Chapter 8

EQUATIONS OF LINES AND PLANES

In this chapter, you will work with vector concepts you learned in the preceding chapters and use them to develop equations for lines and planes. We begin with lines in R^2 and then move to R^3, where lines are once again considered along with planes. The determination of equations for lines and planes helps to provide the basis for an understanding of geometry in R^3. All of these concepts provide the foundation for the solution of systems of linear equations that result from intersections of lines and planes, which are considered in Chapter 9.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the vector and parametric equations of a line in two-space, **Section 8.1**
- make connections between Cartesian, vector, and parametric equations of a line in two-space, **Section 8.2**
- determine the vector, parametric, and symmetric equations of a line in 3-space, **Section 8.3**
- determine the vector, parametric, and Cartesian equations of a plane, **Sections 8.4, 8.5**
- determine some geometric properties of a plane, **Section 8.5**
- determine the equation of a plane in Cartesian, vector, or parametric form, given another form, **Section 8.5**
- sketch a plane in 3-space, **Section 8.6**

Review of Prerequisite Skills

In this chapter, we will develop the equation of a line in two- and three-dimensional space and the equation of a plane in three-dimensional space. You will find it helpful to review the following concepts:

- geometric and algebraic vectors
- the dot product
- the cross product
- plotting points and vectors in three-space

We will begin this chapter by examining equations of lines. Lines are not vectors, but vectors are used to describe lines. The table below shows their similarities and differences.

Lines	Vectors
Lines are bi-directional. A line defines a direction, but there is nothing to distinguish forward from backwards.	Vectors are unidirectional. A vector defines a direction with a clear distinction between forward and backwards.
A line is infinite in extent in both directions. A line segment has a finite length.	Vectors have a finite magnitude.
Lines and line segments have a definite location. The opposite sides of a parallelogram are two different line segments.	A vector has no fixed location. The opposite sides of a parallelogram are described by the same vector.
Two lines are the same when they have the same direction and same location. These lines are said to be coincident.	Two vectors are the same when they have the same direction and the same magnitude. These vectors are said to be equal.

Exercise

1. Determine a single vector that is equivalent to each of the following expressions:

a. $(3, -2, 1) - (1, 7, -5)$

b. $5(2, -3, -4) + 3(1, 1, -7)$

2. Determine if the following sets of points are collinear:

a. $A(1, -3), B(4, 2), C(-8, -18)$

b. $J(-4, 3), K(4, 5), L(0, 4)$

c. $A(1, 2, 1), B(4, 7, 0), C(7, 12, -1)$

d. $R(1, 2, -3), S(4, 1, 3), T(2, 4, 0)$

3. Determine if $\triangle ABC$ is a right-angled triangle, given $A(1, 6, -2)$, $B(2, 5, 3)$, and $C(5, 3, 2)$.

4. Given $\vec{u} = (t, -1, 3)$ and $\vec{v} = (2, t, -6)$, for what values of t are the vectors perpendicular?

5. State a vector perpendicular to each of the following:

   a. $\vec{a} = (1, -3)$
   b. $\vec{b} = (6, -5)$
   c. $\vec{c} = (-7, -4, 0)$

6. Calculate the area of the parallelogram formed by the vectors $(4, 10, 9)$ and $(3, 1, -2)$.

7. Use the cross product to determine a vector perpendicular to each of the following pairs of vectors. Check your answer using the dot product.

   a. $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -5, -2)$
   b. $\vec{a} = (-1, -2, 0)$ and $\vec{b} = (-2, -1, 0)$

8. For each of the following, draw the x-axis, y-axis, and z-axis, and accurately draw the position vectors:

   a. $A(1, 2, 3)$
   b. $B(1, 2, -3)$
   c. $C(1, -2, 3)$
   d. $D(-1, 2, 3)$

9. Determine the position vector that passes from the first point to the second.

   a. $(4, 8)$ and $(-3, 5)$
   b. $(-7, -6)$ and $(3, 8)$
   c. $(1, 2, 4)$ and $(3, -6, 9)$
   d. $(4, 0, -4)$ and $(0, 5, 0)$

10. State the vector that is opposite to each of the vectors you found in question 9.

11. Determine the slope and y-intercept of each of the following linear equations. Then sketch its graph.

    a. $y = -2x - 5$
    b. $4x - 8y = 8$
    c. $3x - 5y + 1 = 0$
    d. $5x = 5y - 15$

12. State a vector that is collinear to each of the following and has the same direction:

    a. $(4, 7)$
    b. $(-5, 4, 3)$
    c. $2\vec{i} + 6\vec{j} - 4\vec{k}$
    d. $-5\vec{i} + 8\vec{j} + 2\vec{k}$

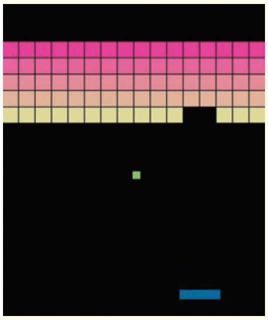
13. If $\vec{u} = (4, -9, -1)$ and $\vec{v} = 4\vec{i} - 2\vec{j} + \vec{k}$, determine each of the following:

    a. $\vec{u} \cdot \vec{v}$
    b. $-\vec{v} \cdot \vec{u}$
    c. $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$
    d. $\vec{u} \times \vec{v}$
    e. $\vec{v} \times \vec{u}$
    f. $(2\vec{u} + \vec{v}) \times (\vec{u} - 2\vec{v})$

14. Both the dot product and the cross product are ways to multiply two vectors. Explain how these products differ.

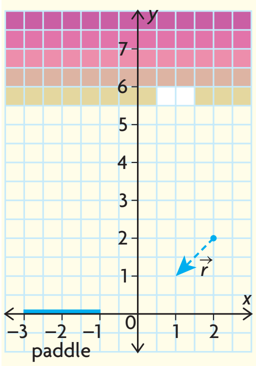
CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

Computer programmers use vectors for a variety of graphics applications. Any time that two- and three-dimensional images are designed, they are represented in the form of vectors. Vectors allow the programmer to move the figure easily to any new location on the screen. If the figure were expressed point by point using coordinate geometry, each and every point would have to be recalculated each time the figure needed to be moved. By using vectors drawn from an anchor point to draw the figure, only the coordinates of the anchor point need to be recalculated on the screen to move the entire figure. This method is used in many different types of software, including games, flight simulators, drafting and architecture tools, and visual design tools.



Case Study—Breakout

Breakout is a classic video game, and a prime example of using vectors in computer graphics. A paddle is used to bounce a ball into a section of bricks to slowly break down a wall. Each time the ball hits the paddle, it bounces off at an angle to the paddle. The path the ball takes from the paddle can be described by a vector, which is dependent upon the angle and speed of the ball. If the wall were not in the path of the ball, the ball would continue along its path at that speed until it "fell" off of the screen.



DISCUSSION QUESTIONS

1. The coordinate plane represents the screen in the game "Breakout." The ball is travelling toward the paddle along the vector $\vec{r}$. Find the equation of the line determined by vector $\vec{r}$ in its current position. Draw a direction vector for the line.
2. Find where the line crosses the x-axis to show where the paddle must move in order to bounce the ball back.
3. Since the angle of entry for the ball is 45°, the ball will bounce off the paddle along a path perpendicular to $\vec{r}$. Draw a vector $\vec{s}$ perpendicular to $\vec{r}$ that emanates from the origin in the direction the ball will travel when it bounces off the paddle. Then draw a line parallel to vector $\vec{s}$ that passes through the point where $\vec{r}$ crosses the x-axis.

Section 8.1—Vector and Parametric Equations of a Line in R^2

In this section, we begin with a discussion about how to find the **vector** and **parametric equations** of a line in R^2. To find the vector and parametric equations of a line, we must be given either two distinct points or one point and a vector that defines the direction of the line. In either situation, a **direction vector** for the line is necessary. A direction vector is defined to be a nonzero vector $\vec{m} = (a, b)$ parallel (collinear) to the given line. The direction vector $\vec{m} = (a, b)$ is represented by a vector with its tail at the origin and its head at the point (a, b). The x and y components of this direction vector are called its **direction numbers**. For the vector (a, b), the direction numbers are a and b.

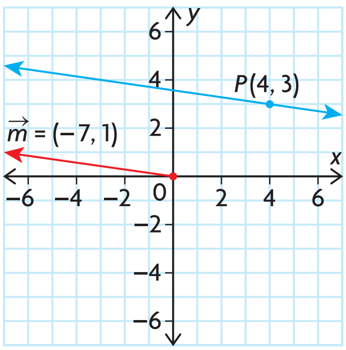
EXAMPLE 1 Representing lines using vectors

a. A line passing through $P(4, 3)$ has $\vec{m} = (-7, 1)$ as its direction vector. Sketch this line.

b. A line passes through the points $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$. Determine a direction vector for this line, and write it using integer components.

Solution

a. The vector $\vec{m} = (-7, 1)$ is a direction vector for the line and is shown on the graph. The required line is parallel to $\vec{m}$ and passes through $P(4, 3)$. This line is drawn through $P(4, 3)$, parallel to $\vec{m}$.



required line with direction vector (−7, 1)

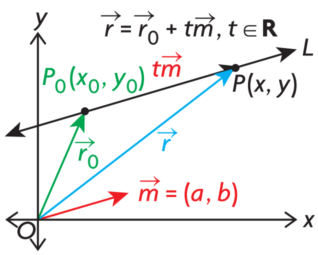
b. When determining a direction vector for the line through $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$, we determine a vector equivalent to either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{AB} = \left(\frac{3}{4} - \frac{1}{2}, \frac{1}{2} - (-3)\right) = \left(\frac{1}{4}, \frac{7}{2}\right) \text{ or } \overrightarrow{BA} = \left(-\frac{1}{4}, -\frac{7}{2}\right)$$

Both of these vectors can be multiplied by 4 to ensure that both direction numbers are integers. As a result, either $\vec{m} = (1, 14)$ or $\vec{m} = (-1, -14)$ are the best choices for a direction vector. When we determine the direction vector, any scalar multiple of this vector of the form $t(1, 14)$ is correct, provided that $t \neq 0$. If $t = 0$, $(0, 0)$ would be the direction vector, meaning that the line would not have a defined direction.

Expressing the Equations of Lines Using Vectors

In general, we would like to determine the equation of a line if we have a direction for the line and a point on it. In the following diagram, the given point $P_0(x_0, y_0)$ is on the line L and is associated with vector $\overrightarrow{OP_0}$, designated as $\vec{r_0}$. The direction of the line is given by $\vec{m} = (a, b)$, where $t\vec{m}$, $t \in \mathbf{R}$ is any vector collinear with $\vec{m}$. $P(x, y)$ represents a general point on the line, where $\overrightarrow{OP}$ is the vector associated with this point.



To find the vector equation of line L, the triangle law of addition is used.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\vec{r} = \overrightarrow{OP}$, $\vec{r_0} = \overrightarrow{OP_0}$, and $t\vec{m} = \overrightarrow{P_0P}$, the vector equation of the line is written as $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

When writing an equation of a line using vectors, the vector form of the line is sometimes modified and put in parametric form. The parametric equations of a line come directly from its vector equation. How to change the equation of a line from vector to parametric form is shown below.

The general vector equation of a line is $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

In component form, this is written as $(x, y) = (x_0, y_0) + t(a, b), t \in \mathbf{R}$. Expanding the right side, $(x, y) = (x_0, y_0) + (ta, tb) = (x_0 + ta, y_0 + tb), t \in \mathbf{R}$. If we equate the respective x and y components, the required parametric form is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^2

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta$, $y = y_0 + tb$, $t \in \mathbf{R}$
where $\vec{r_0}$ is the vector from $(0, 0)$ to the point (x_0, y_0) and $\vec{m}$ is a direction vector with components (a, b).

In either vector or parametric form, t is called a **parameter**. This means that t can be replaced by any real number to obtain the coordinates of points on the line.

EXAMPLE 2 Reasoning about the vector and parametric equations of a line

a. Determine the vector and parametric equations of a line passing through point $A(1, 4)$ with direction vector $\vec{m} = (-3, 3)$.
b. Sketch the line, and determine the coordinates of four points on the line.
c. Is either point $Q(-21, 23)$ or point $R(-29, 34)$ on this line?

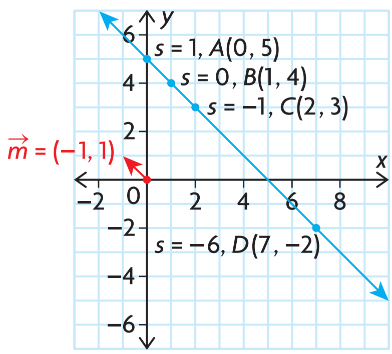
Solution

a. Since $A(1, 4)$ is on the line, $\overrightarrow{OP_0} = \vec{r}_0 = (1, 4)$ and $\vec{m} = (-3, 3)$. The vector equation is $\vec{r} = (1, 4) + t(-3, 3)$, $t \in \mathbf{R}$. The parametric equations are $x = 1 - 3t$, $y = 4 + 3t$, $t \in \mathbf{R}$.

It is also possible to use other scalar multiples of $\vec{m} = (-3, 3)$ as a direction vector, such as $(-1, 1)$, which gives the respective vector and parametric equations $\vec{r} = (1, 4) + s(-1, 1)$, $s \in \mathbf{R}$, and $x = 1 - s$, $y = 4 + s$, $s \in \mathbf{R}$. The vector $(-1, 1)$ has been chosen as our direction vector for the sake of simplicity. Note that we have written the second equation with parameter s to avoid confusion between the two lines. Although the two equations, $\vec{r} = (1, 4) + t(-3, 3)$, $t \in \mathbf{R}$, and $\vec{r} = (1, 4) + s(-1, 1)$, $s \in \mathbf{R}$, appear with different parameters, the lines they represent are identical.

b. To determine the coordinates of points on the line, the parametric equations $x = 1 - s$, $y = 4 + s$, $s \in \mathbf{R}$, were used, with s chosen to be 0, 1, -1, and -6. To find the coordinates of a particular point, such as D, $s = -6$ was substituted into the parametric equations and $x = 1 - (-6) = 7$, $y = 4 + (-6) = -2$.

The required point is $D(7, -2)$. The coordinates of the other points are determined in the same way, using the other values of s.



c. If the point $Q(-21, 23)$ lies on the line, then there must be consistency with the parameter s. We substitute this point into the parametric equations to check for the required consistency. Substituting gives $-21 = 1 - s$ and $23 = 4 + s$.

In the first equation, $s = 22$, and in the second equation, $s = 19$. Since these values are inconsistent, the point Q is not on the line.

If the point $R(-29, 34)$ is on the line, then $-29 = 1 - s$ and $34 = 4 + s$, $s = 30$, for both equations.

Since each of these equations has the same solution, $s = 30$, we conclude that $R(-29, 34)$ is on the line.

Sometimes, the equation of the line must be found when two points are given. This is shown in the following example.

EXAMPLE 3 Connecting vector and parametric equations with two points on a line

a. Determine vector and parametric equations for the line containing points $E(-1, 5)$ and $F(6, 11)$.

b. What are the coordinates of the point where this line crosses the x-axis?

c. Can the equation $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right), t \in \mathbf{R}$, also represent the line containing points E and F?

