we consider various approaches for determining the distance between a point and a line in R^2 and R^3.

Determining a Formula for the Distance between a Point and a Line in R^2



Consider a line in R^2 that has $Ax + By + C = 0$ as its general equation, as shown in the diagram above. The point $P_0(x_0, y_0)$ is a point not on the line and whose coordinates are known. A line from P_0 is drawn perpendicular to $Ax + By + C = 0$ and meets this line at R. The line from P_0 is extended to point Q. The point $P_1(x_1, y_1)$ represents a second point on the line different from R. We wish to determine a formula for $|\overrightarrow{P_0R}|$, the distance from P_0 to the line. (Note that when we are calculating the distance between a point and either a line or a plane, we are always calculating the perpendicular distance, which is always unique. In simple terms, this means that there is only one shortest distance that can be calculated between P_0 and $Ax + By + C = 0$.)

To determine the formula, we are going to take the scalar projection of $\overrightarrow{P_0P_1}$ on $\overrightarrow{P_0Q}$. Since $\overrightarrow{P_0Q}$ is perpendicular to $Ax + By + C = 0$, what we are doing is equivalent to taking the scalar projection of $\overrightarrow{P_0P_1}$ on the normal to the line, $n = (A, B)$, since n and P_0Q are parallel.

We know that $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0)$ and $\vec{n} = (A, B)$. The formula for the dot product is $\overrightarrow{P_0P_1} \times \vec{n} = |\overrightarrow{P_0P_1}||\vec{n}|\cos\theta$, where θ is the angle between $\overrightarrow{P_0P_1}$ and $\vec{n}$. Rearranging this formula gives

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\overrightarrow{P_0P_1} \times \vec{n}}{|\vec{n}|} \qquad \text{(Equation 1)}$$

From triangle P_0RP_1

$$\cos\theta = \frac{|\overrightarrow{P_0R}|}{|\overrightarrow{P_0P_1}|}$$

$$|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$$

Substituting $|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$ into the dot product formula (equation 1 above) gives

$$|\overrightarrow{P_0R}| = \frac{\overrightarrow{P_0P_1} \times \vec{n}}{|\vec{n}|}$$

Since $\overrightarrow{P_0P_1} \times \vec{n} = (x_1 - x_0, y_1 - y_0)(A, B) = Ax_1 - Ax_0 + By_1 - By_0$ and $|\vec{n}| = \sqrt{A^2 + B^2}$, (by substitution) we obtain

$$|\overrightarrow{P_0R}| = \frac{Ax_1 + By_1 - Ax_0 - By_0}{\sqrt{A^2 + B^2}}$$

The point $P_1(x_1, y_1)$ is on the line $Ax + By + C = 0$, meaning that $Ax_1 + By_1 + C = 0$ or $Ax_1 + By_1 = -C$. Substituting this into the formula for $|\overrightarrow{P_0R}|$ gives

$$|\overrightarrow{P_0R}| = \frac{-C - Ax_0 - By_0}{\sqrt{A^2 + B^2}} = \frac{-(C + Ax_0 + By_0)}{\sqrt{A^2 + B^2}}$$

To ensure that this quantity is always positive, it is written as

$$|\overrightarrow{P_0R}| = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

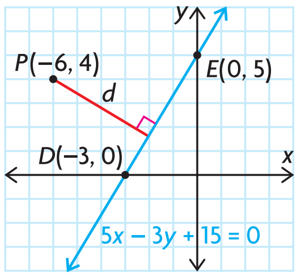
Distance from a Point $P_0(x_0, y_0)$ to the Line with Equation $Ax + By + C = 0$

$d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance between the point $P_0(x_0, y_0)$, and the line defined by $Ax + By + C = 0$, where the point does not lie on the line. But we don't really need this since the formula gives the correct value of 0 when the point *does* lie on the line.

EXAMPLE 1 Calculating the distance between a point and a line in R^2

Determine the distance from point $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$.

Solution



Since $x_0 = -6$, $y_0 = 4$, $A = 5$, $B = -3$, and $C = 15$,

$$d = \frac{|5(-6) - 3(4) + 15|}{\sqrt{5^2 + (-3)^2}} = \frac{|-27|}{\sqrt{34}} \doteq 4.63$$

The distance from $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$ is approximately 4.63.

It is not immediately possible to use the formula for the distance between a point and a line if the line is given in vector form. In the following example, we show how to find the required distance if the line is given in vector form.

EXAMPLE 2 Selecting a strategy to determine the distance between a point and a line in R^2

Determine the distance from point $P(15, -9)$ to the line with equation $\vec{r} = (-2, -1) + s(-4, 3), s \in \mathbf{R}$.

Solution

To use the formula, it is necessary to convert the equation of the line in vector form to its corresponding Cartesian form. The given equation must first be written using parametric form. The parametric equations for this line are $x = -2 - 4s$ and $y = -1 + 3s$. Solving for the parameter s in each equation gives $\frac{x + 2}{-4} = s$ and $\frac{y + 1}{3} = s$. Therefore, $\frac{x + 2}{-4} = \frac{y + 1}{3} = s$. The required equation is $3(x + 2) = -4(y + 1)$ or $3x + 4y + 10 = 0$.

Therefore, $d = \frac{|3(15) + 4(-9) + 10|}{\sqrt{3^2 + 4^2}} = \frac{19}{5} = 3.80$.

The required distance is 3.80.

EXAMPLE 3 Selecting a strategy to determine the distance between two parallel lines

Calculate the distance between the two parallel lines $5x - 12y + 60 = 0$ and $5x - 12y - 60 = 0$.

Solution

To find the required distance, it is necessary to determine the coordinates of a point on one of the lines and then use the distance formula. For the line with equation $5x - 12y - 60 = 0$, we can determine the coordinates of the point where the line crosses either the x-axis or the y-axis. (This point was chosen because it is easy to calculate and it also makes the resulting computation simpler. In practice, however, any point on the chosen line is satisfactory.) If we let $x = 0$, then $5(0) - 12y - 60 = 0$, or $y = -5$. The line crosses the y-axis at $(0, -5)$. To find the required distance, d, it is necessary to find the distance from $(0, -5)$ to the line with equation $5x - 12y + 60 = 0$.

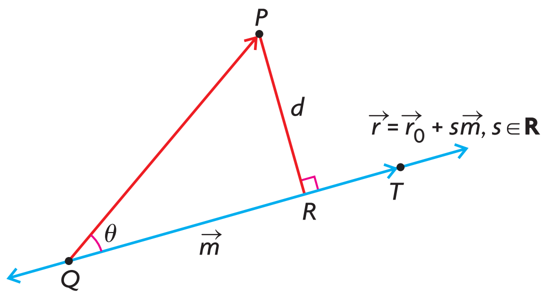
$$d = \frac{|5(0) - 12(-5) + 60|}{\sqrt{5^2 + (-12)^2}} = \frac{|120|}{13} = \frac{120}{13}$$

Therefore, the distance between the two parallel lines is $\frac{120}{13} \doteq 9.23$.

Determining the Distance between a Point and a Line in R^3

It is not possible to use the formula we just developed for finding the distance between a point and a line in R^3 because lines in R^3 are not of the form $Ax + By + C = 0$. We need to use a different approach.

The most efficient way to find the distance between a point and a line in R^3 is to use the cross product. In the following diagram, we would like to find d, which represents the distance between point P, whose coordinates are known, and a line with vector equation $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$. Point Q is any point on the line whose coordinates are also known. Point T is the point on the line such that $\overrightarrow{QT}$ is a vector representing the direction $\vec{m}$, which is known.



The angle between $\overrightarrow{QP}$ and $\overrightarrow{QT}$ is θ. Note that, for computational purposes, it is possible to determine the coordinates of a position vector equivalent to either $\overrightarrow{QP}$ or $\overrightarrow{PQ}$.

In triangle PQR, $\sin\theta = \dfrac{d}{|\overrightarrow{QP}|}$

equivalently $d = |\overrightarrow{QP}|\sin\theta$

From our earlier discussion on cross products, we know that $|\vec{m} \times \overrightarrow{QP}| = |\vec{m}||\overrightarrow{QP}|\sin\theta$.

If we substitute $d = |\overrightarrow{QP}|\sin\theta$ into this formula, we find that $|\vec{m} \times \overrightarrow{QP}| = |\vec{m}|(d)$.

Solving for d gives $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$.

Distance, d, from a Point, P, to the Line $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$

In R^3, $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$, where Q is a point on the line and P is any other point, both of whose coordinates are known, and $\vec{m}$ is the direction vector of the line.

EXAMPLE 4 Selecting a strategy to calculate the distance between a point and a line in R^3

Determine the distance from point $P(-1, 1, 6)$ to the line with equation $\vec{r} = (1, 2, -1) + t(0, 1, 1)$, $t \in \mathbf{R}$.

Solution

Method 1: Using the Formula

Since Q is $(1, 2, -1)$ and P is $(-1, 1, 6)$,
$\overrightarrow{QP} = (-1 - 1, 1 - 2, 6 - (-1)) = (-2, -1, 7)$.

From the equation of the line, we note that $\vec{m} = (0, 1, 1)$.

Thus, $d = \dfrac{|(0, 1, 1) \times (-2, -1, 7)|}{|(0, 1, 1)|}$.

Calculating,
$(0, 1, 1) \times (-2, -1, 7) = (7 - (-1), -2 - 0, 0 + 2) = (8, -2, 2)$
$|(8, -2, 2)| = \sqrt{8^2 + (-2)^2 + 2^2} = \sqrt{72} = 6\sqrt{2}$ and
$|(0, 1, 1)| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

Therefore, the distance from the point to the line is $d = \dfrac{6\sqrt{2}}{\sqrt{2}} = 6$.

This calculation is efficient and gives the required answer quickly. (Note that the vector $(8, -2, 2)$ cannot be reduced by dividing by the common factor 2, or by any other factor.)

Method 2: Using the Dot Product

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1$, $y = 2 + t$, and $z = -1 + t$. We construct a vector from a general point on the line to P and call this vector $\vec{a}$. Thus, $\vec{a} = (-1 - 1, 1 - (2 + t), 6 - (-1 + t)) = (-2, -1 - t, 7 - t)$. What we wish to find is the minimum distance between point $P(-1, 1, 6)$ and the given line. This occurs when $\vec{a}$ is perpendicular to the given line, or when $\vec{m} \cdot \vec{a} = 0$.

Calculating gives
$$\begin{aligned} (0, 1, 1) \cdot (-2, -1 - t, 7 - t) &= 0 \\ 0(-2) + 1(-1 - t) + 1(7 - t) &= 0 \\ -1 - t + 7 - t &= 0 \\ t &= 3 \end{aligned}$$

This means that the minimal distance between $P(-1, 1, 6)$ and the line occurs when $t = 3$, which implies that the point corresponding to $t = 3$ produces the minimal distance between the point and the line. This point has coordinates $x = 1$, $y = 2 + 3 = 5$, and $z = -1 + 3 = 2$. In other words, the minimal distance between the point and the line is the distance between $P(-1, 1, 6)$ and the point, $(1, 5, 2)$. Thus,

$$d = \sqrt{(-1 - 1)^2 + (1 - 5)^2 + (6 - 2)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

This gives the same answer that we found using Method 1. It has the advantage that it also allows us to find the coordinates of the point on the line that produces the minimal distance.

IN SUMMARY

Key Ideas

- In R^2, the distance from point $P_0(x_0, y_0)$ to the line with equation $Ax + By + C = 0$ is $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance.
- In R^3, the formula for the distance d from point P to the line $\vec{r} = \vec{r_0} + s\vec{m}$, $s \in \mathbf{R}$, is $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$, where Q is a point on the line whose coordinates are known.

Exercise 9.5

PART A

1. Determine the distance from $P(-4, 5)$ to each of the following lines:
   a. $3x + 4y - 5 = 0$
   b. $5x - 12y + 24 = 0$
   c. $9x - 40y = 0$

2. Determine the distance between the following parallel lines:
   a. $2x - y + 1 = 0$, $2x - y + 6 = 0$
   b. $7x - 24y + 168 = 0$, $7x - 24y - 336 = 0$

3. Determine the distance from $R(-2, 3)$ to each of the following lines:
   a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$
   b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$
   c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$

PART B

C 4. a. The formula for the distance from a point to a line is $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$. Show that this formula can be modified so the distance from the origin, $O(0, 0)$, to the line $Ax + By + C = 0$ is given by the formula $d = \frac{|C|}{\sqrt{A^2 + B^2}}$.
   b. Determine the distance between $L_1: 3x - 4y - 12 = 0$ and $L_2: 3x - 4y + 12 = 0$ by first finding the distance from the origin to L_1 and then finding the distance from the origin to L_2.
   c. Find the distance between the two lines directly by first determining a point on one of the lines and then using the distance formula. How does this answer compare with the answer you found in part b.?