Solution

a. A direction vector for the line containing points E and F is $\vec{m} = \overrightarrow{EF} = (6 - (-1), 11 - 5) = (7, 6)$. A vector equation for the line is $\vec{r} = (-1, 5) + s(7, 6), s \in \mathbf{R}$, and its parametric equations are $x = -1 + 7s, y = 5 + 6s, s \in \mathbf{R}$.

The equation given for this line is not unique. This is because there are an infinite number of choices for the direction vector, and any point on the line could have been used. In writing a second equation for the line, the parametric equations $x = 6 + 7s, y = 11 + 6s,\ s \in \mathbf{R}$, would also have been correct because $(6, 11)$ is on the line and the direction vector is $(7, 6)$.

b. The line intersects the x-axis at a point with coordinates of the form $(a, 0)$. At the point of intersection, $y = 0$ and, so, $5 + 6s = 0$, $s = \frac{-5}{6}$. Therefore,

$$\begin{aligned} a &= -1 + 7s \\ &= -1 + 7\left(\frac{-5}{6}\right) \\ &= -\frac{41}{6}, \end{aligned}$$

and the line intersects the x-axis at the point $\left(-\frac{41}{6}, 0\right)$.

c. If this equation represents the same line as the equation in part a., it is necessary for the two lines to have the same direction and contain the same set of points.

The line $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right), t \in \mathbf{R}$, has $\left(\frac{14}{3}, 4\right)$ as its direction vector.

The two lines will have the same direction vectors because $\frac{3}{2}\left(\frac{14}{3}, 4\right) = (7, 6)$.

The two lines have the same direction, and if these lines have a point in common, then the equations represent the same line. The easiest approach is to substitute $(-15, -7)$ into the first equation to see if this point is on the line. Substituting gives $(-15, -7) = (-1, 5) + s(7, 6)$ or $-15 = -1 + 7s$ and $-7 = 5 + 6s$. Since the solution to both of these equations is $s = -2$, the point $(-15, -7)$ is on the line, and the two equations represent the same line.

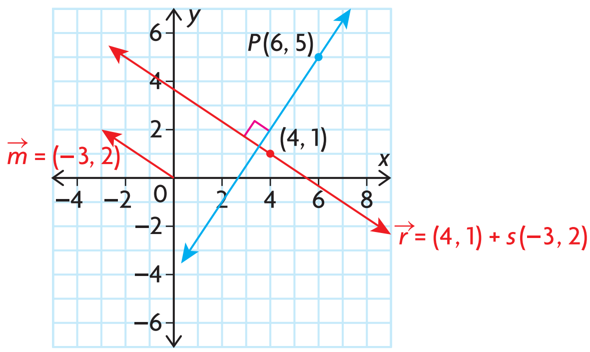
In the next example, vector properties will be used to determine equations for lines that involve perpendicularity.

EXAMPLE 4 Selecting a strategy to determine the vector equation of a perpendicular line

Determine a vector equation for the line that is perpendicular to $\vec{r} = (4, 1) + s(-3, 2), s \in \mathbf{R}$, and passes through point $P(6, 5)$.

Solution

The direction vector for the given line is $\vec{m} = (-3, 2)$, and this line is drawn through $(4, 1)$, as shown in red in the diagram. A sketch of the required line, passing through $(6, 5)$ and perpendicular to the given line, is drawn in blue.



Let the direction vector for the required blue line be $\vec{v} = (a, b)$. Since the direction vector of the given line is perpendicular to that of the required line, $\vec{v} \cdot \vec{m} = 0$.

Therefore, $(a, b) \cdot (-3, 2) = 0$ or $-3a + 2b = 0$.

The simplest integer values for a and b, which satisfy this equation, are $a = 2$ and $b = 3$. This gives the direction vector $(2, 3)$ and the required vector equation for the perpendicular line is $\vec{r} = (6, 5) + t(2, 3),\ t \in \mathbf{R}$.

In this section, the vector and parametric equations of a line in R^2 were discussed. In Section 8.3, the discussion will be extended to R^3, where many of the ideas seen in this section apply to lines in three-space.

The following investigation is designed to aid in understanding the concept of parameter, when dealing with either the vector or parametric equations of a line.

INVESTIGATION

A. i. On graph paper, draw the lines $L_1: \vec{r} = t(0, 1),\ t \in \mathbf{R}$, and $L_2: \vec{r} = p(1, 0)$, $p \in \mathbf{R}$. Make sure that you clearly show a direction vector for each line.

ii. Describe geometrically what each of the two equations represent.

iii. Give a vector equation and corresponding parametric equations for each of the following:

- the line parallel to the x-axis, passing through $P(2, 4)$
- the line parallel to the y-axis, passing through $Q(-2, -1)$

iv. Sketch L_3: $x = -3,\ y = 1 + s,\ s \in \mathbf{R}$, and L_4: $x = 4 + t,\ y = 1,\ t \in \mathbf{R}$, using your own axes.

v. By examining parametric equations of a line, how is it possible to determine by inspection whether the line is parallel to either the x-axis or y-axis?

vi. Write an equation of a line in both vector and parametric form that is parallel to the x-axis.

vii. Write an equation of a line in both vector and parametric form that is parallel to the y-axis.

B. i. Sketch the line $L: \vec{r} = (-3, 0) + s(2, -1), s \in \mathbf{R}$, on graph paper.

ii. On the set of axes used for part i., sketch each of the following:

- $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$
- $L_2: \vec{r} = (-3, 1) + s(2, -1), s \in \mathbf{R}$
- $L_3: \vec{r} = (2, -1) + s(2, -1), s \in \mathbf{R}$
- $L_4: \vec{r} = (4, 2) + s(2, -1), s \in \mathbf{R}$

If you are given the equation $\vec{r} = \vec{r_0} + s(2, -1), s \in \mathbf{R}$, what is the mathematical effect of changing the value of $\vec{r_0}$?

iii. For the line $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$, show that each of the following points are on this line by finding corresponding values of s: $(4, -2), (-4, 2), (198, -99)$, and $(-202, 101)$.

iv. Which part of the equation $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$, indicates that there are an infinite number of points on this line? Explain your answer.

IN SUMMARY

Key Ideas

- The vector equation of a line in R^2 is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, where $\vec{m} = (a, b)$ is the direction vector and $\vec{r_0}$ is the vector from the origin to any point on the line whose general coordinates are (x_0, y_0). This is equivalent to the equation $(x, y) = (x_0, y_0) + t(a, b)$.
- The parametric form of the equation of a line is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Need to Know

- In both the vector and parametric equations, t is a parameter. Every real number for t generates a different point that lies on the line.

Exercise 8.1

PART A

1. A vector equation is given as $\vec{r} = \left(\frac{1}{2}, -\frac{3}{4}\right) + s\left(\frac{1}{3}, \frac{1}{6}\right)$, $s \in \mathbf{R}$. Explain why $\vec{m} = (-2, -1)$, $\vec{m} = (2, 1)$, and $\vec{m} = \left(\frac{2}{7}, \frac{1}{7}\right)$ are acceptable direction vectors for this line.

2. Parametric equations of a line are $x = 1 + 3t$ and $y = 5 - 2t$, $t \in \mathbf{R}$.
   a. Write the coordinates of three points on this line.
   b. Show that the point $P(-14, 15)$ lies on the given line by determining the parameter value of t corresponding to this point.

3. Identify the direction vector and a point on each of the following lines:
   a. $\vec{r} = (3, 4) + t(2, 1)$, $t \in \mathbf{R}$
   b. $x = 1 + 2t$, $y = 3 - 7t$, $t \in \mathbf{R}$
   c. $\vec{r} = (4, 1 + 2t)$, $t \in \mathbf{R}$
   d. $x = -5t$, $y = 6$, $t \in \mathbf{R}$

PART B

4. A line passes through the points $A(2, 1)$ and $B(-3, 5)$. Write two different vector equations for this line.

5. A line is defined by the parametric equations $x = -2 - t$ and $y = 4 + 2t$, $t \in \mathbf{R}$.
   a. Does $R(-9, 18)$ lie on this line? Explain.
   b. Write a vector equation for this line using the given parametric equations.
   c. Write a second vector equation for this line, different from the one you wrote in part b.

6. a. If the equation of a line is $\vec{r} = s(3, 4), s \in \mathbf{R}$, name the coordinates of three points on this line.

   b. Write a vector equation, different from the one given, in part a., that also passes through the origin.

   c. Describe how the line with equation $\vec{r} = (9, 12) + t(3, 4), t \in \mathbf{R}$ relates to the line given in part a.

C 7. A line has $\vec{r} = \left(\frac{1}{3}, \frac{1}{7}\right) + p(-2, 3), p \in \mathbf{R}$, as its vector equation. A student decides to "simplify" this equation by clearing the fractions and multiplies the vector $\left(\frac{1}{3}, \frac{1}{7}\right)$ by 21. The student obtains $\vec{r} = (7, 3) + p(-2, 3), p \in \mathbf{R}$, as a "correct" form of the line. Explain why multiplying a point in this way is incorrect.

8. A line passes through the points $Q(0, 7)$ and $R(0, 9)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

K 9. A line passes through the points $M(4, 5)$ and $N(9, 5)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

10. For the line $L: \vec{r} = (1, -5) + s(3, 5), s \in \mathbf{R}$, determine the following:

    a. an equation for the line perpendicular to L, passing through $P(2, 0)$

    b. the point at which the line in part a. intersects the y-axis

A 11. The parametric equations of a line are given as $x = -10 - 2s$, $y = 8 + s$, $s \in \mathbf{R}$. This line crosses the x-axis at the point with coordinates $A(a, 0)$ and crosses the y-axis at the point with coordinates $B(0, b)$. If O represents the origin, determine the area of the triangle AOB.

T 12. A line has $\vec{r} = (1, 2) + s(-2, 3)$, $s \in \mathbf{R}$, as its vector equation. On this line, the points A, B, C, and D correspond to parametric values $s = 0, 1, 2$, and 3, respectively. Show that each of the following is true:

    a. $\overrightarrow{AC} = 2\overrightarrow{AB}$ b. $\overrightarrow{AD} = 3\overrightarrow{AB}$ c. $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AD}$

PART C

13. The line L has $x = 2 + t, y = 9 + t$, $t \in \mathbf{R}$, as its parametric equations. If L intersects the circle with equation $x^2 + y^2 = 169$ at points A and B, determine the following:

    a. the coordinates of points A and B

    b. the length of the chord AB

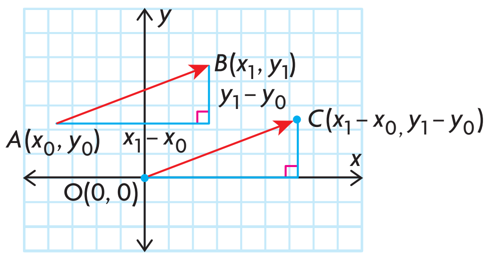
14. Are the lines $2x - 3y + 15 = 0$ and $(x, y) = (1, 6) + t(6, 4)$ parallel? Explain.

Section 8.2—Cartesian Equation of a Line

In the previous section, we discussed the vector and parametric equations of lines in R^2. In this section, we will show how lines of the form $y = mx + b$ (slope–y-intercept form) and $Ax + By + C = 0$ (Cartesian equation of a line, also called a scalar equation of a line) are related to the vector and parametric equations of the line.

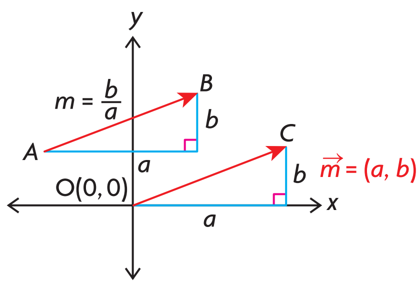
The Relationship between Vector and Scalar Equations of Lines in R^2

The direction, or inclination, of a line can be described in two ways: by its slope and by a direction vector. The slope of the line joining two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$. It is also possible to describe the direction of a line using the vector defined by the two points A and B, $\overrightarrow{AB} = \vec{m} = (x_1 - x_0, y_1 - y_0)$. This direction vector is equivalent to a vector with its tail at the origin and its head at $C(x_1 - x_0, y_1 - y_0)$ and is shown in the diagram below.



Direction Vectors and Slope

In the diagram, a line segment AB with slope $m = \frac{b}{a}$ is shown with a run of a and a rise of b. The vector $\vec{m} = (a, b)$ is used to describe the direction of this line or any line parallel to it, with no restriction on the direction numbers a and b. In practice, a and b can be any two real numbers when describing a direction vector. If the direction vector of a line is $\vec{m} = (a, b)$, this corresponds to a slope of $m = \frac{b}{a}$ except when $a = 0$ (which corresponds to a vertical line).



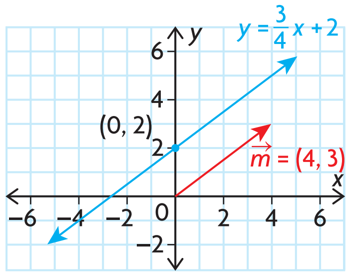
In the following example, we will show how to take a line in slope–y-intercept form and convert it to vector and parametric form.

EXAMPLE 1 Representing the Cartesian equation of a line in vector and parametric form

Determine the equivalent vector and parametric equations of the line $y = \frac{3}{4}x + 2$.

Solution

In the diagram below, the line $y = \frac{3}{4}x + 2$ is drawn. This line passes through $(0, 2)$, has a slope of $m = \frac{3}{4}$, and, as a result, has a direction vector $\vec{m} = (4, 3)$. A vector equation for this line is $\vec{r} = (0, 2) + t(4, 3), t \in \mathbf{R}$, with parametric equations $x = 4t, y = 2 + 3t, t \in \mathbf{R}$.



In the next example, we will show the conversion of a line in vector form to one in slope–y-intercept form.

EXAMPLE 2 Representing a vector equation of a line in Cartesian form

For the line with equation $\vec{r} = (3, -6) + s(-1, -4), s \in \mathbf{R}$, determine the equivalent slope–y-intercept form.

Solution

Method 1:

The direction vector for this line is $\vec{m} = (-1, -4)$, with slope $m = \frac{-4}{-1} = 4$. This line contains the point $(3, -6)$. If $P(x, y)$ represents a general point on this line, then we can use slope–point form to determine the required equation.

Thus, $\frac{y - (-6)}{x - 3} = 4$

$$4(x - 3) = y + 6$$
$$4x - 12 = y + 6$$
$$4x - 18 = y$$

The required equation for this line is $y = 4x - 18$ in slope–y-intercept form.

Method 2:
We start by writing the given line in parametric form, which is $(x, y) = (3, -6) + s(-1, -4)$ or $(x, y) = (3 - s, -6 - 4s)$. This gives the parametric equations $x = 3 - s$ and $y = -6 - 4s$. To find the required equation, we solve for s in each component. Thus, $s = \frac{x - 3}{-1}$ and $s = \frac{y + 6}{-4}$. Since these equations for s are equal,

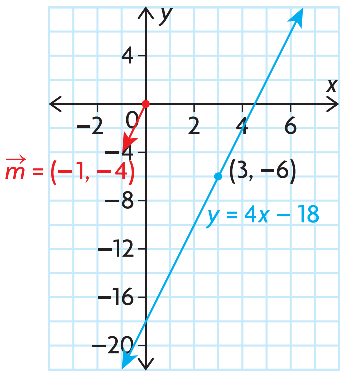
$$\frac{x - 3}{-1} = \frac{y + 6}{-4}$$

$$\frac{-4(x - 3)}{-1} = y + 6$$

$$y + 6 = 4(x - 3)$$

$$y = 4x - 18$$

Therefore, the required equation is $y = 4x - 18$, which is the same answer we obtained using Method 1. The graph of this line is shown below.