K 5. Calculate the distance between the following lines:
   a. $\vec{r} = (-2, 1) + s(3, 4); s \in \mathbf{R}; \vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$
   b. $\frac{x - 1}{4} = \frac{y}{-3}, \frac{x}{4} = \frac{y + 1}{-3}$
   c. $2x - 3y + 1 = 0$, $2x - 3y - 3 = 0$
   d. $5x + 12y = 120$, $5x + 12y + 120 = 0$

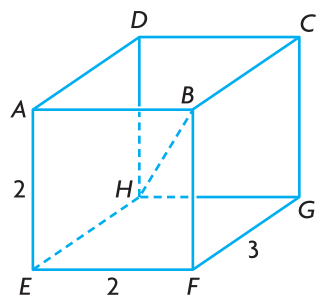
6. Calculate the distance between point P and the given line.
   a. $P(1, 2, -1)$; $\vec{r} = (1, 0, 0) + s(2, -1, 2), s \in \mathbf{R}$
   b. $P(0, -1, 0)$; $\vec{r} = (2, 1, 0) + t(-4, 5, 20), t \in \mathbf{R}$
   c. $P(2, 3, 1)$; $\vec{r} = p(12, -3, 4), p \in \mathbf{R}$
7. Calculate the distance between the following parallel lines.
   a. $\vec{r} = (1, 1, 0) + s(2, 1, 2), s \in \mathbf{R}$; $\vec{r} = (-1, 1, 2) + t(2, 1, 2), t \in \mathbf{R}$
   b. $\vec{r} = (3, 1, -2) + m(1, 1, 3), m \in \mathbf{R}$; $\vec{r} = (1, 0, 1) + n(1, 1, 3), n \in \mathbf{R}$

A 8. a. Determine the coordinates of the point on the line $\vec{r} = (1, -1, 2) + s(1, 3, -1), s \in \mathbf{R}$, that produces the shortest distance between the line and a point with coordinates $(2, 1, 3)$.
   b. What is the distance between the given point and the line?

PART C

9. Two planes with equations $x - y + 2z = 2$ and $x + y - z = -2$ intersect along line L. Determine the distance from $P(-1, 2, -1)$ to L, and determine the coordinates of the point on L that gives this minimal distance.
10. The point $A(2, 4, -5)$ is reflected in the line with equation $\vec{r} = (0, 0, 1) + s(4, 2, 1), s \in \mathbf{R}$, to give the point A'. Determine the coordinates of A'.

T 11. A rectangular box with an open top, measuring 2 by 2 by 3, is constructed. Its vertices are labelled as shown.



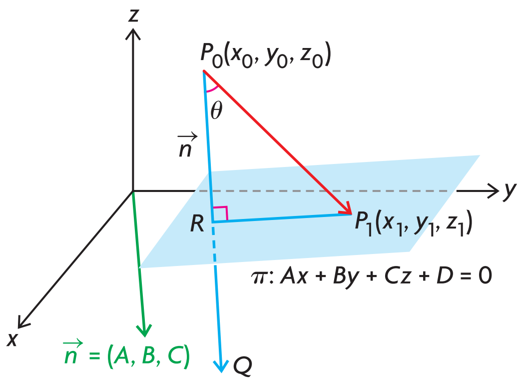
   a. Determine the distance from A to the line segment HB.
   b. What other vertices on the box will give the same distance to HB as the distance you found in part a.?
   c. Determine the area of the $\triangle AHB$.

Section 9.6—The Distance from a Point to a Plane

In the previous section, we developed a formula for finding the distance from a point $P_0(x_0, y_0)$ to the line $Ax + By + C = 0$ in R^2. In this section, we will use the same kind of approach to develop a formula for the distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + \quad + Cz + D = 0$ in R^3.

Determining a Formula for the Distance between a Point and a Plane in R^3

We start by considering a general plane in R^3 that has $Ax + By + Cz + D = 0$ as its equation. The point $P_0(x_0, y_0, z_0)$ is a point whose coordinates are known. A line from P_0 is drawn perpendicular to $Ax + By + Cz + D = 0$ and meets this plane at R. The point $P_1(x_1, y_1, z_1)$ is a point on the plane, with coordinates different from R, and Q is chosen so that $\overrightarrow{P_0Q} = n = (A, B, C)$ is the normal to the plane. The objective is to find a formula for $|\overrightarrow{PR}|$—the perpendicular distance from P_0 to the plane. To develop this formula, we are going to use the fact that $|\overrightarrow{PR}|$ is the scalar projection of $\overrightarrow{P_0P_1}$ on the normal $\vec{n}$.



In $\triangle P_0RP_1$, $\cos\theta = \dfrac{|\overrightarrow{PR}|}{|\overrightarrow{P_0P_1}|}$

$|\overrightarrow{PR}| = |\overrightarrow{P_0P_1}|\cos\theta$, where θ is the angle between the vectors $\vec{n}$ and $\overrightarrow{P_0P_1}$

Since $\vec{n} \cdot \overrightarrow{P_0P_1} = |\vec{n}||\overrightarrow{P_0P_1}|\cos\theta$,

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$$

Therefore, $d = |\overrightarrow{PR}| = \dfrac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$, where d is the distance from the point to the plane.

Since $\vec{n} = (A, B, C)$, $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$d = \left|\overrightarrow{PR}\right| = \frac{(A, B, C) \times (x_1 - x_0, y_1 - y_0, z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}}$$

or

$$d = \frac{Ax_1 - Ax_0 + By_1 - By_0 + Cz_1 - Cz_0}{\sqrt{A^2 + B^2 + C^2}}$$

Since $P_1(x_1, y_1, z_1)$ is a point on $Ax + By + Cz + D = 0$, $Ax_1 + By_1 + Cz_1 + D = 0$ and $Ax_1 + By_1 + Cz_1 = -D$.

Rearranging the formula,

$$d = \frac{-Ax_0 - By_0 - Cz_0 + Ax_1 + By_1 + Cz_1}{\sqrt{A^2 + B^2 + C^2}}$$

Therefore, $d = \dfrac{-Ax_0 - By_0 - Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$

$$d = \frac{-(Ax_0 + By_0 + Cz_0 + D)}{\sqrt{A^2 + B^2 + C^2}}$$

Since the distance d is always positive, the formula is written as

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Distance from a Point $P_0(x_0, y_0, z_0)$ to the Plane with Equation $Ax + By + Cz + D = 0$

In R^3, $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where d is the required distance between the point and the plane.

EXAMPLE 1 Calculating the distance from a point to a plane

Determine the distance from $S(-1, 2, -4)$ to the plane with equation $8x - 4y + 8z + 3 = 0$.

Solution

To determine the required distance, we substitute directly into the formula.

Therefore, $d = \dfrac{|8(-1) - 4(2) + 8(-4) + 3|}{\sqrt{8^2 + (-4)^2 + 8^2}} = \dfrac{|-45|}{12} = \dfrac{45}{12} = 3.75$

The distance between $S(-1, 2, -4)$ and the given plane is 3.75.

It is also possible to use the distance formula to find the distance between two parallel planes, as we show in the following example.

EXAMPLE 2 Selecting a strategy to determine the distance between two parallel planes

a. Determine the distance between the two planes π_1: $2x - y + 2z + 4 = 0$ and π_2: $2x - y + 2z + 16 = 0$.
b. Determine the equation of the plane that is equidistant from π_1 and π_2.

Solution

a. The two given planes are parallel because they each have the same normal, $\vec{n} = (2, -1, 2)$. To find the distance between π_1 and π_2, it is necessary to have a point on one of the planes and the equation of the second plane. If we consider π_1, we can determine the coordinates of its z-intercept by letting $x = y = 0$. Substituting, $2(0) - (0) + 2z + 4 = 0$, or $z = -2$. This means that point $X(0, 0, -2)$ lies on π_1. To find the required distance, apply the formula using $X(0, 0, -2)$ and π_2: $2x - y + 2z + 16 = 0$.

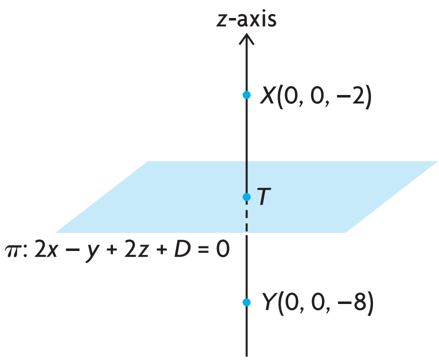
$$d = \frac{|2(0) - (0) + 2(-2) + 16|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|-4 + 16|}{3} = \frac{12}{3} = 4$$

Therefore, the distance between π_1 and π_2 is 4.

When calculating the distance between these two planes, we used the coordinates of the z-intercept as our point on one of the planes. This point was chosen because it is easy to determine and leads to a simple calculation for the distance.

b. The plane that is equidistant from π_1 and π_2 is parallel to both planes and lies midway between them. Since the required plane is parallel to the two given planes, it must have the form π: $2x - y + 2z + D = 0$, with D to be determined. If we follow the same procedure for π_2 that we used in part a., we can find the coordinates of the point associated with its z-intercept. If we substitute $x = y = 0$ into π_2, $2(0) - (0) + 2z + 16 = 0$, or $z = -8$. This means the point $Y(0, 0, -8)$ is on π_2.

The situation can be visualized in the following way:



Point T is on the required plane π: $2x - y + 2z + D = 0$ and is the midpoint of the line segment joining $X(0, 0, -2)$ to $Y(0, 0, -8)$, meaning that point T has coordinates $(0, 0, -5)$. To find D, we substitute the coordinates of point T into π, which gives $2(0) - (0) + 2(-5) + D = 0$, or $D = 10$.

Thus, the required plane has the equation $2x - y + 2z + 10 = 0$. We should note, in this case, that the coordinates of its z-intercept are $(0, 0, -5)$.

We have shown how to use the formula to find the distance from a point to a plane. This formula can also be used to find the distance between two skew lines.

EXAMPLE 3 Selecting a strategy to determine the distance between skew lines

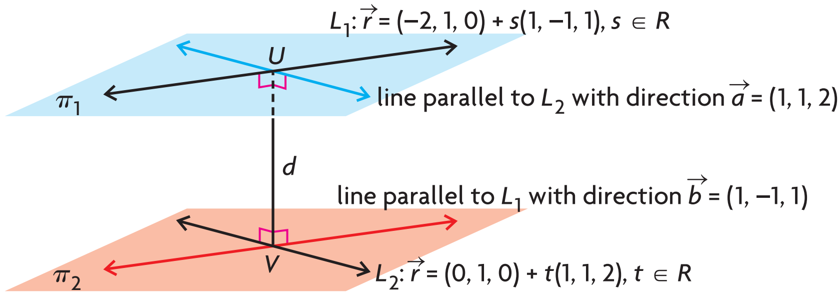
Determine the distance between L_1: $\vec{r} = (-2, 1, 0) + s(1, -1, 1)$, $s \in \mathbf{R}$, and L_2: $\vec{r} = (0, 1, 0) + t(1, 1, 2)$, $t \in \mathbf{R}$.

Solution

Method 1:

These two lines are skew lines because they are not parallel and do not intersect (you should verify this for yourself). To find the distance between the given lines, two parallel planes are constructed. The first plane is constructed so that L_1 lies on it, along with a second line that has direction $\vec{a} = (1, 1, 2)$, the direction vector for L_2.

In the same way, a second plane is constructed containing L_2, along with a second line that has direction $\vec{b} = (1, -1, 1)$, the direction of L_1.



The two constructed planes are parallel because they have two identical direction vectors. By constructing these planes, the problem of finding the distance between the two skew lines has been reduced to finding the distance between the planes π_1 and π_2. Finding the distance between the two planes means finding the Cartesian equation of one of the planes and using the distance formula with a point from the other plane. (This method of constructing planes will not work if the two given lines are parallel because the calculation of the normal for the planes would be (0, 0, 0).)

We first determine the equation of π_2. Since this plane has direction vectors $\vec{m_2} = (1, 1, 2)$ and $\vec{b} = (1, -1, 1)$, we can choose

$\vec{n} = \vec{m_2} \times \vec{b} = (1, 1, 2) \times (1, -1, 1)$.

Thus, $\vec{n} = (1(1) - 2(-1), 2(1) - 1(1), 1(-1) - 1(1)) = (3, 1, -2)$

We must now find D using the equation $3x + y - 2z + D = 0$. Since (0, 1, 0) is a point on this plane, $3(0) + 1 - 2(0) + D = 0$, or $D = -1$. This gives $3x + y - 2z - 1 = 0$ as the equation for π_2.

Using $3x + y - 2z - 1 = 0$ and the point $(-2, 1, 0)$ from the other plane π_1 the distance between the skew lines can be calculated.

$$d = \frac{|3(-2) + 1 - 2(0) - 1|}{\sqrt{3^2 + 1^2 + (-2)^2}} = \frac{6}{\sqrt{14}} = \frac{3}{7}\sqrt{14} \doteq 1.60$$

The distance between the two skew lines is approximately 1.60.

The first method gives one approach for finding the distance between two skew lines. In the second method, we will show how to determine the point on each of these two skew lines that produces this minimal distance.

Method 2:
In Method 1, we constructed two parallel planes and found the distance between them. Since the distance between the two planes is constant, our calculation also gave the distance between the two skew lines. There are points on each of these lines that will produce this minimal distance. Possible points, U and V, are shown

on the diagram for Method 1. To determine the coordinates of these points, we must use the fact that the vector found by joining the two points is perpendicular to the direction vector of each line.

We start by writing each line in parametric form.

For L_1, $x = -2 + s$, $y = 1 - s$, $z = s$, $\overrightarrow{m_1} = (1, -1, 1)$.

For L_2, $x = t$, $y = 1 + t$, $z = 2t$, $\overrightarrow{m_2} = (1, 1, 2)$.

The point with coordinates $U(-2 + s, 1 - s, s)$ represents a general point on L_1, and $V(t, 1 + t, 2t)$ represents a general point on L_2. We next calculate $\overrightarrow{UV}$.

$\overrightarrow{UV} = (t - (-2 + s), (1 + t) - (1 - s), 2t - s) = (t - s + 2, t + s, 2t - s)$

$\overrightarrow{UV}$ represents a general vector with its tail on L_1 and its head on L_2.

To find the points on each of the two lines that produce the minimal distance, we must use equations $\overrightarrow{m_1} \cdot \overrightarrow{UV} = 0$ and $\overrightarrow{m_2} \cdot \overrightarrow{UV} = 0$, since $\overrightarrow{UV}$ must be perpendicular to each of the two planes.

Therefore, $(1, -1, 1)(t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) - 1(t + s) + 1(2t - s) = 0$$

$$\text{or } 2t - 3s = -2 \qquad \text{(Equation 1)}$$

and $(1, 1, 2) \cdot (t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) + 1(t + s) + 2(2t - s) = 0$$

$$\text{or } 3t - s = -1 \qquad \text{(Equation 2)}$$

This gives the following system of equations:

① $2t - 3s = -2$

② $3t - s = -1$

$$-7t = 1 \qquad -3 \times ② + ①$$

$$t = -\frac{1}{7}$$

If we substitute $t = -\frac{1}{7}$ into equation ②, $3\left(-\frac{1}{7}\right) - s = -1$ or $s = \frac{4}{7}$.

We now substitute $s = \frac{4}{7}$ and $t = -\frac{1}{7}$ into the equations for each line to find the required points.

For L_1, $x = -2 + \frac{4}{7} = -\frac{10}{7}$, $y = 1 - \frac{4}{7} = \frac{3}{7}$, $z = \frac{4}{7}$.

Therefore, the required point on L_1 is $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$.

For L_2, $x = -\frac{1}{7}$, $y = 1 + \left(\frac{-1}{7}\right) = \frac{6}{7}$, $z = 2\left(\frac{-1}{7}\right) = -\frac{2}{7}$.

Therefore, the required point on L_2 is $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$.

The required distance between the two lines is the distance between these two points. This distance is $\sqrt{\left(-\frac{10}{7}+\frac{1}{7}\right)^2+\left(\frac{3}{7}-\frac{6}{7}\right)^2+\left(\frac{4}{7}+\frac{2}{7}\right)^2}$

$$= \sqrt{\left(\frac{-9}{7}\right)^2+\left(\frac{-3}{7}\right)^2+\left(\frac{6}{7}\right)^2}$$

$$= \sqrt{\frac{81}{49}+\frac{9}{49}+\frac{36}{49}}$$

$$= \sqrt{\frac{126}{49}}$$

$$= \sqrt{\frac{9 \times 14}{49}}$$

$$= \frac{3}{7}\sqrt{14}$$

Thus, the distance between the two skew lines is $\frac{3}{7}\sqrt{14}$, or approximately 1.60.

The two points that produce this distance are $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$ on L_1 and $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$ on L_2.

INVESTIGATION

A. In this section, we showed that the formula for the distance d from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is
$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

By modifying this formula, show that a formula for finding the distance from $O(0, 0, 0)$ to the plane $Ax + By + Cz + D = 0$ is $d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$.

B. Determine the distance from $O(0, 0, 0)$ to the plane with equation $20x + 4y - 5z + 21 = 0$.

C. Determine the distance between the planes with equations π_1: $20x + 4y - 5z + 21 = 0$ and π_2: $20x + 4y - 5z + 105 = 0$.

D. Determine the coordinates of a point that is equidistant from π_1 and π_2.

E. Determine an equation for a plane that is equidistant from π_1 and π_2.

F. Determine two values of D if the plane with equation $20x + 4y - 5z + D = 0$ is 4 units away from the plane with equation $20x + 4y - 5z = 0$.

G. Determine the distance between the two planes π_3: $20x + 4y - 5z - 105 = 0$ and π_4: $20x + 4y - 5z + 147 = 0$.

H. Determine the distance between the following planes:

a. $2x - 2y + z - 6 = 0$ and $2x - 2y + z - 12 = 0$

b. $6x - 3y + 2z + 14 = 0$ and $6x - 3y + 2z + 35 = 0$

c. $12x + 3y + 4z - 26 = 0$ and $12x + 3y + 4z + 26 = 0$

I. If two planes have equations $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$, explain why the formula for the distance d between these planes is $d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$.

IN SUMMARY

Key Idea

- The distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where d is the required distance.

Need to Know

- The distance between skew lines can be calculated using two different methods.

  Method 1: To determine the distance between the given skew lines, two parallel planes are constructed that are the same distance apart as the skew lines. Determine the distance between the two planes.

  Method 2: To determine the coordinates of the points that produce the minimal distance, use the fact that the general vector found by joining the two points is perpendicular to the direction vector of each line.

Exercise 9.6

PART A

C 1. A student is calculating the distance d between point $A(-3, 2, 1)$ and the plane with equation $2x + y + 2z + 2 = 0$. The student obtains the following answer:

$$d = \frac{|2(-3) + 2 + 2(1) + 2|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{0}{3} = 0$$

a. Has the student done the calculation correctly? Explain.

b. What is the significance of the answer 0? Explain.

PART B

K 2. Determine the following distances:

a. the distance from $A(3, 1, 0)$ to the plane with equation $20x - 4y + 5z + 7 = 0$

b. the distance from $B(0, -1, 0)$ to the plane with equation $2x + y + 2z - 8 = 0$

c. the distance from $C(5, 1, 4)$ to the plane with equation $3x - 4y - 1 = 0$

d. the distance from $D(1, 0, 0)$ to the plane with equation $5x - 12y = 0$

e. the distance from $E(-1, 0, 1)$ to the plane with equation $18x - 9y + 18z - 11 = 0$

3. For the planes π_1: $3x + 4y - 12z - 26 = 0$ and π_2: $3x + 4y - 12z + 39 = 0$, determine

a. the distance between π_1 and π_2

b. an equation for a plane midway between π_1 and π_2

c. the coordinates of a point that is equidistant from π_1 and π_2

4. Determine the following distances:

a. the distance from $P(1, 1, -3)$ to the plane with equation $y + 3 = 0$

b. the distance from $Q(-1, 1, 4)$ to the plane with equation $x - 3 = 0$

c. the distance from $R(1, 0, 1)$ to the plane with equation $z + 1 = 0$

A 5. Points $A(1, 2, 3)$, $B(-3, -1, 2)$, and $C(13, 4, -1)$ lie on the same plane. Determine the distance from $P(1, -1, 1)$ to the plane containing these three points.

T 6. The distance from $R(3, -3, 1)$ to the plane with equation $Ax - 2y + 6z = 0$ is 3. Determine all possible value(s) of A for which this is true.

7. Determine the distance between the lines $\vec{r} = (0, 1, -1) + s(3, 0, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (0, 0, 1) + t(1, 1, 0)$, $t \in \mathbf{R}$.

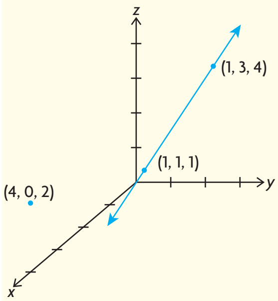
PART C

8. a. Calculate the distance between the lines L_1: $\vec{r} = (1, -2, 5) + s(0, 1, -1)$, $s \in \mathbf{R}$, and L_2: $\vec{r} = (1, -1, -2) + t(1, 0, -1)$, $t \in \mathbf{R}$.

b. Determine the coordinates of points on these lines that produce the minimal distance between L_1 and L_2.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 9: RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES



A pipeline engineer needs to find the line that will allow a new pipeline to intersect and join an existing pipeline at a right angle. The existing line has a pathway determined by the equation L_2: $r = (1, 1, 1) + d(0, 2, 3)$, $d \in \mathbf{R}$. The new pipeline will also need to be exactly 2 units away from the point (4, 0, 2).

a. Determine the vector and parametric equations of L_3, the line that passes through (4, 0, 2) and is perpendicular to L_2.

b. Determine the vector and parametric equations of L_1, the line that is parallel to L_3 and 2 units away from (4, 0, 2). There will be exactly two lines that fulfill this condition.

c. Plot each line on the coordinate axes.

Key Concepts Review

In this chapter, you learned how to solve systems of linear equations using elementary operations. The number of equations and the number of variables in the system are directly related to the geometric interpretation that each system represents.

System of Equations	Geometric Interpretation	Possible Points of Intersection
Two equations and two unknowns	two lines in R^2	zero, one, or an infinite number
Two equations and three unknowns	two planes in R^3	zero or an infinite number
Three equations and three unknowns	three planes in R^3	zero, one, or an infinite number

To make a connection between the algebraic equations and the geometric position and orientation of lines or planes in space, draw graphs or diagrams and compare the direction vectors of the lines and the normals of the planes. This will help you decide whether the system is consistent or inconsistent and which case you are dealing with.

Distances between points, lines and planes can be determined using the formulas developed in this chapter.

Distance between a point and a line in R^2	$d = \dfrac{\lvert Ax_0 + By_0 + C\rvert}{\sqrt{A^2 + B^2}}$
Distance between a point and a line in R^3	$d = \dfrac{\lvert \vec{m} \times \overrightarrow{QP}\rvert}{\lvert \vec{m}\rvert}$
Distance between a point and a plane in R^3	$d = \dfrac{\lvert Ax_0 + By_0 + Cz_0 + D\rvert}{\sqrt{A^2 + B^2 + C^2}}$

Review Exercise

1. The lines $2x - y = 31$, $x + 8y = -34$, and $3x + ky = 38$ all pass through a common point. Determine the value of k.

2. Solve the following system of equations:

   ① $x - y = 13$

   ② $3x + 2y = -6$

   ③ $x + 2y = -19$

3. Solve each system of equations.

   a. ① $x - y + 2z = 3$

   ② $2x - 2y + 3z = 1$

   ③ $2x - 2y + z = 11$

   b. ① $x + y + z = 300$

   ② $x + y - z = 98$

   ③ $x - y + z = 100$

4. a. Show that the points $(1, 2, 6)$, $(7, -5, 1)$, $(1, 1, 4)$, and $(-3, 5, 6)$ all lie on the same plane.

   b. Determine the distance from the origin to the plane you found in part a.