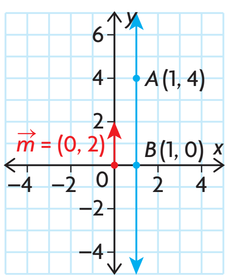
In the example that follows, we examine the situation in which the direction vector of the line is of the form $\vec{m} = (0, b)$.

EXAMPLE 3 Reasoning about equations of vertical lines

Determine the Cartesian form of the line with the equation $\vec{r} = (1, 4) + s(0, 2), s \in \mathbf{R}$.

Solution

The given line passes through the point $(1, 4)$, with direction vector $(0, 2)$, as shown in the diagram below.

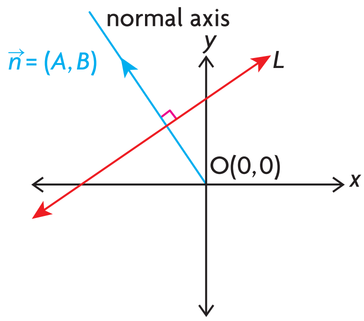


It is not possible, in this case, to calculate the slope because the line has direction vector $(0, 2)$, meaning its slope would be $\frac{2}{0}$, which is undefined. Since the line is parallel to the y-axis, it must have the form $x = a$, where $(a, 0)$ is the point where the line crosses the x-axis. The equation of this line is $x - 1 = 0$ or $x = 1$.

Developing the Cartesian Equation from a Direction Vector

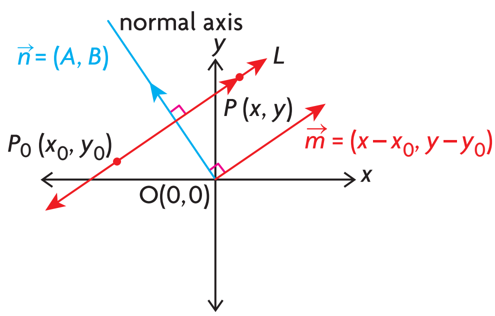
In addition to making the connection between lines in either slope–y-intercept form or Cartesian form with those in vector form, we would like to consider how direction vectors can be used to obtain the equations of lines in Cartesian form.

In the following diagram, the line L represents a general line in R^2. A line has been drawn from the origin, perpendicular to L. This perpendicular line is called the **normal** axis for the line and is the only line that can be drawn from the origin perpendicular to the given line. If the origin is joined to any point on the normal axis, other than itself, the vector formed is described as a normal to the given line. Since there are an infinite number of points on the normal axis, this is a way of saying that any line in R^2 has an infinite number of normals, none of which is the zero vector. A general point on the normal axis is given the coordinates $N(A, B)$, and so a normal vector, denoted by $\vec{n}$, is the vector $\vec{n} = (A, B)$.



The important property of the normal vector is that it is perpendicular to any vector on the given line. This property of normal vectors is what allows us to derive the Cartesian equation of the line.

In the following diagram, the line L is drawn, along with a normal $\vec{n} = (A, B)$, to L. The point $P(x, y)$ represents any point on the line, and the point $P_0(x_0, y_0)$ represents a given point on the line.



To derive the Cartesian equation for this line, we first determine $\overrightarrow{P_0P}$. In coordinate form, this vector is $\overrightarrow{P_0P} = (x - x_0, y - y_0)$, which represents a direction vector for the line. In the diagram, this vector has been shown as $\vec{m} = (x - x_0, y - y_0)$. Since the vectors $\vec{n}$ and $\overrightarrow{P_0P}$ are perpendicular to each other, $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

$(A, B) \cdot (x - x_0, y - y_0) = 0$ (Expand)

$Ax - Ax_0 + By - By_0 = 0$ (Rearrange)

$Ax + By - Ax_0 - By_0 = 0$

Since the point $P_0(x_0, y_0)$ is a point whose coordinates are known, as is $\vec{n} = (A, B)$, we substitute C for the quantity $-Ax_0 - By_0$ to obtain $Ax + By + C = 0$ as the Cartesian equation of the line.

EXAMPLE 4 Connecting the Cartesian equation of a line to its normal

Determine the Cartesian equation of the line passing through $A(4, -2)$, which has $\vec{n} = (5, 3)$ as its normal.

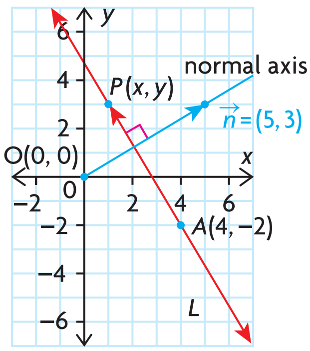
Solution

The required line is sketched by first drawing the normal $\vec{n} = (5, 3)$ and then

Cartesian Equation of a Line in R^2

In R^2, the Cartesian equation of a line (or scalar equation) is given by $Ax + By + C = 0$, where a normal to this line is $\vec{n} = (A, B)$. A normal to this line is a vector drawn from the origin perpendicular to the given line to the point $N(A, B)$.

constructing a line L through $A(4, -2)$ perpendicular to this normal.



Method 1:
Let $P(x, y)$ be any point on the required line L, other than A. Let $\overrightarrow{AP}$ be a vector parallel to L.

$\overrightarrow{AP} = (x - 4, y - (-2)) = (x - 4, y + 2).$

Since $\vec{n}$ and $\overrightarrow{AP}$ are perpendicular, $\vec{n} \cdot \overrightarrow{AP} = 0.$

Therefore, $(5, 3) \cdot (x - 4, y + 2) = 0$ or $5(x - 4) + 3(y + 2) = 0$

Thus, $5x - 20 + 3y + 6 = 0$ or $5x + 3y - 14 = 0$

Method 2:
Since $\vec{n} = (5, 3)$, the Cartesian equation of the line is of the form $5x + 3y + C = 0$, with C to be determined. Since the point $A(4, -2)$ is a point on this line, it must satisfy the following equation:

$5(4) + 3(-2) + C = 0$

So, $C = -14$, and $5x + 3y - 14 = 0$

Using either method, the required Cartesian equation is $5x + 3y - 14 = 0$.

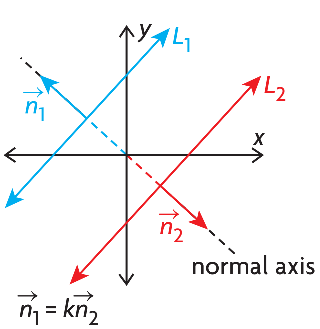
Since it has been established that the line with equation $Ax + By + C = 0$ has a normal vector of $\vec{n} = (A, B)$, this now provides an easy test to determine whether lines are parallel or perpendicular.

Parallel and Perpendicular Lines and their Normals

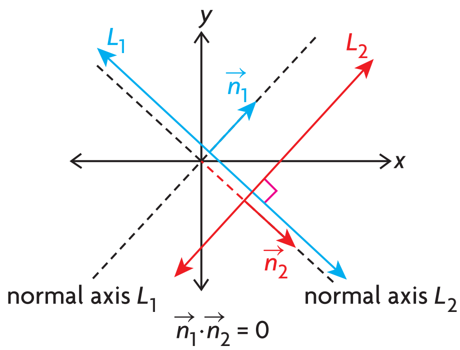
If the lines L_1 and L_2 have normals $\vec{n_1}$ and $\vec{n_2}$, respectively, we know the following:

1. The two lines are parallel if and only if their normals are scalar multiples. $\vec{n_1} = k\vec{n_2}, k \in \mathbf{R}, k \neq 0$ It follows that the lines direction vectors are also scalar multiples in this case.
2. The two lines are perpendicular if and only if their dot product is zero. $\vec{n_1} \cdot \vec{n_2} = 0$ It follows that dot product of the direction vectors is also zero in this case.

Two parallel lines



Two perpendicular lines



The next examples demonstrate these ideas.

EXAMPLE 5 Reasoning about parallel and perpendicular lines in R^2

a. Show that the lines $L_1: 3x - 4y - 6 = 0$ and $L_2: 6x - 8y + 12 = 0$ are parallel and non-coincident.
b. For what value of k are the lines $L_3: kx + 4y - 4 = 0$ and $L_4: 3x - 2y - 3 = 0$ perpendicular lines?

Solution

a. The lines are parallel because when the two normals, $\vec{n_1} = (3, -4)$ and $\vec{n_2} = (6, -8)$, are compared, the two vectors are scalar multiples $\vec{n_2} = (6, -8) = 2(3, -4) = 2\vec{n_1}$. The lines are non-coincident, since there is no value of t such that $6x - 8y + 12 = t(3x - 4y - 6)$. In simple terms, lines can only be coincident if their equations are scalar multiples of each other.
b. If the lines are perpendicular, then the normal vectors $\vec{n_3} = (k, 4)$ and $\vec{n_4} = (3, -2)$ have a dot product equal to zero—that is $(k, 4) \cdot (3, -2) = 0$ or $3k - 8 = 0$, $k = \frac{8}{3}$. This implies that the lines $3x - 2y - 3 = 0$ and $\frac{8}{3}x + 4y - 4 = 0$ are perpendicular.

The following investigation helps in understanding the relationship between normals and perpendicular lines.

EXAMPLE 6 Selecting a strategy to determine the angle between two lines in R^2

Determine the acute angle formed at the point of intersection created by the following pair of lines:
L_1: $(x, y) = (2, 2) + s(-1, 3)$, $s \in \mathbf{R}$
L_2: $(x, y) = (5, 1) + t(3, 4)$, $t \in \mathbf{R}$

Solution

The direction of each line is determined by their respective direction vectors, so the angle formed at the point of intersection is equivalent to the angle formed by the direction vectors when drawn tail to tail. For L_1 its direction vector is $\vec{a} = (-1, 3)$ and for L_2 its direction vectors is $\vec{b} = (3, 4)$. These lines are clearly not parallel as their direction vectors are not scalar multiples. They are also not perpendicular because the dot product of their direction vectors is a nonzero value. The angle between two vectors is determined by:

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right) \quad \text{(Substitute)}$$

$$\theta = \cos^{-1}\left(\frac{(-1, 3) \cdot (3, 4)}{(\sqrt{(-1)^2 + (3)^2})(\sqrt{(3)^2 + (4)^2})}\right) \quad \text{(Simplify)}$$

$$\theta = \cos^{-1}\left(\frac{-3 + 12}{(\sqrt{10})(\sqrt{25})}\right)$$

$$\theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right) \qquad \text{(Evaluate)}$$

$$\theta \doteq 55.3°$$

The acute angle formed at the point of intersection of the given lines is about 55.3°.

INVESTIGATION

A. A family of lines has $kx - 2y - 4 = 0$ as its equation. On graph paper, sketch the three members of this family when $k = 1$, $k = -1$, and $k = 2$.

B. What point do the three lines you sketched in part A have in common?

C. A second family of lines has $4x - ty - 8 = 0$ as its equation. Sketch the three members of this family used in part A for $t = -2$, $t = 2$, and $t = -4$.

D. What points do the lines in part C have in common?

E. Select the three pairs of perpendicular lines from the two families. Verify that you are correct by calculating the dot products of their respective normals.

F. By selecting different values for k and t, determine another pair of lines that are perpendicular.

G. In general, if you are given a line in R^2, how many different lines is it possible to draw through a particular point perpendicular to the given line? Explain your answer.

IN SUMMARY

Key Idea

- The Cartesian (or scalar) equation of a line in R^2 is $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a normal to the line.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$:
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$, where k is any nonzero real number.
  - are perpendicular if and only if $\vec{n_1} \cdot \vec{n_2} = 0$.
- The angle between two lines is defined by the angle between their direction vectors, $\vec{a}$ and $\vec{b}$, where $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$.

Exercise 8.2

PART A

1. A line has $y = -\frac{5}{6}x + 9$ as its equation.
   a. Give a direction vector for a line that is parallel to this line.
   b. Give a direction vector for a line that is perpendicular to this line.
   c. Give the coordinates of a point on the given line.
   d. In both vector and parametric form, give the equations of the line parallel to the given line and passing through $A(7, 9)$.
   e. In both vector and parametric form, give the equations of the line perpendicular to the given line and passing through $B(-2, 1)$.
2. a. Sketch the line defined by the equation $\vec{r} = (2, 1) + s(-2, 5), s \in \mathbf{R}$.
   b. On the same axes, sketch the line $\vec{q} = (-2, 5) + t(2, 1),\ t \in \mathbf{R}$.
   c. Discuss the impact of switching the components of the direction vector with the coordinates of the point on the line in the vector equation of a line in R^2.
3. For each of the given lines, determine the vector and parametric equations.

   a. $y = \frac{7}{8}x - 6$ b. $y = \frac{3}{2}x + 5$ c. $y = -1$ d. $x = 4$

C 4. Explain how you can show that the lines with equations $x - 3y + 4 = 0$ and $6x - 18y + 24 = 0$ are coincident.

5. Two lines have equations $2x - 3y + 6 = 0$ and $4x - 6y + k = 0$.
   a. Explain, with the use of normal vectors, why these lines are parallel.
   b. For what value of k will these lines be coincident?

PART B

6. Determine the Cartesian equation for the line with a normal vector of $(4, 5)$, passing through the point $A(-1, 5)$.
7. A line passes through the points $A(-3, 5)$ and $B(-2, 4)$. Determine the Cartesian equation of this line.
8. A line is perpendicular to the line $2x - 4y + 7 = 0$ and that passes through the point $P(7, 2)$. Determine the equation of this line in Cartesian form.

K 9. A line has parametric equations $x = 3 - t$, $y = -2 - 4t$, $t \in \mathbf{R}$.
   a. Sketch this line.
   b. Determine a Cartesian equation for this line.

A

10. For each pair of lines, determine the size of the acute angle, to the nearest degree, that is created by the intersection of the lines.

   a. $(x, y) = (3, 6) + t(2, -5)$ and $(x, y) = (-3, 4) - t(-4, -1)$

   b. $x = 2 - 5t, y = 3 + 4t$ and $x = -1 + t, y = 2 - 6t$

   c. $y = 0.5x + 6$ and $y = -0.75x - 1$

   d. $(x, y) = (-1, -1) + t(2, 4)$ and $2x - 4y = 8$

   e. $x = 2t, y = 1 - 5t$ and $(x, y) = (4, 0) + t(-4, 1)$

   f. $x = 3$ and $5x - 10y + 20 = 0$

11. The angle between any pair of lines in Cartesian form is also the angle between their normal vectors. For the lines $x - 3y + 6 = 0$ and $x + 2y - 7 = 0$, do the following:

   a. Sketch the lines.

   b. Determine the acute and obtuse angles between these two lines.

T

12. The line segment joining $A(-3, 2)$ and $B(8, 4)$ is the hypotenuse of a right triangle. The third vertex, C, lies on the line with the vector equation $(x, y) = (-6, 6) + t(3, -4)$.

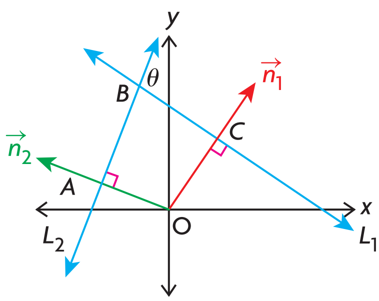
   a. Determine the coordinates of C.

   b. Illustrate with a diagram.

   c. Use vectors to show that $\angle ACB = 90°$.

PART C

13. Lines L_1 and L_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals. Prove that the angle between the two lines is the same as the angle between the two normals.



   (*Hint:* Show that $\angle AOC = \theta$ by using the fact that the sum of the angles in a quadrilateral is 360°.)

14. The lines $x - y + 1 = 0$ and $x + ky - 3 = 0$ have an angle of 60° between them. For what values of k is this true?

Section 8.3—Vector, Parametric, and Symmetric Equations of a Line in R^3

In Section 8.1, we discussed vector and parametric equations of a line in R^2. In this section, we will continue our discussion, but, instead of R^2, we will examine lines in R^3.

The derivation and form of the vector equation for a line in R^3 is the same as in R^2. If we wish to find a vector equation for a line in R^3, it is necessary that either two points or a point and a direction vector be given. If we are given two points and wish to determine a direction vector for the corresponding line, the coordinates of this vector must first be calculated.

EXAMPLE 1

Determining a direction vector of a line in R^3

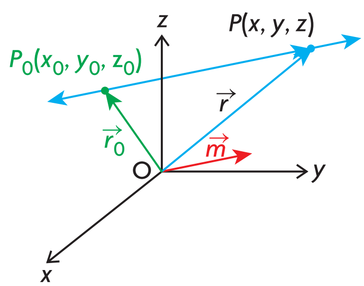
A line passes through the points $A(-1, 3, 5)$ and $B(-3, 3, -4)$. Calculate possible direction vectors for this line.

Solution

A possible direction vector is $\vec{m} = (-1 - (-3), 3 - 3, 5 - (-4)) = (2, 0, 9)$. In general, any vector of the form $t(2, 0, 9)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector for this line. As before, the best choice for a direction vector is one in which the direction numbers are integers, with common divisors removed. This implies that either $(2, 0, 9)$ or $(-2, 0, -9)$ are the best choices for a direction vector for this line. Generally speaking, if a line has $\vec{m} = (a, b, c)$ as its direction vector, then any scalar multiple of this vector of the form $t(a, b, c)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector.

Vector and Parametric Equations of Lines in R^3

Consider the following diagram.



When determining the vector equation of the line passing through P_0 and P, we know that the point $P_0(x_0, y_0, z_0)$ is a given point on the line, and $\vec{m} = (a, b, c)$

is its direction vector. If $P(x, y, z)$ represents a general point on the line, then $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is a direction vector for this line. This allows us to form the vector equation of the line.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\overrightarrow{OP_0} = \vec{r_0}$ and $\overrightarrow{P_0P} = t\vec{m}$, the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$. In component form, this can be written as $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$, $t \in \mathbf{R}$. The parametric equations of the line are found by equating the respective x, y, and z components, giving $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^3

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$
where $\vec{r_0} = (x_0, y_0, z_0)$, the vector from the origin to a point on the line and
$\vec{m} = (a, b, c)$ is a direction vector of the line

EXAMPLE 2 Representing the equation of a line in R^3 in vector and parametric form

Determine the vector and parametric equations of the line passing through $P(-2, 3, 5)$ and $Q(-2, 4, -1)$.

Solution

A direction vector is $\vec{m} = (-2 - (-2), 3 - 4, 5 - (-1)) = (0, -1, 6)$. A vector equation is $\vec{r} = (-2, 3, 5) + s(0, -1, 6)$, $s \in \mathbf{R}$, and its parametric equations are $x = -2$, $y = 3 - s$, $z = 5 + 6s$, $s \in \mathbf{R}$. It would also have been correct to choose any multiple of $(0, -1, 6)$ as a direction vector and any point on the line. For example, the vector equation $\vec{r} = (-2, 4, -1) + t(0, -2, 12)$, $t \in \mathbf{R}$, would also have been correct.

Since a vector equation of a line can be written in many ways, it is useful to be able to tell if different forms are actually equivalent. In the following example, an algebraic approach to this problem is considered.

EXAMPLE 3 Reasoning to establish the equivalence of two lines

a. Show that the following are vector equations for the same line:
   $L_1: \vec{r} = (-1, 0, 4) + s(-1, 2, 5), s \in \mathbf{R}$, and
   $L_2: \vec{r} = (4, -10, -21) + m(-2, 4, 10), m \in \mathbf{R}$
b. Show that the following are vector equations for different lines:
   $L_3: \vec{r} = (1, 6, 1) + l(-1, 1, 2), l \in \mathbf{R}$, and

$$L_4: \vec{r} = (-3, 10, 12) + k\left(\frac{1}{2}, -\frac{1}{2}, -1\right), k \in \mathbf{R}$$

Solution

a. Since the direction vectors are parallel—that is, $2(-1, 2, 5) = (-2, 4, 10)$—this means that the two lines are parallel. To show that the equations are equivalent, we must show that a point on one of the lines is also on the other line. This is based on the logic that, if the lines are parallel and they share a common point, then the two equations must represent the same line. To check whether $(4, -10, -21)$ is also on L_1 substitute into its vector equation.
$(4, -10, -21) = (-1, 0, 4) + s(-1, 2, 5)$

Using the x component, we find $4 = -1 - s$, or $s = -5$. Substituting $s = -5$ into the above equation, $(-1, 0, 4) + \; - 5(-1, 2, 5) = (4, -10, -21)$. This verifies that the point $(4, -10, -21)$ is also on L_1.

Since the lines have the same direction, and a point on one line is also on the second line, the two given equations represent the same line.

b. The first check is to compare the direction vectors of the two lines. Since $-2\left(\frac{1}{2}, -\frac{1}{2}, -1\right) = (-1, 1, 2)$, the lines must be parallel. As in part a, $(-3, 10, 12)$ must be a point on L_3 for the equations to be equivalent. Therefore, $(-3, 10, 12) = (1, 6, 1) + l(-1, 1, 2)$ must give a consistent value of l for each component. If we solve, this gives an inconsistent result since $l = 4$ for the x and y components, and $12 = 1 + 2l$, or $l = \frac{11}{2}$ for the z component. This verifies that the two equations are not equations for the same line.

Symmetric Equations of Lines in R^3

We introduce a new form for a line in R^3, called its **symmetric equation**. The symmetric equation of a line is derived from using its parametric equations and solving for the parameter in each component, as shown below

$$x = x_0 + ta \leftrightarrow t = \frac{x - x_0}{a}, a \neq 0$$

$$y = y_0 + tb \leftrightarrow t = \frac{y - y_0}{b}, b \neq 0$$

$$z = z_0 + tc \leftrightarrow t = \frac{z - z_0}{c}, c \neq 0$$

Combining these statements gives $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$, $a, b, c \neq 0$.

These equations are called the symmetric equations of a line in R^3.

Symmetric Equations of a Line in R^3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad a \neq 0, b \neq 0, c \neq 0$$

where (x_0, y_0, z_0) is the vector from the origin to a point on the line, and (a, b, c) is a direction vector of the line.

EXAMPLE 4 Representing the equation of a line in R^3 in symmetric form

a. Write the symmetric equations of the line passing through the points $A(-1, 5, 7)$ and $B(3, -4, 8)$.
b. Write the symmetric equations of the line passing through the points $P(-2, 3, 1)$ and $Q(4, 3, -5)$.
c. Write the symmetric equations of the line passing through the points $X(-1, 2, 5)$ and $Y(-1, 3, 9)$.

Solution

a. A direction vector for this line is $\vec{m} = (-1 - 3, 5 - (-4), 7 - 8) = (-4, 9, -1)$. Using the point $A(-1, 5, 7)$, the parametric equations of the line are $x = -1 - 4t$, $y = 5 + 9t$, and $z = 7 - t$, $t \in \mathbf{R}$. Solving each equation for t gives the required symmetric equations, $\frac{x + 1}{-4} = \frac{y - 5}{9} = \frac{z - 7}{-1}$. It is usually not necessary to find the parametric equations before finding the symmetric equations. The symmetric equations of a line can be written by inspection if the direction vector and a point on the line are known. Using point B and the direction vector found above, the symmetric equations of this line by inspection are $\frac{x - 3}{-4} = \frac{y + 4}{9} = \frac{z - 8}{-1}$.

b. A direction vector for the line is $\vec{m} = (-2 - 4, 3 - 3, 1 - (-5)) = (-6, 0, 6)$. The vector $(1, 0, -1)$ will be used as the direction vector. In a situation like this, where the y direction number is 0, using point P the equation is written as $\frac{x + 2}{1} = \frac{z - 1}{-1}$, $y = 3$.

c. A direction vector for the line is $\vec{m} = (-1 - (-1), 2 - 3, 5 - 9) = (0, -1, -4)$. Using point X, possible symmetric equations are $\frac{y - 2}{-1} = \frac{z - 5}{-4}$, $x = -1$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a line and $\vec{m} = (a, b, c)$ is a direction vector of the same line, then
  - the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$
  - the parametric form of the equation of the line is $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$, $t \in \mathbf{R}$
  - the symmetric form of the equation of the line is $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} (= t)$, $a \neq 0$, $b \neq 0$, $c \neq 0$

Need to Know

- Knowing one of these forms of the equation of a line enables you to find the other two, since all three forms depend on the same information about the line.

Exercise 8.3

PART A

1. State the coordinates of a point on each of the given lines.

   a. $\vec{r} = (-3, 1, 8) + s(-1, 1, 9),\ s \in \mathbf{R}$

   b. $\dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-3}{-1}$

   c. $x = -2 + 3t,\ y = 1 + (-4t),\ z = 3 - t,\ t \in \mathbf{R}$

   d. $\dfrac{x+2}{-1} = \dfrac{z-1}{2},\ y = -3$

   e. $x = 3,\ y = -2,\ z = -1 + 2k,\ k \in \mathbf{R}$

   f. $\dfrac{x-\frac{1}{3}}{\frac{1}{2}} = \dfrac{y+\frac{3}{4}}{\frac{-1}{4}} = \dfrac{z-\frac{2}{5}}{\frac{1}{2}}$

2. State a direction vector for each line in question 1, making certain that the components for each are integers.

PART B

3. A line passes through the points $A(-1, 2, 4)$ and $B(3, -3, 5)$.

   a. Write two vector equations for this line.

   b. Write the two sets of parametric equations associated with the vector equations you wrote in part a.

4. A line passes through the points $A(-1, 5, -4)$ and $B(2, 5, -4)$.

   a. Write a vector equation for the line containing these points.

   b. Write parametric equations corresponding to the vector equation you wrote in part a.

   c. Explain why there are no symmetric equations for this line.

K 5. State where possible vector, parametric, and symmetric equations for each of the following lines.

   a. the line passing through the point $P(-1, 2, 1)$ with direction vector $(3, -2, 1)$

   b. the line passing through the points $A(-1, 1, 0)$ and $B(-1, 2, 1)$

   c. the line passing through the point $B(-2, 3, 0)$ and parallel to the line passing through the points $M(-2, -2, 1)$ and $N(-2, 4, 7)$

   d. the line passing through the points $D(-1, 0, 0)$ and $E(-1, 1, 0)$

   e. the line passing through the points $X(-4, 3, 0)$ and $O(0, 0, 0)$

   f. the line passing through the point $Q(1, 2, 4)$ and parallel to the z-axis

6. a. Determine parametric equations for each of the following lines:
$\frac{x+6}{1} = \frac{y-10}{-1} = \frac{z-7}{1}$ and $\frac{x+7}{1} = \frac{y-11}{-1}, z = 5$

b. Determine the angle between the two lines.

C 7. Show that the following two sets of symmetric equations represent the same straight line: $\frac{x+7}{8} = \frac{y+1}{2} = \frac{z-5}{-2}$ and $\frac{x-1}{-4} = \frac{y-1}{-1} = \frac{z-3}{1}$

8. a. Show that the points $A(6, -2, 15)$ and $B(-15, 5, -27)$ lie on the line that passes through (0, 0, 3) and has the direction vector $(-3, 1, -6)$.

b. Use parametric equations with suitable restrictions on the parameter to describe the line segment from A to B.

A 9. Determine the value of k for which the direction vectors of the lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z+1}{k-1}$ and $\frac{x+3}{-2} = \frac{z}{1}, y = -1$ are perpendicular.

10. Determine the coordinates of three different points on each line.

a. $(x, y, z) = (4, -2, 5) + t(-4, -6, 8)$

b. $x = -4 + 5s, y = 2 - s, z = 9 - 6s$

c. $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}$

d. $x = -4, \frac{y-2}{3} = \frac{z-3}{5}$

11. Express each equation in question 10 in two other equivalent forms. (i.e. vector, parametric or symmetric form)

PART C

12. Determine the parametric equations of the line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x}{-4} = \frac{y+10}{-7} = \frac{z+2}{3}$ and $\frac{x-5}{3} = \frac{y-5}{2} = \frac{z+5}{4}$ and passes through the point $(2, -5, 0)$.

T 13. A line with parametric equations $x = 10 + 2s, y = 5 + s, z = 2, s \in \mathbf{R}$, intersects a sphere with the equation $x^2 + y^2 + z^2 = 9$ at the points A and B. Determine the coordinates of these points.

14. You are given the two lines $L_1: x = 4 + 2t, y = 4 + t, z = -3 - t, t \in \mathbf{R}$, and $L_2: x = -2 + 3s, y = -7 + 2s, z = 2 - 3s, s \in \mathbf{R}$. If the point P_1 lies on L_1 and the point P_2 lies on L_2, determine the coordinates of these two points if $\overrightarrow{P_1P_2}$ is perpendicular to each of the two lines. (*Hint*: The vector $\overrightarrow{P_1P_2}$ is perpendicular to the direction vector of each of the two lines.)

15. Determine the angle formed by the intersection of the lines defined by $\frac{x-1}{2} = \frac{y+3}{1}, z = -3$ and $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z}{1}$.

Mid-Chapter Review

1. Name three points on each of the following lines:

   a. $x = 2t - 5, y = 3t + 1, t \in \mathbf{R}$

   b. $\vec{r} = (2, 3) + s(3, -2), s \in \mathbf{R}$

   c. $3x + 5y - 8 = 0$

   d. $\frac{x - 1}{3} = \frac{y + 2}{2} = \frac{z - 5}{1}$

2. Find x- and y-intercepts for each of the following lines:

   a. $\vec{r} = (3, 1) + t(-3, 5), t \in \mathbf{R}$

   b. $x = -6 + 2s$ and $y = 3 - 2s, s \in \mathbf{R}$

3. Two lines $L_1: \vec{r} = (5, 3) + p(-4, 7), p \in \mathbf{R}$, and $L_2: \vec{r} = (5, 3) + q(2, 1), q \in \boldsymbol{R}$, intersect at the point with coordinates $(5, 3)$. What is the angle between L_1 and L_2?

4. Determine the angle that the line with equation $\vec{r} = t(4, -5), t \in \mathbf{R}$, makes with the x-axis and y-axis.

5. Determine a Cartesian equation for the line that passes through the point $(4, -3)$ and is perpendicular to the line $\vec{r} = (2, -3) + t(5, -7), t \in \mathbf{R}$.