5. Determine the following distances:

   a. the distance from $A(-1, 1, 2)$ to the plane with equation $3x - 4y - 12z - 8 = 0$

   b. the distance from $B(3, 1, -2)$ to the plane with equation $8x - 8y + 4z - 7 = 0$

6. Determine the intersection of the plane $3x - 4y - 5z = 0$ with $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$.

7. Solve the following systems of equations:

   a. ① $3x - 4y + 5z = 9$

   ② $6x - 9y + 10z = 9$

   ③ $9x - 12y + 15z = 9$

   b. ① $2x + 3y + 4z = 3$

   ② $4x + 6y + 8z = 4$

   ③ $5x + y - z = 1$

   c. ① $4x - 3y + 2z = 2$

   ② $8x - 6y + 4z = 4$

   ③ $12x - 9y + 6z = 1$

8. Solve each system of equations.

a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$

9. Solve each of the following systems:

a. ① $3x - 5y + 2z = 4$
② $6x + 2y - z = 2$
③ $6x - 3y + 8z = 6$

b. ① $2x - 5y + 3z = 1$
② $4x + 2y + 5z = 5$
③ $2x + 7y + 2z = 4$

10. Determine the intersection of each set of planes, and show your answer geometrically.

a. $2x + y + z = 6, x - y - z = -9, 3x + y = 2$

b. $2x - y + 2z = 2, 3x + y - z = 1, x - 3y + 5z = 4$

c. $2x + y - z = 0, x - 2y + 3z = 0,\ 9x + 2y - z = 0$

11. The line $\vec{r} = (2, -1, -2) + s(1, 1, -2), s \in \mathbf{R}$, intersects the xz-plane at point P and the xy-plane at point Q. Calculate the length of the line segment PQ.

12. a. Given the line $\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$, and the plane $x - 2y + z + 4 = 0$, verify that the line lies on the plane.

b. Determine the point of intersection between the line $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$, and the line given in part a.

c. Show that the point of intersection of the lines is a point on the plane given in part a.

d. Determine the Cartesian equation of the plane that contains the line $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$ and is perpendicular to the plane given in part a.

13. a. Determine the distance from point $A(-2, 1, 1)$ to the line with equation $\vec{r} = (3, 0, -1) + t(1, 1, 2), t \in \mathbf{R}$.

b. What are the coordinates of the point on the line that produces this shortest distance?

14. You are given the lines $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$, and $\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$.

    a. Determine the coordinates of their point of intersection.

    b. Determine a vector equation for the line that is perpendicular to both of the given lines and passes through their point of intersection.

15. a. Determine the equation of the plane that contains $L: \vec{r} = (1, 2, -3) + s(1, 2, -1), s \in \mathbf{R}$, and point $K(3, -2, 4)$.

    b. Determine the distance from point $S(1, 1, -1)$ to the plane you found in part a.

16. Consider the following system of equations:

    ① $x + y - z = 1$

    ② $2x - 5y + z = -1$

    ③ $7x - 7y - z = k$

    a. Determine the value(s) of k for which the solution to this system is a line.

    b. Determine the vector equation of the line.

17. Determine the solution to each system of equations.

    a. ① $x + 2y + z = 1$
       ② $2x - 3y - z = 6$
       ③ $3x + 5y + 4z = 5$
       ④ $4x + y + z = 8$

    b. ① $x - 2y + z = 1$
       ② $2x - 5y + z = -1$
       ③ $3x - 7y + 2z = 0$
       ④ $6x - 14y + 4z = 0$

18. Solve the following system of equations for a, b, and c:

    ① $\dfrac{9a}{b} - 8b + \dfrac{3c}{b} = 4$

    ② $\dfrac{-3a}{b} + 4b + \dfrac{4c}{b} = 3$

    ③ $\dfrac{3a}{b} + 4b - \dfrac{4c}{b} = 3$

    $\left(\textit{Hint}\text{: Let } x = \frac{a}{b}, y = b, \text{ and } z = \frac{c}{b}.\right)$

19. Determine the point of intersection of the line $\frac{x + 1}{-4} = \frac{y - 2}{3} = \frac{z - 1}{-2}$ and the plane with equation $x + 2y - 3z + 10 = 0$.

20. Point $A(1, 0, 4)$ is reflected in the plane with equation $x - y + z - 1 = 0$. Determine the coordinates of the image point.

21. The three planes with equations $3x + y + 7z + 3 = 0$, $4x - 12y + 4z - 24 = 0$, and $x + 2y + 3z - 4 = 0$ do not simultaneously intersect.

   a. Considering the planes in pairs, determine the three lines of intersection.

   b. Show that these three lines are parallel.

22. Solve for a, b, and c in the following system of equations:

   ① $\frac{2}{a^2} + \frac{5}{b^2} + \frac{3}{c^2} = 40$

   ② $\frac{3}{a^2} - \frac{6}{b^2} - \frac{1}{c^2} = -3$

   ③ $\frac{9}{a^2} - \frac{5}{b^2} + \frac{4}{c^2} = 67$

23. Determine the equation of a parabola that has its axis parallel to the y-axis and passes through the points $(-1, 2)$, $(1, -1)$, and $(2, 1)$. (Note that the general form of the parabola that is parallel to the y-axis is $y = ax^2 + bx + c$.)

24. A perpendicular line is drawn from point $X(3, 2, -5)$ to the plane $4x - 5y + z - 9 = 0$ and meets the plane at point M. Determine the coordinates of M.

25. Determine the values of A, B, and C if the following is true:

   $$\frac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1}$$

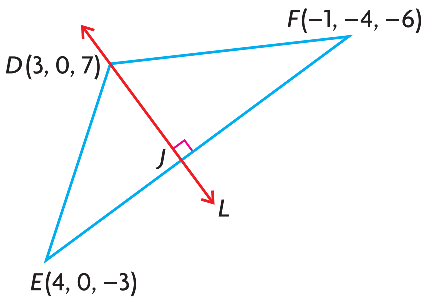
   (*Hint*: Simplify the right side by combining fractions and comparing numerators.)

26. A line L is drawn through point D, perpendicular to the line segment EF, and meets EF at point J.

   a. Determine an equation for the line containing the line segment EF.

   b. Determine the coordinates of point J on EF.

   c. Determine the area of $\triangle DEF$.



27. Determine the equation of the plane that passes through $(5, -5, 5)$ and is perpendicular to the line of intersection of the planes $3x - 2z + 1 = 0$ and $4x + 3y + 7 = 0$.

Chapter 9 Test

1. a. Determine the point of intersection for the lines having equations $\vec{r} = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$, and $\vec{r} = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$.

   b. Verify that the intersection point of these two lines is on the plane $x - y + z + 1 = 0$.

2. a. Determine the distance from point $A(3, 2, 3)$ to π: $8x - 8y + 4z - 7 = 0$.

   b. Determine the distance between the planes π_1: $2x - y + 2z - 16 = 0$ and π_2: $2x - y + 2z + 24 = 0$.

3. a. Determine the equation of the line of intersection L between the planes π_1: $2x + 3y - z = 3$ and π_2: $-x + y + z = 1$.

   b. Determine the point of intersection between L and the xz-plane.

4. a. Solve the following system of equations:

   ① $x - y + z = 10$

   ② $2x + 3y - 2z = -21$

   ③ $\frac{1}{2}x + \frac{2}{5}y + \frac{1}{4}z = -\frac{1}{2}$

   b. Explain what your solution means geometrically.

5. a. Solve the following system of equations:

   ① $x - y + z = -1$

   ② $2x + 2y - z = 0$

   ③ $x - 5y + 4z = -3$

   b. Explain what your solution means geometrically.

6. The three planes $x + y + z = 0$, $x + 2y + 2z = 1$, and $2x - y + mz = n$ intersect in a line.

   a. Determine the values of m and n for which this is true.

   b. What is the equation of the line?

7. Determine the distance between the skew lines with equations
   L_1: $\vec{r} = (-1, -3, 0) + s(1, 1, 1), s \in \mathbf{R}$, and
   L_2: $\vec{r} = (-5, 5, -8) + t(1, 2, 5), t \in \mathbf{R}$.

Cumulative Review of Vectors

1. For the vectors $\vec{a} = (2, -1, -2)$ and $\vec{b} = (3, -4, 12)$, determine the following:
   a. the angle between the two vectors
   b. the scalar and vector projections of $\vec{a}$ on $\vec{b}$
   c. the scalar and vector projections of $\vec{b}$ on $\vec{a}$

2. a. Determine the line of intersection between $\pi_1: 4x + 2y + 6z - 14 = 0$ and $\pi_2: x - y + z - 5 = 0$.
   b. Determine the angle between the two planes.

3. If $\vec{x}$ and $\vec{y}$ are unit vectors, and the angle between them is 60°, determine the value of each of the following:
   a. $|\vec{x} \cdot \vec{y}|$
   b. $|2\vec{x} \cdot 3\vec{y}|$
   c. $|(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})|$

4. Expand and simplify each of the following, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors in R^3:
   a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$
   b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$

5. Determine the angle that the vector $\vec{a} = (4, -2, -3)$ makes with the positive x-axis, y-axis, and z-axis.

6. If $\vec{a} = (1, -2, 3)$, $\vec{b} = (-1, 1, 2)$, and $\vec{c} = (3, -4, -1)$, determine each of the following:
   a. $\vec{a} \times \vec{b}$
   b. $2\vec{a} \times 3\vec{b}$
   c. the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$
   d. $\vec{c} \cdot (\vec{b} \times \vec{a})$

7. Determine the coordinates of the unit vector that is perpendicular to $\vec{a} = (1, -1, 1)$ and $\vec{b} = (2, -2, 3)$.

8. a. Determine vector and parametric equations for the line that contains $A(2, -3, 1)$ and $B(1, 2, 3)$.
   b. Verify that $C(4, -13, -3)$ is on the line that contains A and B.

9. Show that the lines $L_1: \vec{r} = (2, 0, 9) + t(-1, 5, 2)$, $t \in \mathbf{R}$, and $L_2: x - 3 = \frac{y + 5}{-5} = \frac{z - 10}{-2}$ are parallel and distinct.

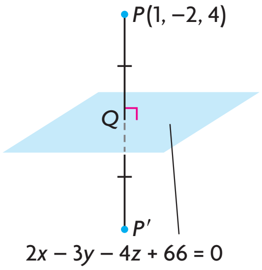
10. Determine vector and parametric equations for the line that passes through $(0, 0, 4)$ and is parallel to the line with parametric equations $x = 1$, $y = 2 + t$, and $z = -3 + t$, $t \in \mathbf{R}$.

11. Determine the value of c such that the plane with equation $2x + 3y + cz - 8 = 0$ is parallel to the line with equation $\frac{x - 1}{2} = \frac{y - 2}{3} = z + 1$.

12. Determine the intersection of the line $\frac{x-2}{3} = y + 5 = \frac{z-3}{5}$ with the plane $5x + y - 2z + 2 = 0$.

13. Sketch the following planes, and give two direction vectors for each.

a. $x + 2y + 2z - 6 = 0$ b. $2x - 3y = 0$ c. $3x - 2y + z = 0$



14. If $P(1, -2, 4)$ is reflected in the plane with equation $2x - 3y - 4z + 66 = 0$, determine the coordinates of its image point, P'. (Note that the plane $2x - 3y - 4z + 66 = 0$ is the right bisector of the line joining $P(1, -2, 4)$ with its image.)

15. Determine the equation of the line that passes through the point $A(1, 0, 2)$ and intersects the line $\vec{r} = (-2, 3, 4) + s(1, 1, 2)$, $s \in \mathbf{R}$, at a right angle.

16. a. Determine the equation of the plane that passes through the points $A(1, 2, 3)$, $B(-2, 0, 0)$, and $C(1, 4, 0)$.

b. Determine the distance from $O(0, 0, 0)$ to this plane.

17. Determine a Cartesian equation for each of the following planes:

a. the plane through the point $A(-1, 2, 5)$ with $\vec{n} = (3, -5, 4)$

b. the plane through the point $K(4, 1, 2)$ and perpendicular to the line joining the points $(2, 1, 8)$ and $(1, 2, -4)$

c. the plane through the point $(3, -1, 3)$ and perpendicular to the z-axis

d. the plane through the points $(3, 1, -2)$ and $(1, 3, -1)$ and parallel to the y-axis

18. An airplane heads due north with a velocity of 400 km/h and encounters a wind of 100 km/h from the northeast. Determine the resultant velocity of the airplane.