6. Determine an equation in symmetric form of a line parallel to $\frac{x - 3}{3} = \frac{y - 5}{-4} = \frac{z + 7}{4}$ and passing through $(0, 0, 2)$.

7. Determine parametric equations of the line passing through $(1, 2, 5)$ and parallel to the line passing through $K(2, 4, 5)$ and $L(3, -5, 6)$.

8. Determine direction angles (the angles the direction vector makes with the x-axis, y-axis, and z-axis) for the line with parametric equations $x = 5 + 2t$, $y = 12 - 8t$, $z = 5 + 7t$, $t \in \mathbf{R}$.

9. Determine an equation in symmetric form for the line passing through $P(3, -4, 6)$ and having direction angles 60°, 90°, and 30°.

10. Write an equation in parametric form for each of the three coordinate axes in R^3.

11. The two lines with equations $\vec{r} = (1, 2, -4) + t(k + 1, 3k + 1, k - 3)$, $t \in \mathbf{R}$, and $x = 2 - 3s$, $y = 1 - 10s$, $z = 3 - 5s$, $s \in \mathbf{R}$, are given.

    a. Determine a value for k if these lines are parallel.

    b. Determine a value for k if these lines are perpendicular.

12. Determine the perimeter and area of the triangle whose vertices are the origin and the x- and y-intercepts of the line $\frac{x - 6}{3} = \frac{y + 8}{-2}$.

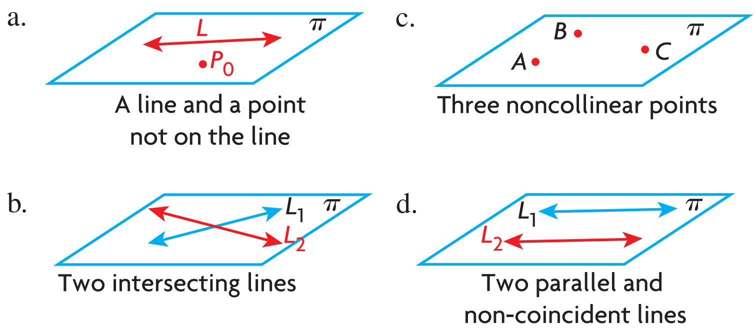
13. The Cartesian equation of a line is given by $3x + 4y - 24 = 0$.
    a. Determine a vector equation for this line.
    b. Determine the parametric equations of this line.
    c. Determine the acute angle that this line makes with the x-axis.
    d. Determine a vector equation of the line that is perpendicular to the given line and passes through the origin.
14. Determine the scalar, vector, and parametric equations of the line that passes through points $A(-4, 6)$ and $B(8, 4)$.
15. Determine a unit vector normal to the line defined by the parametric equations $x = 1 + 2t$ and $y = -5 - 4t$.
16. Determine the parametric equations of each line.
    a. the line that passes through $(-5, 10)$ and has a slope of $-\frac{2}{3}$
    b. the line that passes through $(1, -1)$ and is perpendicular to the line $(x, y) = (4, -6) + t(2, -2)$
    c. the line that passes through $(0, 7)$ and $(0, 10)$
17. Given the line $(x, y, z) = (12, -8, -4) + t(-3, 4, 2)$,
    a. determine the intersections with the coordinate planes, if any
    b. determine the intercepts with the coordinate axes, if any
    c. graph the line in an x-, y-, z-coordinate system.
18. For each of the following, determine vector, parametric, and, if possible, symmetric equations of the line that passes through P_0 and has direction vector $\vec{d}$.
    a. $P_0 = (1, -2, 8)$, $\vec{d} = (-5, -2, 1)$
    b. $P_0 = (3, 6, 9)$, $\vec{d} = (2, 4, 6)$
    c. $P_0 = (0, 0, 6)$, $\vec{d} = (-1, 5, 1)$
    d. $P_0 = (2, 0, 0)$, $\vec{d} = (0, 0, -2)$
19. Determine a vector equation of the line that passes through the origin and is parallel to the line through the points $(-4, 5, 6)$ and $(6, -5, 4)$.
20. Determine the parametric equations of the line through $(0, -8, 1)$ and which passes through the midpoint of the segment joining $(2, 6, 10)$ and $(-4, 4, -8)$.
21. The symmetric equations of two lines are given. Show that these lines are parallel.
    $L_1: \frac{x-2}{1} = \frac{y+3}{3} = \frac{z-4}{-5}$ and $L_2: \frac{x+1}{-3} = \frac{y-2}{-9} = \frac{z+1}{15}$
22. Does the point $D(7, -1, 8)$ lie on the line with symmetric equations $\frac{x-4}{3} = \frac{y+2}{1} = \frac{z-6}{2}$? Explain.

Section 8.4—Vector and Parametric Equations of a Plane

In the previous section, the vector, parametric, and symmetric equations of lines in R^3 were developed. In this section, we will develop vector and parametric equations of planes in R^3. Planes are flat surfaces that extend infinitely far in all directions. To represent planes, parallelograms are used to represent a small part of the plane and are designated with the Greek letter π. This is the usual method for representing planes. In real life, part of a plane might be represented by the top of a desk, by a wall, or by the ice surface of a hockey rink.

Before developing the equation of a plane, we start by showing that planes can be determined in essentially four ways. That is, a plane can be determined if we are given any of the following:

a. a line and a point not on the line
b. three noncollinear points (three points not on a line)
c. two intersecting lines
d. two parallel and non-coincident lines



If we are given any one of these conditions, we are guaranteed that we can form a plane, and the plane formed will be unique. For example, in condition a, we are given line L and point P_0 not on this line; there is just one plane that can be formed using this point and this line.

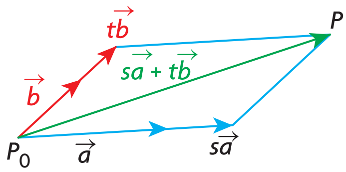
Linear Combinations and their Relationship to Planes

The ideas of linear combination and spanning sets are the two concepts needed to understand how to obtain the vector and parametric equations of planes. For example, suppose that vectors $\vec{a} = (1, 2, -1)$ and $\vec{b} = (0, 2, 1)$ and a linear combination of these vectors—that is, $\overrightarrow{P_0P} = s(1, 2, -1) + t(0, 2, 1),\ s, t \in \mathbf{R}$— are formed. As different values are chosen for s and t, a new vector is formed. Different values for these parameters have been selected, and the corresponding calculations have been done in the table shown, with vector $\overrightarrow{P_0P}$ also calculated.

s	t	$s(1, 2, -1) + t(0, 2, 1), s, t \in \mathbf{R}$	$\overrightarrow{P_0P}$
−2	1	$-2(1, 2, -1) + 1(0, 2, 1)$	$(-2, -4, 2) + (0, 2, 1) = (-2, -2, 3)$
4	−3	$4(1, 2, -1) - 3(0, 2, 1)$	$(4, 8, -4) + (0, -6, -3) = (4, 2, -7)$
10	−7	$10(1, 2, -1) - 7(0, 2, 1)$	$(10, 20, -10) + (0, -14, -7) = (10, 6, -17)$
−2	−1	$-2(1, 2, -1) - 1(0, 2, 1)$	$(-2, -4, 2) + (0, -2, -1) = (-2, -6, 1)$

$\overrightarrow{P_0P}$ is on the plane determined by the vectors $\vec{a}$ and $\vec{b}$, as is its head. The parameters s and t are chosen from the set of real numbers, meaning that there are an infinite number of vectors formed by selecting all possible combinations of s and t. Each one of these vectors is different, and every point on the plane can be obtained by choosing appropriate parameters. This observation is used in developing the vector and parametric equations of a plane.

In the following diagram, two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given. The linear combinations of these vectors, $s\vec{a} + t\vec{b}$, form a diagonal of the parallelogram determined by $s\vec{a}$ and $t\vec{b}$.

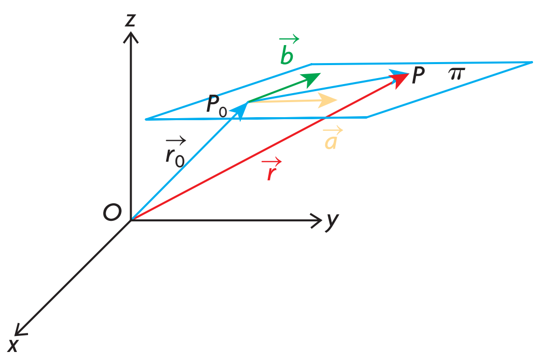


EXAMPLE 1 Developing the vector and parametric equations of a plane

Two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given, and the point P_0. Determine the vector and parametric equations of the plane π formed by taking all linear combinations of these vectors.

Solution

The vectors $\vec{a}$ and $\vec{b}$ can be translated anywhere in R^3. When drawn tail to tail they form and infinite number of parallel planes, but only one such plane contains the point P_0. We start by drawing a parallelogram to represent part of this plane π. This plane contains P_0, $\vec{a}$, and $\vec{b}$.



From the diagram, it can be seen that vector $\vec{r_0}$ represents the vector for a particular point P_0 on the plane, and $\vec{r}$ represents the vector for any point P on the plane. $\overrightarrow{P_0P}$ is on the plane and is a linear combination of $\vec{a}$ and $\vec{b}$—that is, $\overrightarrow{P_0P} = s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$. Using the triangle law of addition in $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$. Thus, $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$.

The vector equation for the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$, and can be used to generate parametric equations for the plane.

If $\vec{r} = (x, y, z)$, $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$, these expressions can be substituted into the vector equation to obtain $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$, $s, t \in \mathbf{R}$.

Expanding, $(x, y, z) = (x_0, y_0, z_0) + (sa_1, sa_2, sa_3) + (tb_1, tb_2, tb_3)$

Simplifying, $(x, y, z) = (x_0 + sa_1 + tb_1, y_0 + sa_2 + tb_2, z_0 + sa_3 + tb_3)$

Equating the respective components gives the parametric equations $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2$, $z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$.

Vector and Parametric Equations of a Plane in R^3

In R^3, a plane is determined by a vector $\vec{r_0} = (x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point on the plane, and two noncollinear vectors vector $\vec{a} = (a_1, a_2, a_3)$ and vector $\vec{b} = (b_1, b_2, b_3)$.

Vector Equation of a Plane: $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$.

Parametric Equations of a Plane: $x = x_0 + sa_1 + tb_1$,
$y = y_0 + sa_2 + tb_2$,
$z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$

The vectors $\vec{a}$ and $\vec{b}$ are the direction vectors for the plane. When determining the equation of a plane, it is necessary to have two direction vectors. As will be seen in the examples, any pair of noncollinear vectors are coplanar, so they can be used as direction vectors for a plane. The vector equation of a plane always requires two parameters, s and t, each of which are real numbers. Because two parameters are required to define a plane, the plane is described as two-dimensional. Earlier, we saw that the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, required just one parameter. For this reason, a line is described as one-dimensional. A second observation about the vector equation of the plane is that there is a one-to-one correspondence between the chosen parametric values (s, t) and points on the plane. Each time values for s and t are selected, this generates a different point on the plane, and because s and t can be any real number, this will generate all points on the plane.

After deriving vector and parametric equations of lines, a symmetric form was also developed. Although it is possible to derive vector and parametric equations of planes, it is not possible to derive a corresponding symmetric equation of a plane.

The next example shows how to derive an equation of a plane passing through three points.

EXAMPLE 2 Selecting a strategy to represent the vector and parametric equations of a plane

a. Determine a vector equation and the corresponding parametric equations for the plane that contains the points $A(-1, 3, 8)$, $B(-1, 1, 0)$, and $C(4, 1, 1)$.
b. Do either of the points $P(14, 1, 3)$ or $Q(14, 1, 5)$ lie on this plane?

Solution

a. In determining the required vector equation, it is necessary to have two direction vectors for the plane. The following shows the calculations for each of the direction vectors.

Direction Vector 1:
When calculating the first direction vector, any two points can be used and a position vector determined. If the points $A(-1, 3, 8)$ and $B(-1, 1, 0)$ are used, then $\overrightarrow{AB} = (-1 - (-1), 1 - 3, 0 - 8) = (0, -2, -8)$.

Since, $(0, -2, -8) = -2(0, 1, 4)$, a possible first direction vector is $\vec{a} = (0, 1, 4)$.

Direction Vector 2:
When finding the second direction vector, any two points (other than A and B) can be chosen. If B and C are used, then
$\overrightarrow{BC} = (4 - (-1), 1 - 1, 1 - 0) = (5, 0, 1)$.

A second direction vector is $\vec{b} = (5, 0, 1)$.

To determine the equation of the plane, any of the points A, B, or C can be used. An equation for the plane is $\vec{r} = (-1, 3, 8) + s(0, 1, 4) + t(5, 0, 1)$, $s, t \in \mathbf{R}$.

Writing the vector equation in component form will give the parametric equations. Thus, $(x, y, z) = (-1, 3, 8) + (0, s, 4s) + (5t, 0, t)$.

The parametric equations are $x = -1 + 5t$, $y = 3 + s$, and $z = 8 + 4s + t$, $s, t \in \mathbf{R}$.

b. If the point $P(14, 1, 3)$ lies on the plane, there must be parameters that correspond to this point. To find these parameters, $x = 14$ and $y = 1$ are substituted into the corresponding parametric equations.

Thus, $14 = -1 + 5t$ and $1 = 3 + s$.

Solving for s and t, we find that $14 + 1 = 5t$, or $t = 3$ and $1 - 3 = s$, or $s = -2$. Using these values, consistency will be checked with the z component. If $s = -2$ and $t = 3$ are substituted into $z = 8 + 4s + t$, then $z = 8 + 4(-2) + 3 = 3$. Since the z value for the point is also 3, this tells us that the point with coordinates $P(14, 1, 3)$ is on the given plane.

From this, it can be seen that the parametric values used for the x and y components, $s = -2$ and $t = 3$, will not produce consistent values for $z = 5$. So, the point $Q(14, 1, 5)$ is not on the plane.

In the following example, we will show how to use vector and parametric equations to find the point of intersection of planes with the coordinate axes.

EXAMPLE 3 Solving a problem involving a plane

A plane π has $\vec{r} = (6, -2, -3) + s(1, 3, 0) + t(2, 2, -1), s, t \in \mathbf{R}$, as its equation. Determine the point of intersection between π and the z-axis.

Solution

We start by writing this equation in parametric form. The parametric equations of the plane are $x = 6 + s + 2t$, $y = -2 + 3s + 2t$, and $z = -3 - t$.

The plane intersects the z-axis at a point with coordinates of the form $P(0, 0, c)$—that is, where $x = y = 0$. Substituting these values into the parametric equations for x and y gives $0 = 6 + s + 2t$ and $0 = -2 + 3s + 2t$. Simplifying gives the following system of equations:

① $s + 2t = -6$

② $3s + 2t = 2$

Subtracting equation ① from equation ② gives $2s = 8$, so $s = 4$.

The value of t is found by substituting into either of the two equations. Using equation ①, $4 + 2t = -6$, or $t = -5$.

Solving for z using the equation of the third component, we find that $z = -3 - (-5) = 2$.

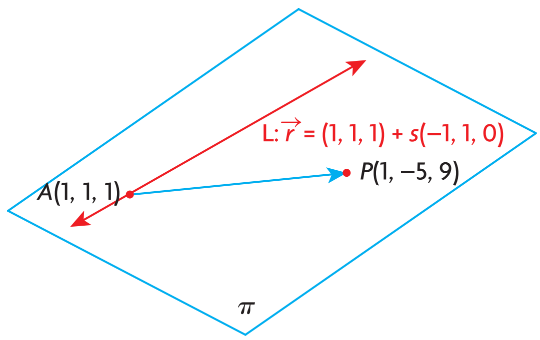
Thus, the plane cuts the z-axis at the point $P(0, 0, 2)$.

EXAMPLE 4 Representing the equations of a plane from a point and a line

Determine the vector and parametric equations of the plane containing the point $P(1, -5, 9)$ and the line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$.

Solution

In the following diagram, a representation of the line L and the point P are given.



To find the equation of the plane, it is necessary to find two direction vectors and a point on the plane. The line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$, gives a point and one direction vector, so all that is required is a second direction vector. Using $A(1, 1, 1)$ and $P(1, -5, 9)$, $\overrightarrow{AP} = (1 - 1, -5 - 1, 9 - 1) = (0, -6, 8) = -2(0, 3, -4)$. The equation of the plane is $\vec{r} = (1, 1, 1) + s(-1, 1, 0) + t(0, 3, -4),\ s, t \in \mathbf{R}$.

The corresponding parametric equations are $x = 1 - s, y = 1 + s + 3t$, and $z = 1 - 4t, s, t \in \mathbf{R}$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a plane and $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are direction vectors, then
  - the vector equation of the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}, s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2\ a_3) + t(b_1, b_2, b_3)$
  - the parametric form of the equation of the plane is $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2, z = z_0 + sa_3 + tb_3, s, t \in \mathbf{R}$
  - there are no symmetric equations of the plane

Need to Know

- Replacing the parameters in the vector and parametric equations of a plane with numbers generates points on the plane. Because there are an infinite number of real numbers that can be used for s and t, there are an infinite number of points that lie on a plane.

Exercise 8.4

PART A

1. State which of the following equations define lines and which define planes. Explain how you made your decision.
   a. $\vec{r} = (1, 2, 3) + s(1, 1, 0) + t(3, 4, -6), s, t \in \mathbf{R}$
   b. $\vec{r} = (-2, 3, 0) + m(3, 4, 7),\ m \in \mathbf{R}$
   c. $x = -3 - t, y = 5, z = 4 + t, t \in \mathbf{R}$
   d. $\vec{r} = m(4, -1, 2) + t(4, -1, 5), m, t \in \mathbf{R}$
2. A plane has vector equation $\vec{r} = (2, 1, 3) + s\left(\frac{1}{3}, -2, \frac{3}{4}\right) + t(6, -12, 30)$, $s, t \in \mathbf{R}$.
   a. Express the first direction vector with only integers.
   b. Reduce the second direction vector.
   c. Write a new equation for the plane using the calculations from parts a. and b.
3. A plane has $x = 2m$, $y = -3m + 5n$, $z = -1 - 3m - 2n$, $m, n \in \mathbf{R}$, as its parametric equations.
   a. By inspection, identify the coordinates of a point that is on this plane.
   b. What are the direction vectors for this plane?
   c. What point corresponds to the parameter values of $m = -1$ and $n = -4$?
   d. What are the parametric values corresponding to the point $A(0, 15, -7)$?
   e. Using your answer for part d., explain why the point $B(0, 15, -8)$ cannot be on this plane.
4. A plane passes through the points $P(-2, 3, 1)$, $Q(-2, 3, 2)$, and $R(1, 0, 1)$.
   a. Using $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as direction vectors, write a vector equation for this plane.
   b. Using $\overrightarrow{QR}$ and one other direction vector, write a second vector equation for this plane.

C 5. Explain why the equation $\vec{r} = (-1, 0, -1) + s(2, 3, -4) + t(4, 6, -8)$, $s, t \in \mathbf{R}$, does not represent the equation of a plane. What does this equation represent?

PART B

6. Determine vector equations and the corresponding parametric equations of each plane.
   a. the plane with direction vectors $\vec{a} = (4, 1, 0)$ and $\vec{b} = (3, 4, -1)$, passing through the point $A(-1, 2, 7)$
   b. the plane passing through the points $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$
   c. the plane passing through points $A(1, 1, 0)$ and $B(4, 5, -6)$, with direction vector $\vec{a} = (7, 1, 2)$

7. a. Determine parameters corresponding to the point $P(5, 3, 2)$, where P is a point on the plane with equation $\pi: \vec{r} = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2), s, t \in \mathbf{R}$.
   b. Show that the point $A(0, 5, -4)$ does not lie on π.
8. A plane has $\vec{r} = (-3, 5, 6) + s(-1, 1, 2) + v(2, 1, -3), s, v \in \mathbf{R}$ as its equation.
   a. Give the equations of two intersecting lines that lie on this plane.
   b. What point do these two lines have in common?

A

9. Determine the coordinates of the point where the plane with equation $\vec{r} = (4, 1, 6) + s(11, -1, 3) + t(-7, 2, -2), s, t \in \mathbf{R}$, crosses the z-axis.
10. Determine the equation of the plane that contains the point $P(-1, 2, 1)$ and the line $\vec{r} = (2, 1, 3) + s(4, 1, 5), s \in \mathbf{R}$.
11. Determine the equation of the plane that contains the point $A(-2, 2, 3)$ and the line $\vec{r} = m(2, -1, 7), m \in \mathbf{R}$.
12. a. Determine two pairs of direction vectors that can be used to represent the xy-plane in R^3.
    b. Write a vector and parametric equations for the xy-plane in R^3.

K

13. a. Determine a vector equation for the plane containing the points $O(0, 0, 0)$, $A(-1, 2, 5)$, and $C(3, -1, 7)$.
    b. Determine a vector equation for the plane containing the points $P(-2, 2, 3)$,$Q(-3, 4, 8)$, and $R(1, 1, 10)$.
    c. How are the planes found in parts a. and b. related? Explain your answer.
14. Show that the following equations represent the same plane:
    $\vec{r} = u(-3, 2, 4) + v(-4, 7, 1), u, v \in \mathbf{R}$, and
    $\vec{r} = s(-1, 5, -3) + t(-1, -5, 7), s, t \in \mathbf{R}$
    (*Hint*: Express each direction vector in the first equation as a linear combination of the direction vectors in the second equation.)

T

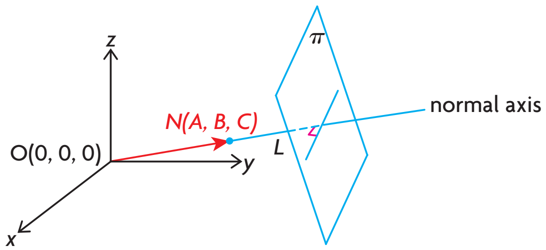
15. The plane with equation $\vec{r} = (1, 2, 3) + m(1, 2, 5) + n(1, -1, 3)$ intersects the y- and z-axes at the points A and B, respectively. Determine the equation of the line that contains these two points.

PART C

16. Suppose that the lines L_1 and L_2 are defined by the equations $\vec{r} = \overrightarrow{OP_0} + s\vec{a}$ and $\vec{r} = \overrightarrow{OP_0} + t\vec{b}$, respectively, where $s, t \in \mathbf{R}$, and $\vec{a}$ and $\vec{b}$ are noncollinear vectors. Prove that the plane defined by the equation $\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of these lines.

Section 8.5—The Cartesian Equation of a Plane

In the previous section, the vector and parametric equations of a plane were found. In this section, we will show how to derive the Cartesian (or scalar) equation of a plane. The process is very similar to the process used to find the Cartesian equation of a line in R^2.

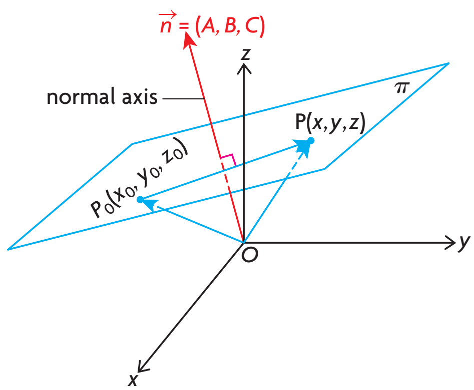


In the diagram above, a plane π is shown, along with a line L drawn from the origin, so that L is perpendicular to the given plane. For any plane in R^3, there is only one possible line that can be drawn through the origin perpendicular to the plane. This line is called the normal axis for the plane. The direction of the normal axis is given by a vector joining the origin to any point on the normal axis. The direction vector is called a normal to the plane. In the diagram, $\overrightarrow{ON}$ is a normal to the plane because it joins the origin to $N(A, B, C)$, a point on the normal axis. Any nonzero vector is a normal to a plane if it lies along the normal axis. This implies that an infinite number of normals exist for all planes.

A plane is completely determined when we know a point $(P_0(x_0, y_0, z_0))$ on the plane and a normal to the plane. This single idea can be used to determine the Cartesian equation of a plane.

Deriving the Cartesian Equation of a Plane

Consider the following diagram:



To derive the equation of this plane, we need two points on the plane, $P_0(x_0, y_0, z_0)$ (its coordinates given) and a general point, $P(x, y, z)$, different from P_0. The vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ represents any vector on the plane. If $\vec{n} = \overrightarrow{ON} = (A, B, C)$ is a known normal to the plane, then the relationship, $\vec{n} \cdot \overrightarrow{P_0P} = 0$ can be used to derive the equation of the plane, since $\vec{n}$ and any vector on the plane are perpendicular.

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$$

Or, $Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$

Since the quantities in the expression $-Ax_0 - By_0 - Cz_0$ are known, we'll replace this with D to make the equation simpler. The Cartesian equation of the plane is, thus, $Ax + By + Cz + D = 0$.

Cartesian Equation of a Plane

The Cartesian (or scalar) equation of a plane in R^3 is of the form $Ax + By + Cz + D = 0$ with normal $\vec{n} = (A, B, C)$. The normal $\vec{n}$ is a nonzero vector perpendicular to all vectors in the plane.

EXAMPLE 1 Representing a plane by its Cartesian equation

The point $A(1, 2, 2)$ is a point on the plane with normal $\vec{n} = (-1, 2, 6)$. Determine the Cartesian equation of this plane.

Solution

Two different methods can be used to determine the Cartesian equation of this plane. Both methods will give the same answer.

Method 1:
Let $P(x, y, z)$ be any point on the plane.

Therefore, $\overrightarrow{AP} = (x - 1, y - 2, z - 2)$ represents any vector on the plane. Since $\vec{n} = (-1, 2, 6)$ and $\vec{n} \cdot \overrightarrow{AP} = 0$,

$$\begin{aligned} (-1, 2, 6) \cdot (x - 1, y - 2, z - 2) &= 0 \\ -1(x - 1) + 2(y - 2) + 6(z - 2) &= 0 \\ -x + 1 + 2y - 4 + 6z - 12 &= 0 \\ -x + 2y + 6z - 15 &= 0 \end{aligned}$$

Multiplying each side by -1, $x - 2y - 6z + 15 = 0$.

Either $-x + 2y + 6z - 15 = 0$ or $x - 2y - 6z + 15 = 0$ is a correct equation for the plane, but usually we write the equation with integer coefficients and with a positive coefficient for the x-term.

Method 2:
Since the required equation has the form $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C) = (-1, 2, 6)$, the direction numbers for the normal can be substituted directly into the equation. This gives $-x + 2y + 6z + D = 0$, with D to be determined. Since the point $A(1, 2, 2)$ is on the plane, it satisfies the equation.

Substituting the coordinates of this point into the equation gives $-(1) + 2(2) + 6(2) + D = 0$, and thus $D = -15$.

If $D = -15$ is substituted into $-x + 2y + 6z + D = 0$, the equation will be $-x + 2y + 6z + (-15) = 0$ or $x - 2y - 6z + 15 = 0$.

To find the Cartesian equation of a plane, either Method 1 or Method 2 can be used.

The Cartesian equation of a plane is simpler than either the vector or the parametric form and is used most often.

EXAMPLE 2 Determining the Cartesian equation of a plane from three coplanar points

Determine the Cartesian equation of the plane containing the points $A(-1, 2, 5)$, $B(3, 2, 4)$, and $C(-2, -3, 6)$.

Solution

A normal to this plane is determined by calculating the cross product of the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This results in a vector perpendicular to the plane in which both these vectors lie.

$\overrightarrow{AB} = (3 - (-1), 2 - 2, 4 - 5) = (4, 0, -1)$ and

$\overrightarrow{AC} = (-2 - (-1), -3 - 2, 6 - 5) = (-1, -5, 1)$

$$\begin{aligned}\text{Thus, } \overrightarrow{AB} \times \overrightarrow{AC} &= (0(1) - (-1)(-5), -1(-1) - (4)(1), 4(-5) - (0)(-1)) \\ &= (-5, -3, -20) \\ &= -1(5, 3, 20)\end{aligned}$$

If we let $P(x, y, z)$ be any point on the plane, then $\overrightarrow{AP} = (x + 1, y - 2, z - 5)$, and since a normal to the plane is $(5, 3, 20)$,

$$(5, 3, 20) \cdot (x + 1, y - 2, z - 5) = 0$$

$$5x + 5 + 3y - 6 + 20z - 100 = 0$$

After simplifying, the required equation of the plane is $5x + 3y + 20z - 101 = 0$.

A number of observations can be made about this calculation. If we had used $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as direction vectors, for example, we would have found that $\overrightarrow{BC} = (-5, -5, 2)$ and $\overrightarrow{AB} \times \overrightarrow{BC} = (4, 0, -1) \times (-5, -5, 2) = -1(5, 3, 20)$. When finding the equation of a plane, it is possible to use any pair of direction vectors on the plane to find a normal to the plane. Also, when finding the value for D, if we had used the method of substitution, it would have been possible to substitute any one of the three given points in the equation.

In the next example, we will show how to convert from vector or parametric form to Cartesian form. We will also show how to obtain the vector form of a plane if given its Cartesian form.

EXAMPLE 3 Connecting the various forms of the equation of a plane

a. Determine the Cartesian form of the plane whose equation in vector form is $\vec{r} = (1, 2, -1) + s(1, 0, 2) + t(-1, 3, 4)$, $s, t \in \mathbf{R}$.

b. Determine the vector and parametric equations of the plane with Cartesian equation $x - 2y + 5z - 6 = 0$.

Solution

a. To find the Cartesian equation of the plane, two direction vectors are needed so that a normal to the plane can be determined. The two given direction vectors for the plane are $(1, 0, 2)$ and $(-1, 3, 4)$. Their cross product is

$$\begin{aligned}(1, 0, 2) \times (-1, 3, 4) &= (0(4)-2(3), 2(-1)-1(4), 1(3)-0(-1)) \\ &= -3(2, 2, -1)\end{aligned}$$

A normal to the plane is $(2, 2, -1)$, and the Cartesian equation of the plane is of the form $2x + 2y - z + D = 0$. Substituting the point $(1, 2, -1)$ into this equation gives $2(1) + 2(2) - (-1) + D = 0$, or $D = -7$.

Therefore, the Cartesian equation of the plane is $2x + 2y - z - 7 = 0$.

b. *Method 1:*

To find the corresponding vector and parametric equations of a plane, the equation of the plane is first converted to its parametric form. The simplest way to do this is to choose any two of the variables and replace them with a parameter. For example, if we substitute $y = s$ and $z = t$ and solve for x, we obtain $x - 2s + 5t - 6 = 0$ or $x = 2s - 5t + 6$.