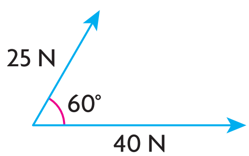
19. a. Determine a vector equation for the plane with Cartesian equation $3x - 2y + z - 6 = 0$, and verify that your vector equation is correct.

b. Using coordinate axes you construct yourself, sketch this plane.

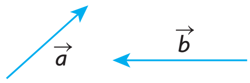
20. a. A line with equation $\vec{r} = (1, 0, -2) + s(2, -1, 2)$, $s \in \mathbf{R}$, intersects the plane $x + 2y + z = 2$ at an angle of θ degrees. Determine this angle to the nearest degree.

b. Show that the planes π_1: $2x - 3y + z - 1 = 0$ and π_2: $4x - 3y - 17z = 0$ are perpendicular.

c. Show that the planes π_3: $2x - 3y + 2z - 1 = 0$ and π_4: $2x - 3y + 2z - 3 = 0$ are parallel but not coincident.



21. Two forces, 25 N and 40 N, have an angle of 60° between them. Determine the resultant and equilibrant of these two vectors.



22. You are given the vectors $\vec{a}$ and $\vec{b}$, as shown at the left.
    a. Sketch $\vec{a} - \vec{b}$.
    b. Sketch $2\vec{a} + \frac{1}{2}\vec{b}$.
23. If $\vec{a} = (6, 2, -3)$, determine the following:
    a. the coordinates of a unit vector in the same direction as $\vec{a}$
    b. the coordinates of a unit vector in the opposite direction to $\vec{a}$
24. A parallelogram $OBCD$ has one vertex at $O(0, 0)$ and two of its remaining three vertices at $B(-1, 7)$ and $D(9, 2)$.
    a. Determine a vector that is equivalent to each of the two diagonals.
    b. Determine the angle between these diagonals.
    c. Determine the angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.
25. Solve the following systems of equations:
    a. ① $x - y + z = 2$
       ② $-x + y + 2z = 1$
       ③ $x - y + 4z = 5$
    b. ① $-2x - 3y + z = -11$
       ② $x + 2y + z = 2$
       ③ $-x - y + 3z = -12$
    c. ① $2x - y + z = -1$
       ② $4x - 2y + 2z = -2$
       ③ $2x + y - z = 5$
    d. ① $x - y - 3z = 1$
       ② $2x - 2y - 6z = 2$
       ③ $-4x + 4y + 12z = -4$
26. State whether each of the following pairs of planes intersect. If the planes do intersect, determine the equation of their line of intersection.
    a. $x - y + z - 1 = 0$
       $x + 2y - 2z + 2 = 0$
    b. $x - 4y + 7z = 28$
       $2x - 8y + 14z = 60$
    c. $x - y + z - 2 = 0$
       $2x + y + z - 4 = 0$
27. Determine the angle between the line with symmetric equations $x = -y$, $z = 4$ and the plane $2x - 2z = 5$.
28. a. If $\vec{a}$ and $\vec{b}$ are unit vectors, and the angle between them is 60°, calculate $(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b})$.
    b. Calculate the dot product of $4\vec{x} - \vec{y}$ and $2\vec{x} + 3\vec{y}$ if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 60°.

29. A line that passes through the origin is perpendicular to a plane π and intersects the plane at $(-1, 3, 1)$. Determine an equation for this line and the cartesian equation of the plane.

30. The point $P(-1, 0, 1)$ is reflected in the plane $\pi: y - z = 0$ and has P' as its image. Determine the coordinates of the point P'.

31. A river is 2 km wide and flows at 4 km/h. A motorboat that has a speed of 10 km/h in still water heads out from one bank, which is perpendicular to the current. A marina lies directly across the river, on the opposite bank.
    a. How far downstream from the marina will the motorboat touch the other bank?
    b. How long will it take for the motorboat to reach the other bank?

32. a. Determine the equation of the line passing through $A(2, -1, 3)$ and $B(6, 3, 4)$.
    b. Does the line you found lie on the plane with equation $x - 2y + 4z - 16 = 0$? Justify your answer.

33. A sailboat is acted upon by a water current and the wind. The velocity of the wind is 16 km/h from the west, and the velocity of the current is 12 km/h from the south. Find the resultant of these two velocities.

34. A crate has a mass of 400 kg and is sitting on an inclined plane that makes an angle of 30° with the level ground. Determine the components of the *weight* of the mass, perpendicular and parallel to the plane. (Assume that a 1 kg mass exerts a force of 9.8 N.)

35. State whether each of the following is true or false. Justify your answer.
    a. Any two non-parallel lines in R^2 must always intersect at a point.
    b. Any two non-parallel planes in R^3 must always intersect on a line.
    c. The line with equation $x = y = z$ will always intersect the plane with equation $x - 2y + 2z = k$, regardless of the value of k.
    d. The lines $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ and $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ are parallel.

36. Consider the lines $L_1: x = 2, \frac{y - 2}{3} = z$ and $L_2: x = y + k = \frac{z + 14}{k}$.
    a. Explain why these lines can never be parallel, regardless of the value of k.
    b. Determine the value of k that makes these two lines intersect at a single point, and find the actual point of intersection.

25. A plane has two parameters, because a plane goes in two different directions, unlike a line that goes only in one direction.

26. This equation will always pass through the origin, because you can always set $s = 0$ and $t = -1$ to obtain (0, 0, 0).

27. **a.** They do not form a plane, because these three points are collinear.
$\vec{r} = (-1, 2, 1) + t(3, 1, -2)$
b. They do not form a plane, because the point lies on the line.
$\vec{r} = (4, 9, -3) + t(1, -4, 2)$
$\vec{r} = (4, 9, -3) + 4(1, -4, 2)$
$= (8, -7, 5)$

28. $bcx + acy + abz - abc = 0$

29. $6x - 5y + 12z + 46 = 0$

30. **a., b.** $\vec{r} = (1, -3, 2) + t(-3, 7, -4) + s(5, -2, 3)\ t, s \in \mathbf{R}$;
$x = 1 - 3t + 5s$,
$y = -3 + 7t - 2s$,
$z = 2 - 4t + 3s$
c. $13x - 11y - 29z + 12 = 0$
d. no

31. **a.** $4x - 2y + 5z = 0$
b. $4x - 2y + 5z + 19 = 0$
c. $4x - 2y + 5z - 22 = 0$

32. **a.** These lines are coincident. The angle between them is 0°.
b. $\left(\frac{3}{2}, 5\right)$, 86.82°

33. **a.** $\vec{r} = (1, 3, 5) + t(-2, -4, -10)$, $t \in \mathbf{R}$;
$x = 1 - 2t$, $y = 3 - 4t$, $z = 5 - 10t$;
$\frac{x-1}{-2} = \frac{y-3}{-4} = \frac{z-5}{-10}$
b. $\vec{r} = (1, 3, 5) + t(-8, 6, -2)$, $t \in \mathbf{R}$;
$x = 1 - 8t$, $y = 3 + 6t$,
$z = 5 - 2t$;
$\frac{x-1}{-8} = \frac{x-3}{6} = \frac{x-5}{-2}$
c. $\vec{r} = (1, 3, 5) + t(-6, -13, 14)$, $t \in \mathbf{R}$;
$x = 1 - 6t$, $y = 3 - 13t$,
$z = 5 + 14t$;
$\frac{x-1}{-6} = \frac{x-3}{-13} = \frac{x-5}{14}$
d. $\vec{r} = (1, 3, 5) + t(1, 0, 0)$, $t \in \mathbf{R}$;
$x = 1 + t$, $y = 3$, $z = 5$
e. $a = 0$, $b = 6$, $c = 4$;
$\vec{r} = (1, 3, 5) + t(0, 6, 4)$, $t \in \mathbf{R}$
f. $\vec{r} = (1, 3, 5) + t(0, 1, 6)$;
$x = 1$, $y = 3 + t$, $z = 5 + 6t$

34. **a.** $2x - 4y + 5z + 23 = 0$
b. $29x + 27y + 24z - 86 = 0$
c. $z - 3 = 0$
d. $3x + y - 4z + 26 = 0$
e. $y - 2z - 4 = 0$
f. $-5x + y + 7z + 18 = 0$

Chapter 8 Test, p. 484

1. **a.** **i.** $\vec{r} = (1, 2, 4) + s(1, -2, -1) + t(3, 2, 0)$, $s, t \in \mathbf{R}$;
$x = 1 + s + 3t$,
$y = 2 - 2s + 2t$, $z = 4 - s$,
$s, t \in \mathbf{R}$
ii. $2x - 3y + 8z - 28 = 0$
b. no

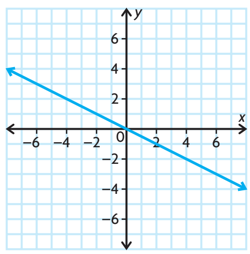
2. **a.** $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$
b. (6, 4, 3)

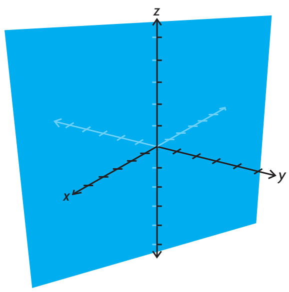
3. **a.** $\vec{r} = s(2, 1, 3) + t(1, 2, 5)$, $s, t \in \mathbf{R}$
b. $-x - 7y + 3z = 0$

4. **a.** $\vec{r} = (4, -3, 5) + s(2, 0, -3) + t(5, 1, -1)$, $s, t \in \mathbf{R}$
b. $3x - 13y + 2z - 61 = 0$

5. **a.** $\left(0, 5, -\frac{1}{2}\right)$
b. $\frac{x}{4} = \frac{y-5}{-2} = \frac{1}{2}$

6. **a.** about 70.5°
b. **i.** 4
ii. $-\frac{1}{5}$
c. The y-intercepts are different and the planes are parallel.

7. **a.**


b.


c. The equation for the plane can be written as $Ax + By + 0z = 0$. For any real number t, $A(0) + B(0) + 0(t) = 0$, so $(0, 0, t)$ is on the plane. Since this is true for all real numbers, the z-axis is on the plane.

Chapter 9

Review of Prerequisite Skills, p. 487

1. **a.** yes **c.** yes
b. no **d.** no

2. Answers may vary. For example:
a. $\vec{r} = (2, 5) + t(5, -2)$, $t \in \mathbf{R}$;
$x = 2 + 5t$, $y = 5 - 2t$, $t \in \mathbf{R}$
b. $\vec{r} = (-3, 7) + t(7, -14)$, $t \in \mathbf{R}$;
$x = -3 + 7t$, $y = 7 - 14t$, $t \in \mathbf{R}$
c. $\vec{r} = (-1, 0) + t(-2, -11)$, $t \in \mathbf{R}$;
$x = -1 + -2t$, $y = -11t$, $t \in \mathbf{R}$
d. $\vec{r} = (1, 3, 5) + t(5, -10, -5)$, $t \in \mathbf{R}$;
$x = 1 + 5t$, $y = 3 - 10t$, $z = 5 - 5t$, $t \in \mathbf{R}$
e. $\vec{r} = (2, 0, -1) + t(-3, 5, 3)$, $t \in \mathbf{R}$;
$x = 2 - 3t$, $y = 5t$, $z = -1 + 3t$, $t \in \mathbf{R}$
f. $\vec{r} = (2, 5, -1) + t(10, -10, -6)$, $t \in \mathbf{R}$;
$x = 2 + 10t$, $y = 5 - 10t$, $z = -1 - 6t$, $t \in \mathbf{R}$

3. **a.** $2x + 6y - z - 17 = 0$
b. $y = 0$
c. $4x - 3y - 15 = 0$
d. $6x - 5y + 3z = 0$
e. $11x - 6y - 38 = 0$
f. $x + y - z - 6 = 0$

4. $5x + 11y + 2z - 21 = 0$

5. L_1 is not parallel to the plane. L_1 is on the plane.
L_2 is parallel to the plane.
L_3 is not parallel to the plane.

6. **a.** $x - y - z - 2 = 0$
b. $x + 6y - 10z - 30 = 0$

7. $\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0)$, $s, t \in \mathbf{R}$

8. $3y + z = 13$

Section 9.1, pp. 496–498

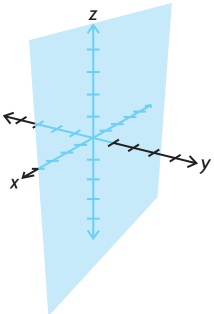
1. **a.** π: $x - 2y - 3z = 6$,
$\vec{r} = (1, 2, -3) + s(5, 1, 1)\ s \in \mathbf{R}$
b. This line lies on the plane.