This gives us the required parametric equations $x = 2s - 5t + 6$, $y = s$, and $z = t$. The vector form of the plane can be found by rearranging the parametric form.

$$\text{Therefore, } (x, y, z) = (2s - 5t + 6, s, t)$$
$$(x, y, z) = (6, 0, 0) + (2s, s, 0) + (-5t, 0, t)$$
$$\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1),\ s, t \in \mathbf{R}$$

The parametric equations of this plane are $x = 2s - 5t + 6$, $y = s$, and $z = t$, and the corresponding vector form is $\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1)$, $s, t \in \mathbf{R}$.

Check:
This vector equation of the plane can be checked by converting to Cartesian form. A normal to the plane is $(2, 1, 0) \times (-5, 0, 1) = (1, -2, 5)$. The plane has the form $x - 2y + 5z + D = 0$. If $(6, 0, 0)$ is substituted into the equation to find D, we find that $6 - 2(0) + 5(0) + D = 0$, so $D = -6$ and the equation is the given equation $x - 2y + 5z - 6 = 0$.

Method 2:
We rewrite the given equation as $x = 2y - 5z + 6$. We are going to find the coordinates of three points on the plane, and writing the equation in this way allows us to choose integer values for y and z that will give an integer value for x. The values in the table are chosen to make the computation as simple as possible.

The following table shows our choices for y and z, along with the calculation for x.

y	z	$x = 2y - 5z + 6$	Resulting Point
0	0	$x = 2(0) - 5(0) + 6 = 6$	$A(6, 0, 0)$
-1	0	$x = 2(-1) - 5(0) + 6 = 4$	$B(4, -1, 0)$
-1	1	$x = 2(-1) - 5(1) + 6 = -1$	$C(-1, -1, 1)$

$\overrightarrow{AB} = (4 - 6, -1 - 0, 0 - 0) = (-2, -1, 0)$ and
$\overrightarrow{AC} = (-1 - 6, -1 - 0, 1 - 0) = (-7, -1, 1)$

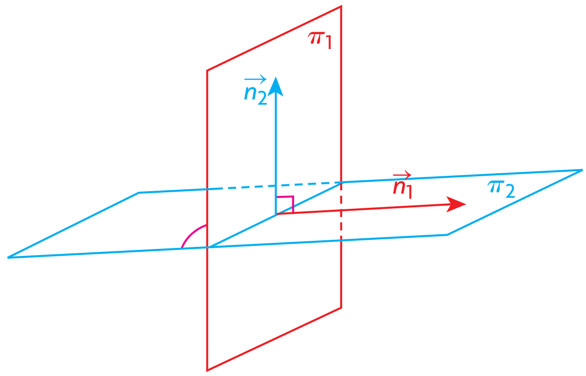
A vector equation is $\vec{r} = (6, 0, 0) + p(-2, -1, 0) + q(-7, -1, 1)$, $p, q \in \mathbf{R}$.

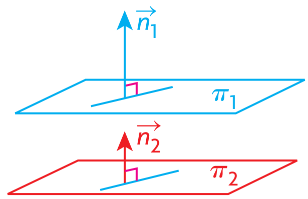
The corresponding parametric form is $x = 6 - 2p - 7q$, $y = -p - q$, and $z = q$.

Check*:*
To check that these equations are correct, the same procedure shown in Method 1 is used. This gives the identical Cartesian equation, $x - 2y + 5z - 6 = 0$.

When we considered lines in R^2, we showed how to determine whether lines were parallel or perpendicular. It is possible to use the same formula to determine whether planes are parallel or perpendicular.

Parallel and Perpendicular Planes





1. If π_1 and π_2 are two perpendicular planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are perpendicular (that is, $\vec{n_1} \cdot \vec{n_2} = 0$).
2. If π_1 and π_2 are two parallel planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are parallel (that is, $\vec{n_1} = k\vec{n_2}$) for all nonzero real numbers k.

EXAMPLE 4 Reasoning about parallel and perpendicular planes

a. Show that the planes $\pi_1 : 2x - 3y + z - 1 = 0$ and $\pi_2 : 4x - 3y - 17z = 0$ are perpendicular.

b. Show that the planes $\pi_3 : 2x - 3y + 2z - 1 = 0$ and $\pi_4 : 2x - 3y + 2z - 3 = 0$ are parallel but not coincident.

Solution

a. For π_1: $\vec{n_1} = (2, -3, 1)$ and for $\pi_2 : \vec{n_2} = (4, -3, -17)$.

$$\begin{aligned}\vec{n_1} \cdot \vec{n_2} &= (2, -3, 1) \cdot (4, -3, -17) \\ &= 2(4) - 3(-3) + 1(-17) \\ &= 8 + 9 - 17 \\ &= 0\end{aligned}$$

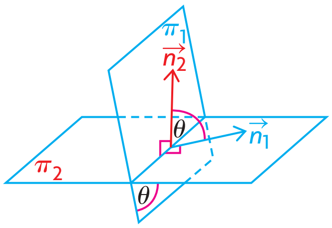
Since $\vec{n_1} \cdot \vec{n_2} = 0$, the two planes are perpendicular to each other.

b. For π_3 and π_4, $\vec{n_3} = \vec{n_4} = (2, -3, 2)$, so the planes are parallel. Because the planes have different constants (that is, -1 and -3), the planes are not coincident.

In general, if planes are coincident, it means that the planes have equations that are scalar multiples of each other. For example, the two planes $2x - y + z - 13 = 0$ and $-6x + 3y - 3z + 39 = 0$ are coincident because $-6x + 3y - 3z + 39 = -3(2x - y + z - 13)$.

It is also possible to find the angle between intersecting planes using their normals and the dot product formula for calculating the angle between two vectors. The angle between two planes is the same as the angle between their normals.

Angle between Intersecting Planes



The angle θ between two planes, π_1 and π_2, with normals of $\vec{n_1}$ and $\vec{n_2}$, respectively, can be calculated using the formula $\cos\theta = \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

EXAMPLE 5 Calculating the angle formed between two intersecting planes

Determine the angle between the two planes $\pi_1: x - y - 2z + 3 = 0$ and $\pi_2: 2x + y - z + 2 = 0$.

Solution

For $\pi_1: \vec{n_1} = (1, -1, -2)$. For $\pi_2: \vec{n_2} = (2, 1, -1)$.

Since $|\vec{n_1}| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$ and

$|\vec{n_2}| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$,

$$\cos\theta = \frac{(1, -1, -2) \cdot (2, 1, -1)}{\sqrt{6}\sqrt{6}}$$

$$\cos\theta = \frac{2 - 1 + 2}{6}$$

$$\cos\theta = \frac{1}{2}$$

Therefore, the angle between the two planes is 60°. Normally, the angle between planes is given as an acute angle, but it is also correct to express it as 120°.

IN SUMMARY

Key Idea

- The Cartesian equation of a plane in R^3 is $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal to the plane and $\vec{n} = \vec{a} \times \vec{b}$. $\vec{a}$ and $\vec{b}$ are any two noncollinear direction vectors of the plane.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$ for any nonzero real number k.
  - are perpendicular if and only if $\vec{n_1} . \vec{n_2} = 0$.
  - have an angle θ between the planes determined by $\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$.

Exercise 8.5

PART A

1. A plane is defined by the equation $x - 7y - 18z = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
2. A plane is defined by the equation $2x - 5y = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
3. A plane is defined by the equation $x = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
4. a. A plane is determined by a normal, $\vec{n} = (15, 75, -105)$, and passes through the origin. Write the Cartesian equation of this plane, where the normal is in reduced form.
   b. A plane has a normal of $\vec{n} = \left(-\frac{1}{2}, \frac{3}{4}, \frac{7}{16}\right)$ and passes through the origin. Determine the Cartesian equation of this plane.

PART B

5. A plane is determined by a normal, $\vec{n} = (1, 7, 5)$, and contains the point $P(-3, 3, 5)$. Determine a Cartesian equation for this plane using the *two* methods shown in Example 1.

K 6. The three noncollinear points $P(-1, 2, 1)$, $Q(3, 1, 4)$, and $R(-2, 3, 5)$ lie on a plane.

a. Using $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as direction vectors and the point $R(-2, 3, 5)$, determine the Cartesian equation of this plane.

b. Using $\overrightarrow{QP}$ and $\overrightarrow{PR}$ as direction vectors and the point $P(-1, 2, 1)$, determine the Cartesian equation of this plane.

c. Explain why the two equations must be the same.

7. Determine the Cartesian equation of the plane that contains the points $A(-2, 3, 1)$, $B(3, 4, 5)$, and $C(1, 1, 0)$.

8. The line with vector equation $\vec{r} = (2, 0, 1) + s(-4, 5, 5)$, $s \in \mathbf{R}$, lies on the plane π, as does the point $P(1, 3, 0)$. Determine the Cartesian equation of π.

9. Determine unit vectors that are normal to each of the following planes:

a. $2x + 2y - z - 1 = 0$

b. $4x - 3y + z - 3 = 0$

c. $3x - 4y + 12z - 1 = 0$

10. A plane contains the point $A(2, 2, -1)$ and the line $\vec{r} = (1, 1, 5) + s(2, 1, 3)$, $s \in \mathbf{R}$. Determine the Cartesian equation of this plane.

A 11. Determine the Cartesian equation of the plane containing the point $(-1, 1, 0)$ and perpendicular to the line joining the points $(1, 2, 1)$ and $(3, -2, 0)$.

C 12. a. Explain the process you would use to determine the angle formed between two intersecting planes.

b. Determine the angle between the planes $x - z + 7 = 0$ and $2x + y - z + 8 = 0$.

13. a. Determine the angle between the planes $x + 2y - 3z - 4 = 0$ and $x + 2y - 1 = 0$.

b. Determine the Cartesian equation of the plane that passes through the point $P(1, 2, 1)$ and is perpendicular to the line $\dfrac{x - 3}{-2} = \dfrac{y + 1}{3} = \dfrac{z + 4}{1}$.

14. a. What is the value of k that makes the planes $4x + ky - 2z + 1 = 0$ and $2x + 4y - z + 4 = 0$ parallel?

b. What is the value of k that makes these two planes perpendicular?

c. Can these two planes ever be coincident? Explain.

T 15. Determine the Cartesian equation of the plane that passes through the points $(1, 4, 5)$ and $(3, 2, 1)$ and is perpendicular to the plane $2x - y + z - 1 = 0$.

PART C

16. Determine an equation of the plane that is perpendicular to the plane $x + 2y + 4 = 0$, contains the origin, and has a normal that makes an angle of $30°$ with the z-axis.

17. Determine the equation of the plane that lies between the points $(-1, 2, 4)$ and $(3, 1, -4)$ and is equidistant from them.

Section 8.6—Sketching Planes in R^3

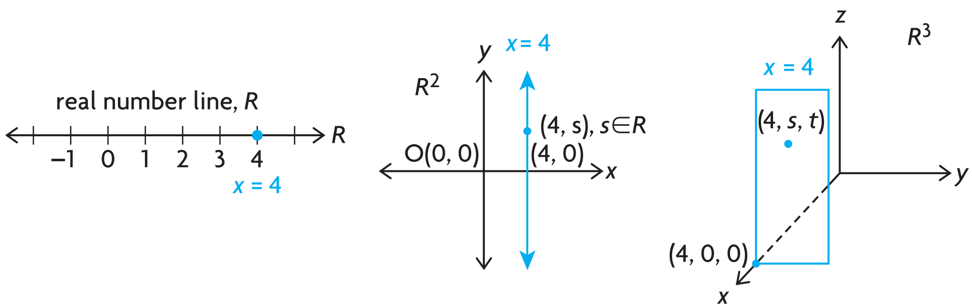
In previous sections, we developed methods for finding the equation of planes in both vector and Cartesian form. In this section, we examine how to sketch a plane if the equation is given in Cartesian form. When graphing planes in R^3, many of the same methods used for graphing a line in R^2 will be used.

An important first observation is that, if we are given an equation such as $x = 4$ and are asked to find its related graph, then a different graph is produced depending on the dimension in which we are working.

1. On the real number line, this equation refers to a point at $x = 4$.
2. In R^2, this equation represents a line parallel to the y-axis and 4 units to its right.
3. In R^3, the equation $x = 4$ represents a plane that intersects the x-axis at $(4, 0, 0)$ and is 4 units in front of the plane formed by the y- and z-axes.

We can see that the equation $x = 4$ results in a different graph depending on whether it is drawn on the number line, in R^2, or in R^3.

Different interpretations of the graph with equation $x = 4$



Varying the Coefficients in the Cartesian Equation

In the following situations, the graph of $Ax + By + Cz + D = 0$ in R^3 is considered for different cases.

Case 1– The equation contains one variable

Case 1a: Two of A, B, or C equal zero, and D equals zero.

In this case, the resulting equation will be of the form $x = 0$, $y = 0$, or $z = 0$. If $x = 0$, for example, this equation represents the yz-plane, since every point on this plane has an x-value equal to 0. Similarly, $y = 0$ represents the xz-plane, and $z = 0$ represents the xy-plane.

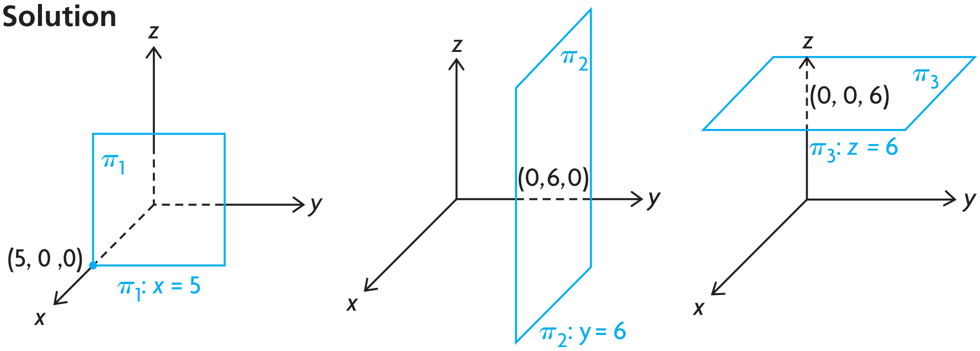
Case 1b: Two of A, B, or C equal zero, and D does not equal zero.

If two of the three coefficients are equal to zero, the resulting equation will be of the form $x = a$, $y = b$, or $z = c$. The following examples show that these equations represent planes parallel to the yz-, xz-, and xy-planes, respectively.

EXAMPLE 1 Representing the graphs of planes in R^3 whose Cartesian equations involve one variable

Draw the planes with equations $\pi_1 : x = 5$, $\pi_2 : y = 6$, and $\pi_3 : z = 6$.

Solution



Descriptions of the planes in Example 1 are given in the following table.

Plane	Description	Generalization
π_1: $x = 5$	A plane parallel to the yz-plane crosses the x-axis at (5, 0, 0). This plane has an x-intercept of 5.	A plane with equation $x = a$ is parallel to the yz-plane and crosses the x-axis at the point $(a, 0, 0)$. The plane $x = 0$ is the yz-plane.
π_2: $y = 6$	A plane parallel to the xz-plane crosses the y-axis at (0, 6, 0). This plane has a y-intercept of 6.	A plane with equation $y = b$ is parallel to the xz-plane and crosses the y-axis at the point $(0, b, 0)$. The plane $y = 0$ is the xz-plane.
π_3: $z = 6$	A plane parallel to the xy-plane crosses the x-axis at the point (0, 0, 6). This plane has a z-intercept of 6.	A plane with equation $z = c$ is parallel to the xy-plane and crosses the z-axis at the point $(0, 0, c)$. The plane $z = 0$ is the xy-plane.