2. **a.** A line and a plane can intersect in three ways: (1) The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
(2) The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
(3) The line and the plane have an infinite number of intersections. This occurs when the line is

coincident with the plane, meaning the line lies on the plane.

b. Assume that the line and the plane have more than one intersection, but not an infinite number. For simplicity, assume two intersections. At the first intersection, the line crosses the plane. In order for the line to continue on, it must have the same direction vector. If the line has already crossed the plane, then it continues to move away from the plane, and can not intersect again. So, the line and the plane can only intersect zero, one, or infinitely many times.

3. **a.** The line $\vec{r} = s(1, 0, 0)$ is the x-axis.
b. The plane is parallel to the xz-plane, but just one unit away to the right.
c.



d. There are no intersections between the line and the plane.

4. **a.** For $x + 4y + z - 4 = 0$, if we substitute our parametric equations, we have $(-2 + t) + 4(1 - t) + (2 + 3t) + 4 = 0$
All values of t give a solution to the equation, so all points on the line are also on the plane.
b. For the plane $2x - 3y + 2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$:
$2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11 = 0$
All values of t give a solution to this equation, so all points on the line are also on the plane.

5. **a.** $2(-1 - s) - 2(1 + 2s) + 3(2s) - 1 = -5$
Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.
b. $2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13 = 1$
Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. **a.** The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, $\vec{m} \cdot \vec{n} = 0$. So the line is parallel to the plane, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So, the point on the line is not on the plane.
b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, $\vec{m} \cdot \vec{n}$, $= 0$, so the line is parallel to the plane. and
$2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$
So, the point on the line is not on the plane.

7. **a.** $(-19, 0, 10)$
b. $(-11, 1, 0)$

8. **a.** There is no intersection and the lines are skew.
b. $(4, 1, 2)$

9. **a.** not skew
b. not skew
c. not skew
d. skew

10. 8

11. **a.** Comparing components results in the equation $s - t = -4$ for each component.
b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.

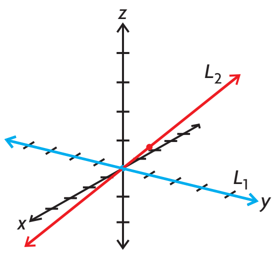
12. **a.** 3
b. $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$

13. 3

14. **a.** $(-6, 1, 3)$
b. 3

15. **a.** $(4, 1, 12)$
b. $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$

16. **a.**



b. (0, 0, 0)
c. If $p = 0$ and $q = 0$, the intersection occurs at (0, 0, 0).

17. **a.** Represent the lines parametrically, and then substitute into the equation for the plane. For the first equation, $x = t$, $y = 7 - 8t$, $z = 1 + 2t$. Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$. Simplifying, $0t = 0$. So, the line lies on the plane.
For the second line, $x = 4 + 3s$, $y = -1$, $z = 1 - 2s$. Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$. Simplifying, $0s = 0$. This line also lies on the plane.
b. $(1, -1, 3)$

18. Answers may vary. For example, $\vec{r} = (2, 0, 0) + p(2, 0, 1)$, $p \in \mathbf{R}$.

Section 9.2, pp. 507–509

1. **a.** linear
b. not linear
c. linear
d. not linear

2. Answers may vary. For example:
a. $x + y + 2z = -15$
$x + 2y + z = -3$
$2x + y + z = -10$
b. $(-3, 4, -8)$

3. **a.** yes
b. no

4. **a.** $(-2, -3)$
b. $(-2, -3)$
The two systems are equivalent because they have the same solution.

5. **a.** $(6, 1)$
b. $(-3, 5)$
c. $(-4, 3)$

6. **a.** These two lines are parallel, and therefore cannot have an intersection.
b. The second equation is five times the first; therefore, the lines are coincident.

7. **a.** $x = t$, $y = 2t - 3$, $t \in \mathbf{R}$
b. $x = t$, $y = s$, $z = 2s - t$, $t \in \mathbf{R}$

8. **a.** $2x + y = -11$
b. $2x + y = -11$
$2(3t + 3) + (-6t - 17) = 6t - 6t + 6 - 17 = -11$

9. **a.** $k \neq 12$
b. not possible
c. $k = 12$

10. **a** infinitely many

b. $x = t$,

$y = \dfrac{11}{4} - \dfrac{1}{2}t, t \in \mathbf{R}$

c. This equation will not have any integer solutions because the left side is an even function and the right side is an odd function.

11. **a.** $x = -a + b, y = -\frac{1}{3}b + \frac{2}{3}a$

b. Since they have different direction vectors, these two equations are not parallel or coincident and will intersect somewhere.

12. **a.** $(-1, -2, 3)$

b. $(3, 4, 12)$

c. $(4, 6, -8)$

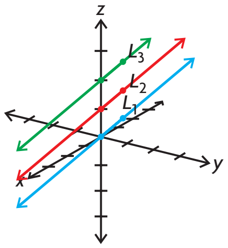
d. $(60, 120, -180)$

e. $(2, 4, 1)$

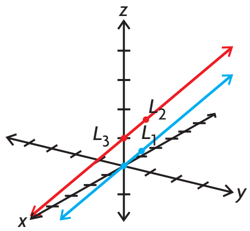
f. $(-2, 3, 6)$

13. Answers may vary. For example:

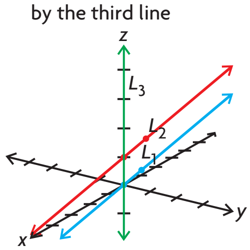
a. Three lines parallel



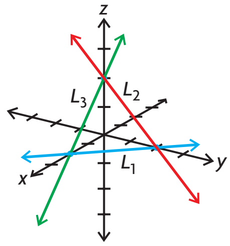
Two lines coincident and the third parallel



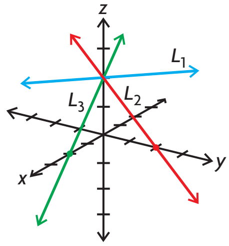
Two parallel lines cut by the third line



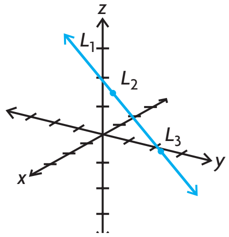
The lines form a triangle



b. Lines meet in a point



c. Three coincident lines



14. $(a - c, -a + b + c, a - b)$

15. **a.** $k = 2$

b. $k = -2$

c. $k \neq \pm 2$

Section 9.3, pp. 516–517

1. **a.** The two equations represent planes that are parallel and not coincident.

b. Answers may vary. For example: $x - y + z = 1, x - y + z = -2$

2. **a.** $x = \dfrac{1}{2} + \dfrac{1}{2}s - t, y = s, z = t$;

$s, t \in \mathbf{R}$; the two planes are coincident.

b. Answers may vary. For example:
$x - y + z = -1$,
$2x - 2y + 2z = -2$

3. **a.** $x = 1 + s, y = s, z = -2, s \in \mathbf{R}$; the two planes intersect in a line.

b. Answers may vary. For example: $x - y + z = -1, x - y - z = 3$

4. **a.** $m = \frac{1}{2}, p = 2q, q = 1$, and $p = 2$; The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

b. $m = \frac{1}{2}, q = 1$, and $p = 3$; The value for m is unique, but p and q can be arbitrarily chosen as long as $p \neq 2q$.

c. $m = -20$; This value is unique, since only one value was found to satisfy the given conditions.

d. $m = -20, p = 1, q = 1$; The value for m is unique from the solution to **c.**, but the values for p and q can be arbitrary since the only value which can change the angle between the planes is m.

5. **a.** $x = 9s, y = -3s, z = s, s \in \mathbf{R}$

b. $x = -3t, y = t, z = -\dfrac{1}{3}t, t \in \mathbf{R}$

c. Since t is an arbitrary real number, we can express t as part b. $t = -3s$, $s \in \mathbf{R}$.

6. **a.** yes; plane

b. no

c. yes; line

d. yes; line

e. yes; line

f. yes; line

7. **a.** $x = 1 - s - t, y = s, z = t, s, t \in \mathbf{R}$

b. no solution

c. $x = -2s, y = -2, z = s, s \in \mathbf{R}$

d. $x = -s + 5, y = -s - 1, z = s$, $s \in \mathbf{R}$

e. $x = \dfrac{5}{4}s, y = s, z = 1 - \dfrac{3}{4}s, s \in \mathbf{R}$

f. $x = s - 8, y = s, z = 4, s \in \mathbf{R}$

8. **a.** The system will have an infinite number of solutions for any value of k.

b. No, there is no value of k for which the system will not have a solution.

9. $\vec{r}_2 = (-2, 3, 6) + s(-5, -8, 2)$, $s \in \mathbf{R}$

10. The line of intersection of the two planes,
$x = 1 - 2s, y = 2 - 2s, z = s; s \in \mathbf{R}$;
$5x + 3y + 16z - 11 = 0$
$5(1 - 2s) + 3(2 - 2s) +$
$16(s) - 11 = 0$
$5 + 6 - 11 - 10s - 6s + 16s = 0$
$0 = 0$
Since this is true, the line is contained in the plane.

11. **a.** $x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}$

b. about 1.73

12. $8x + 14y - 3z - 8 = 0$

Mid-Chapter Review, pp. 518–519

1. **a.** $(-2, 6, 0)$
b. $(2, 0, 10)$
c. $(0, 3, 5)$

2. **a, c.** Answers may vary. For example:
$x = 2 + 3t, y = 1 + 3t,$
$z = 3 - 2t, t \in \mathbf{R};$
$x = 3 + t, y = -2, z = 5\ t \in \mathbf{R};$
$x = -8 + 7t, y = -5 + 3t,$
$z = 7 - 2t, t \in \mathbf{R}$
b. $(-1, -2, 5)$
d. $C: x = -8 + 7t, y = -5 + 3t,$
$z = 7 - 2t, t \in \mathbf{R}$
$t = 1$
$x = -8 + 7(1), y = -5 + 3(1),$
$z = 7 - 2(1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
e. $(-1, -2, 5)$

3. **a.** $\vec{r} = (-7, 20, 0) + t(0, -2, 1),$
$t \in \mathbf{R}$
b. $\vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7),$
$t \in \mathbf{R}$
c. $(-7, 0, 10)$

4. **a.** $x = -\frac{11t}{5} - \frac{1}{40}, y = -\frac{2t}{5} - \frac{117}{40},$
$z = t, t \in \mathbf{R}$
b. $x = -\frac{11}{5}s + \frac{227}{5}, y = -\frac{2}{5}s + \frac{94}{5},$
$z = s, t \in \mathbf{R}$
c. The lines found in 4.a. and 4.b. do not intersect, because they are in parallel and distinct planes.

5. **a.** $a = 3$
b. $a = -3$
c. $a \neq \pm 3, a \in \mathbf{R}$

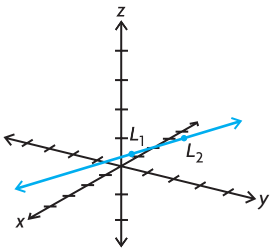
6. Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

7. **a.** no intersection
b. The lines are skew.

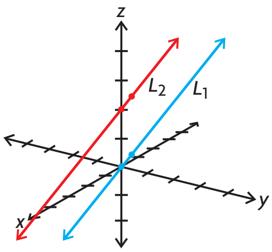
8. $(-3, 6, 6)$

9. **a.** $(3, 1, 2)$
b. $t \in \mathbf{R}$
These lines are the same, so either one of these lines can be used as their intersection.

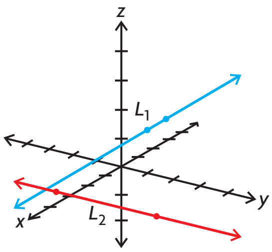
10. **a.** Answers may vary. For example:
i. coincident



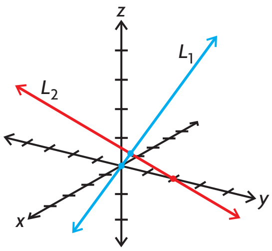
ii. parallel and distinct



iii. skew



iv. intersect in a point



b. **i.** When lines are the same, they are a multiple of each other.
ii. When lines are parallel, one equation is a multiple of the other equation, except for the constant term.
iii. When lines are skew, there are no common solutions to make each equation consistent.
iv. When the solution meets in a point, there is only one unique solution for the system.

11. **a.** when the line lies in the plane
b. Answers may vary. For example:
$\vec{r} = t(3, -5, -3), t \in \mathbf{R};$
$\vec{r} = t(3, -5, -3) + s(1, 1, 1),$
$t, s \in \mathbf{R}$

12. **a.** $(3, 8)$
b. no solution
c. $(2, 1, 4)$

13. **a.** The two lines intersect at a point.
b. The two planes are parallel and do not meet.
c. The three planes intersect at a point.

14. **a.** $\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$
b. $\theta = 90°$
c. $2x - y + z + 1 = 0$

Section 9.4, pp. 530–533

1. **a.** $(-9, -5, -4)$
b. This solution is the point at which all three planes meet.

2. **a.** Answers may vary. For example,
$3x - 3y + 3z = 12$ and
$2x - 2y + 2z = 8.$
b. These three planes are intersecting in one single plane because all three equations can be changed into one equivalent equation. They are coincident planes.
c. $x = t, y = s, z = s - t + 4, s, t \in \mathbf{R}$
d. $y = t, z = s, x = t - s + 4, s, t \in \mathbf{R}$

3. **a.** Answers may vary. For example,
$2x - y + 3z = -2, x - y + 4z = 3,$
and $3x - 2y + 7z = 2;$
$2x - y + 3z = -2, x - y + 4z = 3,$
and $2x - 2y + 8z = 5.$
b. no solutions

4. **a.** $\left(-3, \frac{11}{4}, -\frac{3}{2}\right)$
b. This solution is the point at which all three planes meet.

5. **a.** Since equation ③ = equation ②, equation ② and equation ③ are consistent or lie in the same plane. Equation ① meets this plane in a line.
b. $x = 0, y = t,$ and $z = 1 + t, t \in \mathbf{R}$

6. If you multiply equation ② by 5, you obtain a new equation, $5x - 5y + 15z = -1005$, which is inconsistent with equation ③.

7. **a.** Yes, when this equation is alone, this is true.
b. Answers may vary. For example:
$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 12$

8. a. $(-1, -1, 0)$ is the point at which the three planes meet.
b. $\left(-6, \frac{1}{2}, 3\right)$ is the point at which the three planes meet.
c. $(-99, 100, -101)$ is the point at which the three planes meet.
d. $(4, 2, 3)$ is the point at which the three planes meet.

9. a. $x = -\frac{1}{7}t - \frac{9}{7}, y = -\frac{15}{7} + \frac{3}{7}t$, and $z = t, t \in \mathbf{R}$; the planes intersect in a line.
b. no solution
c. $x = -t, y = 2$, and $z = t, t \in \mathbf{R}$; the planes intersect in a line.

10. a. $x = 0, y = t - 2$, and $z = t, t \in \mathbf{R}$
b. $x = \frac{t - 3s}{2}, y = t$, and $z = s$, $s, t \in \mathbf{R}$

11. a. ① $x + y + z = 1$
② $x - 2y + z = 0$
③ $x - y + z = 0$
Equation ① − equation ③ =
Equation ④ $= 2y = 1$ or $y = \frac{1}{2}$
Equation ② − equation ③ =
Equation ⑤ $= -y = 0$ or $y = 0$
Since the y-variable is different in Equation ④ and Equation ⑤, the system is inconsistent and has no solution.
b. Answers may vary. For example:
$\vec{n_1} = (1, 1, 1)$
$\vec{n_2} = (1, -2, 1)$
$\vec{n_3} = (1, -1, 1)$
$m_1 = \vec{n_1} \times \vec{n_2} = (3, 0, -3)$
$m_2 = \vec{n_1} \times \vec{n_3} = (2, 0, -2)$
$m_3 = \vec{n_2} \times \vec{n_3} = (-1, 0, 1)$
c. The three lines of intersection are parallel and coplanar, so they form a triangular prism.
d. Since $(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3} = 0$, a triangular prism forms.

12. a. Equation ① and equation ② have the same set of coefficients and variables; however, equations ① equals 3, while equation ② equals 6, which means there is no possible solution.
b. All three equations equal different numbers, so there is no possible solution.
c. Equation ② equals 18, while equation ③ equals 17, which means there is no possible solution.
d. The coefficients of equation ① are half the coefficients of equation ③, but the constant term is not half the other constant term.

13. a. $(4, 3, -5)$
b. $x = \frac{t - 2}{3}, y = \frac{5t + 5}{3}, z = t, t \in \mathbf{R}$
c. $x = 0, y = t, z = t, t \in \mathbf{R}$
d. no solution
e. $x = -t, y = 2, z = t, t \in \mathbf{R}$
f. $(0, 0, 0)$

14. a. $p = q = 5$
b. $x = -\frac{2}{3}t + 3, y = \frac{1}{3}t - 2, z = t$, $t \in \mathbf{R}$

15. a. $m = 2$
b. $m \neq \pm 2, m \in \mathbf{R}$
c. $m = -2$

16. $(3, 6, 2)$

Section 9.5, pp. 540–541

1. a. $\frac{3}{5}$
b. $\frac{56}{13}$ or 4.31
c. $\frac{236}{\sqrt{1681}}$ or 5.76

2. a. $\frac{5}{\sqrt{5}}$ or 2.24
b. $\frac{504}{25}$ or 20.16

3. a. 1.4
b. about 3.92
c. about 2.88

4. a. $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$
If you substitute the coordinates $(0, 0)$, the formula changes to $d = \frac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}}$, which reduces to $d = \frac{|C|}{\sqrt{A^2 + B^2}}$.
b. $\frac{24}{5}$
c. $\frac{24}{5}$; the answers are the same

5. a. 3
b. $\frac{7}{5}$ or 1.4
c. $\frac{4}{\sqrt{13}}$ or 1.11
d. $\frac{240}{13}$ or 18.46

6. a. about 1.80
b. about 2.83
c. about 3.44

7. a. about 2.83
b. about 3.28

8. a. $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$
b. about 1.65

9. about 3.06; $\left(-\frac{11}{14}, \frac{5}{14}, \frac{22}{14}\right)$

10. $\left(\frac{38}{21}, -\frac{44}{21}, \frac{167}{21}\right)$

11. a. about 1.75
b. D and G
c. about 3.61 units^2

Section 9.6, pp. 549–550

1. a. Yes, the calculations are correct. Point A lies in the plane.
b. The answer 0 means that the point lies in the plane.

2. a. 3
b. 3
c. 2
d. $\frac{5}{13}$ or 0.38
e. $\frac{11}{27}$ or 0.41

3. a. 5
b. $6x + 8y - 24z + 13 = 0$
c. Answers may vary. For example: $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$

4. a. 4 b. 4 c. 2

5. $\frac{2}{3}$ or 0.67

6. 3

7. about 1.51

8. a. about 3.46
b. $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$.

Review Exercise, pp. 552–555

1. $-\frac{4}{99}$

2. no solution

3. a. no solution
b. $(99, 100, 101)$

4. a. All four points lie on the plane $3x + 4y - 2z + 1 = 0$
b. about 0.19

5. a. 3
b. $\frac{1}{12}$ or 0.08

6. $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$

7. a. no solution
b. no solution
c. no solution

8. **a.** $x = -\frac{5}{7}t, y = 1 + \frac{2}{7}t, z = t, t \in \mathbf{R}$

b. $x = 3, y = \frac{1}{4}, z = -\frac{1}{2}$

c. $x = 3t - 3s + 7, y = t, z = s,$ $s, t \in \mathbf{R}$

9. **a.** $x = \frac{1}{2} + \frac{1}{36}t, y = -\frac{1}{2} + \frac{5}{12}t,$ $z = t, t \in \mathbf{R}$ $\left(\frac{42}{83}, -\frac{31}{83}, \frac{18}{83}\right)$

b. $x = \frac{9}{8} - \frac{31}{24}t, y = \frac{1}{4} + \frac{1}{12}t, z = t,$ $t \in \mathbf{R}$

10. **a.** These three planes meet at the point $(-1, 5, 3)$.

b. The planes do not intersect. Geometrically, the planes form a triangular prism.

c. The planes meet in a line through the origin, with equation $x = t,$ $y = -7t, z = -5t, t \in \mathbf{R}$

11. 4.90

12. **a.** $x - 2y + z + 4 = 0$

$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$

$\vec{m} \times \vec{n} = (2, 1, 0)(1, -2, 1) = 0$

Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.

b. $(-1, -1, -5)$

c. $x - 2y + z + 4 = 0$

$-1 - 2(-1) + (-5) + 4 = 0$

The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.

d. $7x - 2y - 11z - 50 = 0$

13. **a.** 5.48

b. $(3, 0, -1)$

14. **a.** $(-2, -3, 0)$.

b. $\vec{r} = (-2, -3, 0) + t(1, -2, 1),$ $t \in \mathbf{R}$

15. **a.** $-10x + 9y + 8z + 16 = 0$

b. about 0.45

16. **a.** 1

b. $\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$

17. **a.** $x = 2, y = -1, z = 1$

b. $x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. $a = \frac{2}{3}, b = \frac{3}{4}, c = \frac{1}{2}$

19. $\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. $\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$

21. **a.** $\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right)$ $+ t(11, 2, -5), t \in \mathbf{R};$

$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right)$ $+ t(11, 2, -5), t \in \mathbf{R};$

$\vec{r} = (7, 0, -1) + t(11, 2, -5),$ $t \in \mathbf{R}; z = -1 - 5t, t \in \mathbf{R}$

b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. $\left(\frac{1}{2}, 1, \frac{1}{3}\right), \left(\frac{1}{2}, 1, -\frac{1}{3}\right), \left(\frac{1}{2}, -1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right), \left(-\frac{1}{2}, 1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right) \left(-\frac{1}{2}, 1, -\frac{1}{3}\right)$, and $\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$

23. $y = \frac{7}{6}x^2 - \frac{3}{2}x - \frac{2}{3}$

24. $\left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$

25. $A = 5, B = 2, C = -4$

26. **a.** $\vec{r} = (-1, -4, -6)$ $+ t(-5, -4, -3), t \in \mathbf{R}$

b. $\left(\frac{13}{2}, 2, -\frac{3}{2}\right)$

c. about 33.26 units^2

27. $6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. **a.** $(3, -1, -5)$

b. $3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. **a.** $\frac{13}{12}$ or 1.08

b. $\frac{40}{3}$ or 13.33

3. **a.** $x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$

b. $(4, 0, 5)$

4. **a.** $(1, -5, 4)$

b. The three planes intersect at the point $(1, -5, 4)$.

5. **a.** $x = -\frac{1}{2} - \frac{t}{4}, y = \frac{3t}{4} + \frac{1}{2}, z = t,$ $t \in \mathbf{R}$

b. The three planes intersect at this line.

6. **a.** $m = -1, n = -3$

b. $x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. 10.20

Cumulative Review of Vectors, pp. 557–560

1. **a.** about 111.0°

b. scalar projection: $-\frac{14}{13}$,

vector projection: $\left(-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169}\right)$

c. scalar projection: $-\frac{14}{3}$,

vector projection: $\left(-\frac{28}{9}, \frac{14}{9}, \frac{28}{9}\right)$

2. **a.** $x = 8 + 4t, y = t, z = -3 - 3t,$ $t \in \mathbf{R}$

b. about 51.9°

3. **a.** $\frac{1}{2}$

b. 3

c. $\frac{3}{2}$

4. **a.** $-7\vec{i} - 19\vec{j} - 14\vec{k}$

b. 18

5. x-axis: about 42.0°, y-axis: about 111.8°, z-axis: about 123.9°

6. **a.** $(-7, -5, -1)$

b. $(-42, -30, -6)$

c. about 8.66 square units

d. 0

7. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

8. **a.** vector equation: Answers may vary. $\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R};$ parametric equation: $x = 2 - t, y = -3 + 5t,$ $z = 1 + 2t, t \in \mathbf{R}$

b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2)$ $= (4, -13, -3)$. Hence, $C(4, -13, -3)$ is a point on the line.

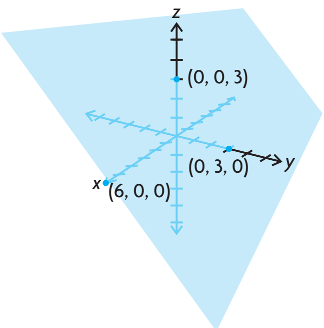
9. The direction vector of the first line is $(-1, 5, 2)$ and of the second line is $(1, -5, -2) = -(-1, 5, 2)$. So they are collinear and hence parallel. The lines coincide if and only if for any point on the first line and second line, the vector connecting the two points is a multiple of the direction vector for the lines. $(2, 0, 9)$ is a point on the first line and $(3, -5, 10)$ is a point on the second line. $(2, 0, 9) - (3, -5, 10) = (-1, 5, -1)$ $\neq k(-1, 5, 2)$ for $k \in \mathbf{R}$. Hence, the lines are parallel and distinct.

10. vector equation:
$\vec{r} = (0, 0, 4) + t(0, 1, 1), t \in \mathbf{R}$;
parametric equation: $x = 0, y = t,$
$z = 4 + t, t \in \mathbf{R}$

11. -13

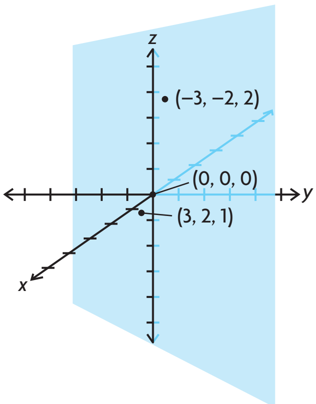
12. $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$

13. **a.**



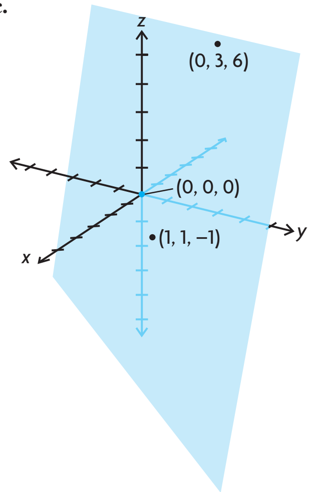
Answers may vary. For example, $(0, 3, -3)$ and $(6, 0, -3)$.

b.



Answers may vary. For example, $(-3, -2, 2)$ and $(3, 2, 1)$.

c.



Answers may vary. For example, $(0, 3, 6)$ and $(1, 1, -1)$.

14. $(-7, 10, 20)$

15. $\vec{q} = (1, 0, 2) + t(-11, 7, 2), t \in \mathbf{R}$

16. **a.** $12x - 9y - 6z + 24 = 0$
b. about 1.49 units

17. **a.** $3x - 5y + 4z - 7 = 0$
b. $x - y + 12z - 27 = 0$
c. $z - 3 = 0$
d. $x + 2z + 1 = 0$

18. 336.80 km/h, N 12.1° W

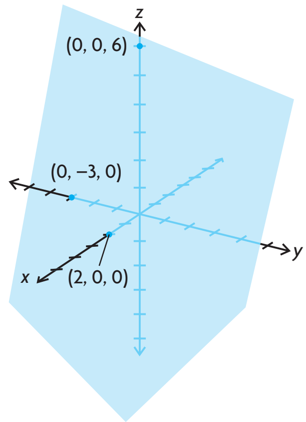
19. **a.** $\vec{r} = (0, 0, 6) + s(1, 0, -3) + t(0, 1, 2), s, t \in \mathbf{R}$. To verify, find the Cartesian equation corresponding to the above vector equation and see if it is equivalent to the Cartesian equation given in the problem. A normal vector to this plane is the cross product of the two directional vectors.

$\vec{n} = (1, 0, -3) \times (0, 1, 2)$

$= (0(2) - (-3)(1), -3(0) - 1(2), 1(1) - 0(0))$
$= (3, -2, 1)$

So the plane has the form $3x + 2y + z + D = 0$, for some constant D. To find D, we know that $(0, 0, 6)$ is a point on the plane, so $3(0) - 2(0) + (6) + D = 0$. So, $6 + D = 0$, or $D = -6$. So, the Cartesian equation for the plane is $3x - 2y + z - 6 = 0$. Since this is the same as the initial Cartesian equation, the vector equation for the plane is correct.

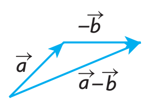
b.



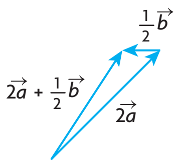
20. **a.** 16°
b. The two planes are perpendicular if and only if their normal vectors are also perpendicular. A normal vector for the first plane is $(2, -3, 1)$ and a normal vector for the second plane is $(4, -3, -17)$. The two vectors are perpendicular if and only if their dot product is zero.
$(2, -3, 1) \cdot (4, -3, -17)$
$= 2(4) - 3(-3) + 1(-17)$
$= 0$
Hence, the normal vectors are perpendicular. Thus, the planes are perpendicular.
c. The two planes are parallel if and only if their normal vectors are also parallel. A normal vector for the first plane is $(2, -3, 2)$ and a normal vector for the second plane is $(2, -3, 2)$. Since both normal vectors are the same, the planes are parallel. Since $2(0) - 3(-1) + 2(0) - 3 = 0$, the point $(0, -1, 0)$ is on the second plane. Yet since $2(0) - 3(-1) + 2(0) - 1 = 2 \neq 0$, $(0, -1, 0)$ is not on the first plane. Thus, the two planes are parallel but not coincident.

21. resultant: about 56.79 N, 37.6° from the 25 N force toward the 40 N force, equilibrant: about 56.79 N, 142.4° from the 25 N force away from the 40 N force

22. **a.**



b.



23. **a.** $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

b. $\left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$

24. **a.** $\overrightarrow{OC} = (8, 9)$, $\overrightarrow{BD} = (10, -5)$

b. about 74.9°

c. about 85.6°

25. **a.** $x = t, y = -1 + t, z = 1, t \in \mathbf{R}$

b. $(1, 2, -3)$

c. $x = 1, y = t, z = -3 + t, t \in \mathbf{R}$

d. $x = 1 + 3s + t, y = t, z = s,$ $s, t \in \mathbf{R}$

26. **a.** yes; $x = 0, y = -1 + t, z = t, t \in \mathbf{R}$

b. no

c. yes; $x = 2 - 2t, y = t, z = 3t, t \in \mathbf{R}$

27. 30°

28. **a.** $-\frac{3}{2}$

b. 84

29. $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$, $-x + 3y + z - 11 = 0$

30. $(-1, 1, 0)$

31. **a.** 0.8 km

b. 12 min

32. **a.** Answers may vary. $\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$

b. The line found in part a will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane. We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. 20 km/h at N 53.1° E

34. parallel: 1960 N, perpendicular: about 3394.82 N

35. **a.** True; all non-parallel pairs of lines intersect in exactly one point in $\mathbf{R}^2$. However, this is not the case for lines in $\mathbf{R}^3$ (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in $\mathbf{R}^3$.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k is any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. **a.** A direction vector for $L_1: x = 2, \frac{y - 2}{3} = z$ is $(0, 3, 1)$, and a direction vector for $L_2: x = y + k = \frac{z + 14}{k}$ is $(1, 1, k)$. But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k, since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. 6; $(2, -4, -2)$

Calculus Appendix

Implicit Differentiation, p. 564

1. The chain rule states that if y is a composite function, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. To differentiate an equation implicitly, first differentiate both sides of the equation with respect to x, using the chain rule for terms involving y, then solve for $\frac{dy}{dx}$.

2. **a.** $-\frac{x}{y}$

b. $\frac{x^2}{5y}$

c. $\frac{-y^2}{2xy + y^2}$

d. $\frac{9x}{16y}$

e. $-\frac{13x}{48y}$

f. $-\frac{2x}{2y + 5}$

3. **a.** $y = \frac{2}{3}x - \frac{13}{3}$

b. $y = \frac{2}{3}(x + 8) + 3$

c. $y = -\frac{3\sqrt{3}}{5}x - 3$

d. $y = \frac{11}{10}(x + 11) - 4$

4. $(0, 1)$

5. **a.** 1

b. $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$

6. -10

7. $7x - y - 11 = 0$

8. $y = \frac{1}{2}x - \frac{3}{2}$

9. **a.** $\frac{4}{(x + y)^2} - 1$

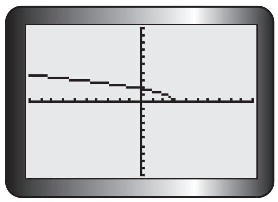
b. $4\sqrt{x + y} - 1$

10. **a.** $\frac{3x^2 - 8xy}{4x^2 - 3}$

b. $y = \frac{x^3}{4x^2 - 3}; \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$

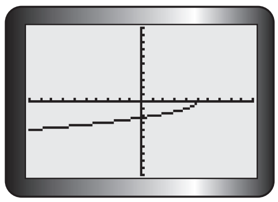
c.
$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$$
$$y = \frac{x^3}{4x^2 - 3}$$
$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$
$$= \frac{3x^2 - (4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$$

11. **a.**



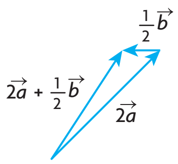
one tangent

b.



one tangent

b.



23. **a.** $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

b. $\left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$

24. **a.** $\overrightarrow{OC} = (8, 9)$,
$\overrightarrow{BD} = (10, -5)$

b. about 74.9°

c. about 85.6°

25. **a.** $x = t, y = -1 + t, z = 1, t \in \mathbf{R}$

b. $(1, 2, -3)$

c. $x = 1, y = t, z = -3 + t, t \in \mathbf{R}$

d. $x = 1 + 3s + t, y = t, z = s,$
$s, t \in \mathbf{R}$

26. **a.** yes; $x = 0, y = -1 + t, z = t, t \in \mathbf{R}$

b. no

c. yes;
$x = 2 - 2t, y = t, z = 3t, t \in \mathbf{R}$

27. 30°

28. **a.** $-\frac{3}{2}$

b. 84

29. $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$,
$-x + 3y + z - 11 = 0$

30. $(-1, 1, 0)$

31. **a.** 0.8 km

b. 12 min

32. **a.** Answers may vary.
$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$

b. The line found in part a will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane. We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. 20 km/h at N 53.1° E

34. parallel: 1960 N,
perpendicular: about 3394.82 N

35. **a.** True; all non-parallel pairs of lines intersect in exactly one point in $\mathbf{R}^2$. However, this is not the case for lines in $\mathbf{R}^3$ (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in $\mathbf{R}^3$.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k is any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z+1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x-1}{-4} = \frac{y-1}{-2} = \frac{z+1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. **a.** A direction vector for
L_1: $x = 2, \frac{y-2}{3} = z$ is $(0, 3, 1)$,
and a direction vector for
L_2: $x = y + k = \frac{z + 14}{k}$ is $(1, 1, k)$.
But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k, since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. 6; $(2, -4, -2)$

Calculus Appendix

Implicit Differentiation, p. 564

1. The chain rule states that if y is a composite function, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. To differentiate an equation implicitly, first differentiate both sides of the equation with respect to x, using the chain rule for terms involving y, then solve for $\frac{dy}{dx}$.

2. **a.** $-\frac{x}{y}$

b. $\frac{x^2}{5y}$

c. $\frac{-y^2}{2xy + y^2}$

d. $\frac{9x}{16y}$

e. $-\frac{13x}{48y}$

f. $-\frac{2x}{2y + 5}$

3. **a.** $y = \frac{2}{3}x - \frac{13}{3}$

b. $y = \frac{2}{3}(x + 8) + 3$

c. $y = -\frac{3\sqrt{3}}{5}x - 3$

d. $y = \frac{11}{10}(x + 11) - 4$

4. $(0, 1)$

5. **a.** 1

b. $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$

6. -10

7. $7x - y - 11 = 0$

8. $y = \frac{1}{2}x - \frac{3}{2}$

9. **a.** $\frac{4}{(x + y)^2} - 1$

b. $4\sqrt{x + y} - 1$

10. **a.** $\frac{3x^2 - 8xy}{4x^2 - 3}$

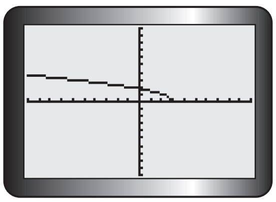
b. $y = \frac{x^3}{4x^2 - 3}$; $\frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$

c. $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$

$y = \frac{x^3}{4x^2 - 3}$

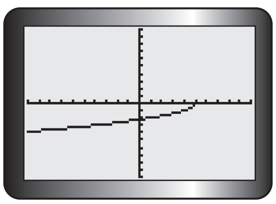
$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$
$$= \frac{3x^2 - (4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$$

11. **a.**



one tangent

b.



one tangent