Case 2– The equation contains two variables

Case 2a: One of A, B, or C equals zero, and D equals zero.

In this case, the resulting equation will have the form $Ax + By = 0$, $Ax + Cz = 0$, or $By + Cz = 0$. The following example demonstrates how to graph an equation of this type.

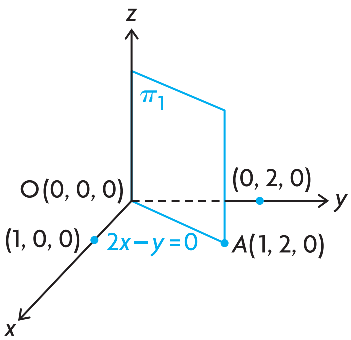
EXAMPLE 2 Representing the graph of a plane in R^3 whose Cartesian equation involves two variables, $D = 0$

Sketch the plane defined by the equation $\pi_1: 2x - y = 0$.

Solution

For $\pi_1: 2x - y = 0$, we note that the origin $O(0, 0, 0)$ lies on the plane, and it also contains the z-axis. We can see that π_1 contains the z-axis because, if it is written in the form $2x - y + 0z = 0$, $(0, 0, t)$ is on the plane because $2(0) - 0 + 0(t) = 0$. Since $(0, 0, t)$, $t \in \mathbf{R}$, represents any point on the z-axis, this means that the plane contains the z-axis.

In addition, the plane cuts the xy-plane along the line $2x - y = 0$. All that is necessary to graph this line is to select a point on the xy-plane that satisfies the equation and join that point to the origin. Since the point with coordinates $A(1, 2, 0)$ satisfies the equation, we draw the parallelogram determined by the z-axis and the line joining $O(0, 0, 0)$ to $A(1, 2, 0)$, and we have a sketch of the plane $2x - y = 0$.



EXAMPLE 3 Describing planes whose Cartesian equations involve two variables, $D = 0$

Write descriptions of the planes $\pi_1: 2x - z = 0$ and $\pi_2: y + 2z = 0$.

Solution

These equations can be written as $\pi_1: 2x + 0y - z = 0$ and $\pi_2: 0x + y + 2z = 0$.

Using the same reasoning as above, this implies that π_1 contains the origin and the y-axis, and cuts the xz-plane along the line with equation $2x - z = 0$. Similarly, π_2 contains the origin and the x-axis, and cuts the yz-plane along the line with equation $y + 2z = 0$.

Case 2b: One of A, B, or C equals zero, and D does not equal zero.

If one of the coefficients equals zero and $D \neq 0$, the resulting equations can be written as $Ax + By + D = 0$, $Ax + Cz + D = 0$, or $By + Cz + D = 0$.

EXAMPLE 4 Graphing planes whose Cartesian equations involve two variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1 : 2x - 5y - 10 = 0$.

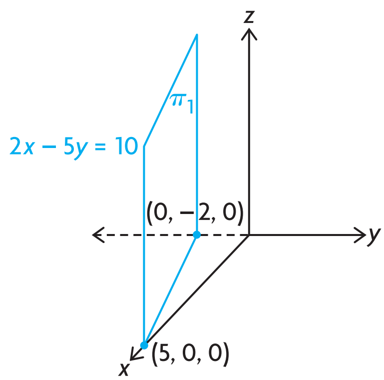
Solution

It is best to write this equation as $2x - 5y = 10$ so that we can easily calculate the intercepts.

x-intercept: We calculate the x-intercept for this plane in exactly the same way that we would calculate the x-intercept for the line $2x - 5y = 10$. If we let $y = 0$, then $2x - 5(0) = 10$ or $x = 5$. This means that the plane has an x-intercept of 5 and that it crosses the x-axis at $(5, 0, 0)$.

y-intercept: To calculate the y-intercept, we let $x = 0$. Thus, $-5y = 10, y = -2$. This means that the plane has a y-intercept of -2 and it crosses the y-axis at $(0, -2, 0)$.

To complete the analysis for the plane, we write the equation as $2x - 5y + 0z = 10$. If the plane did cross the z-axis, it would do so at a point where $x = y = 0$. Substituting these values into the equation, we obtain $2(0) - 5(0) + 0z = 10$ or $0z = 10$. This implies that the plane has no z-intercept because there is no value of z that will satisfy the equation. Thus, the plane passes through the points (5, 0, 0) and $(0, -2, 0)$ and is parallel to the z-axis. The plane is sketched in the diagram below. Possible direction vectors for the plane are $\overrightarrow{m_1} = (5 - 0, 0 - (-2), 0 - 0) = (5, 2, 0)$ and $\overrightarrow{m_2} = (0, 0, 1)$.



Using the same line of reasoning as above, if A, C and D are nonzero when $B = 0$, the resulting plane is parallel to the y-axis. If B, C and D are nonzero when $A = 0$, the resulting plane is parallel to the x-axis.

Case 3– The equation contains three variables

Case 3b: A, B, and C do not equal zero, and D equals zero.

This represents an equation of the form, $Ax + By + Cz = 0$, which is a plane most easily sketched using the fact that a plane is uniquely determined by three points. The following example illustrates this approach.

EXAMPLE 5 Graphing planes whose Cartesian equations involve three variables, $D = 0$

Sketch the plane $\pi_1 : x + 3y - z = 0$.

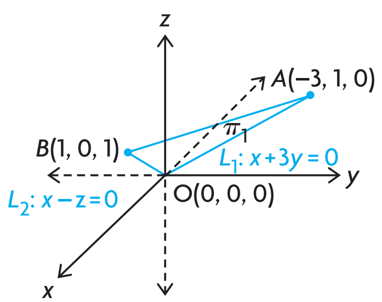
Solution

Since there is no constant in the equation, the point $(0, 0, 0)$ is on the plane. To sketch the plane, we require two other points. We first find a point on the xy-plane and a second point on the xz-plane.

Point on xy-plane: Any point on the xy-plane has $z = 0$. If we first substitute $z = 0$ into $x + 3y - z = 0$, then $x + 3y - 0 = 0$, or $x + 3y = 0$, which means that the given plane cuts the xy-plane along the line $x + 3y = 0$. Using this equation, we can now select convenient values for x and y to obtain the coordinates of a point on this line. The easiest values are $x = -3$ and $y = 1$, implying that the point $A(-3, 1, 0)$ is on the plane.

Point on xz-plane: Any point on the xz-plane has $y = 0$. If we substitute $y = 0$ into $x + 3y - z = 0$, then $x + 3(0) - z = 0$, or $x = z$, which means that the given plane cuts the xz-plane along the line $x = z$. As before, we choose convenient values for x and z. The easiest values are $x = z = 1$, implying that $B(1, 0, 1)$ is a point on the plane.

Since three points determine a plane, we locate these points in R^3 and form the related triangle. This triangle, OAB, represents part of the required plane.



Case 3b: A, B, and C do not equal zero, and D does not equal zero.

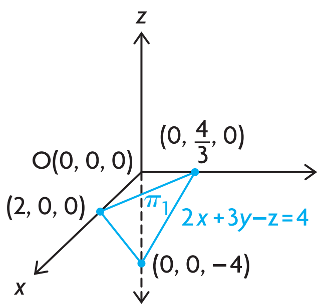
This represents the plane with equation $Ax + By + Cz + D = 0$, which is most easily sketched by finding its intercepts. Since we know that three noncollinear points determine a plane, knowing these three intercepts will allow us to graph the plane.

EXAMPLE 6 Graphing planes whose Cartesian equations involve three variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1 : 2x + 3y - z = 4$.

Solution

To sketch the plane, we calculate the coordinates of the points where the plane intersects each of the three coordinate axes (that is, we determine the three intercepts for the plane). This is accomplished by setting 2 of the 3 variables equal to zero and solving for the remaining variable. The x-, y-, and z-intercepts are 2, $\frac{4}{3}$, and -4, respectively. These three points form a triangle that forms part of the required plane.



EXAMPLE 7 Reasoning about direction vectors of planes

Determine two direction vectors for the planes $\pi_1 : 3x + 4y = 12$ and $\pi_2 : x - y - 5z = 0$.

Solution

The plane $\pi_1 : 3x + 4y = 12$ crosses the x-axis at the point $(4, 0, 0)$ and the y-axis at the point $(0, 3, 0)$. One direction vector is thus $\overrightarrow{m_1} = (4 - 0, 0 - 3, 0 - 0) = (4, -3, 0)$. Since the plane can be written as $3x + 4y + 0z = 12$, this implies that it does not intersect the z-axis, and therefore has $\overrightarrow{m_2} = (0, 0, 1)$ as a second direction vector.

The plane $\pi_2 : x - y - 5z = 0$ passes through $O(0, 0, 0)$ and cuts the xz-plane along the line $x - 5z = 0$. Convenient choices for x and z are 5 and 1, respectively. This means that $A(5, 0, 1)$ is on π_2. Similarly, the given plane cuts the xy-plane along the line $x - y = 0$. Convenient values for x and y are 1 and 1. This means that $B(1, 1, 0)$ is on π_2.

Possible direction vectors for π_2 are $\overrightarrow{m_1} = (5 - 0, 0 - 0, 1 - 0) = (5, 0, 1)$ and $\overrightarrow{m_2} = (1, 1, 0)$.

IN SUMMARY

Key Idea

- A sketch of a plane in R^3 can be created by using a combination of points and lines that help to define the plane.

Need to Know

- To sketch the graph of a plane, consider each of the following cases as it relates to the Cartesian equation $Ax + By + Cz + D = 0$:

  Case 1: The equation contains one variable.

  a. Two of the coefficients (two of A, B, or C) equal zero, and D equals zero.

     These are the three coordinate planes—xy-plane, xz-plane, and yz-plane.

  b. Two of the coefficients (two of A, B, or C) equal zero, and D does not equal zero.

     These are parallel to the three coordinate planes.

  Case 2: The equation contains two variables.

  a. One of the coefficients (one of A, B, or C) equals zero, and D equals zero.

     Find a point with missing variable set equal to 0. Join this point to (0, 0, 0), and draw a plane containing the missing variable axis and this point.

  b. One of the coefficients (one of A, B, or C) equals zero, and D does not equal zero.

     Find the two intercepts, and draw a plane parallel to the missing variable axis.

  Case 3: The equation contains three variables.

  a. None of the coefficients (none of A, B, or C) equals zero, and D equals zero.

     Determine two points in addition to (0, 0, 0), and draw the plane through these points.

  b. None of the coefficients (none of A, B, or C) equals zero, and D does not equal zero.

     Find the three intercepts, and draw a plane through these three points.

Exercise 8.6

PART A

C 1. Describe each of the following planes in words:

a. $x = -2$ b. $y = 3$ c. $z = 4$

2. For the three planes given in question 1, what are coordinates of their point of intersection?

3. On which of the planes π_1: $x = 5$ or π_2: $y = 6$ could the point $P(5, -3, -3)$ lie? Explain.

PART B

A 4. Given that $x^2 - 1 = (x - 1)(x + 1)$, sketch the two graphs associated with $x^2 - 1 = 0$ in R^2 and R^3.

5. a. State the x-, y-, and z-intercepts for each of the following three planes:
   i. $\pi_1 : 2x + 3y = 18$
   ii. $\pi_2 : 3x - 4y + 5z = 120$
   iii. $\pi_3 : 13y - z = 39$
   b. State two direction vectors for each plane.

6. a. For the plane with equation $\pi : 2x - y + 5z = 0$, determine
   i. the coordinates of three points on this plane
   ii. the equation of the line where this plane intersects the xy-plane
   b. Sketch this plane.

7. Name the three planes that the equation $xyz = 0$ represents in R^3.

K 8. For each of the following equations, sketch the corresponding plane:
   a. $\pi_1 : 4x - y = 0$
   b. $\pi_2 : 2x + y - z = 4$
   c. $\pi_3 : z = 4$
   d. $\pi_4 : y - z = 4$

9. a. Write the expression $xy + 2y = 0$ in factored form.
   b. Sketch the lines corresponding to this expression in R^2.
   c. Sketch the planes corresponding to this expression in R^3.

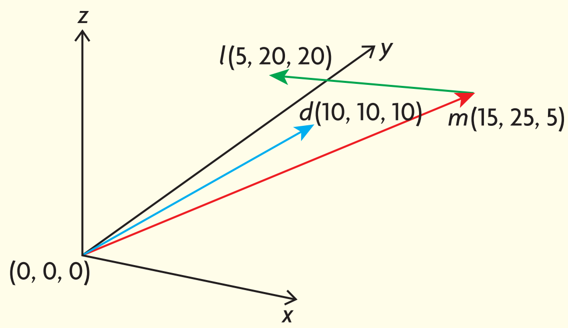
10. For each given equation, sketch the corresponding plane.
   a. $2x + 2y + z - 4 = 0$
   b. $3x - 4z = 12$
   c. $5y - 15 = 0$

PART C

T 11. It is sometimes useful to be able to write an equation of a plane in terms of its intercepts. If a, b, and c represent the x-, y-, and z-intercepts, respectively, then the resulting equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
   a. Determine the equation of the plane that has x-, y-, and z-intercepts of 3, 4, and 6, respectively.
   b. Determine the equation of the plane that has x- and z-intercepts of 5 and –7, respectively, and is parallel to the y-axis.
   c. Determine the equation of the plane that has no x- or y-intercept, but has a z-intercept of 8.

CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

A computer programmer is designing a 3-D space game. She wants to have an asteroid fly past a spaceship along the path of vector m, collide with another asteroid, and be deflected along vector path $\overrightarrow{ml}$. The spaceship is treated as the origin and is travelling along vector d.



a. Determine the vector and parametric equations for the line determined by vector $\overrightarrow{ml}$ in its current position.

b. Determine the vector and parametric equations for the line determined by vector $\vec{d}$ in its current position.

c. By using the previous parts, can you determine if the asteroid and spaceship could possibly collide as they travel along their respective trajectories? Explain in detail all that would have to take place for this collision to occur (if, indeed, a collision is even possible).

Key Concepts Review

In Chapter 8, you examined how the algebraic description of a straight line could be represented using vectors in both two and three dimensions. The form of the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{d}$, is the same whether the line lies in a two-dimensional plane or a three-dimensional space. The following table summarizes the various forms of the equation of a line, where the coordinates of a point on the line are known as well as a direction vector. t is a parameter where $t \in \mathbf{R}$.

Form	R^2	R^3
Vector equation	$(x, y) = (x_0, y_0) + t(a, b)$	$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$
Parametric equations	$x = x_0 + at, y = y_0 + bt$	$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$
Symmetric equations	$\frac{x - x_0}{a} = \frac{y - y_0}{b}$	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
Cartesian equation	$Ax + BY + C = 0: \vec{n} = (A, B)$	not applicable

This concept was then extended to planes in R^3. The following table summarizes the various forms of the equation of a plane, where the coordinates of a point on the plane are known as well as two direction vectors. s and t are parameters where $s, t \in \mathbf{R}$.

Form	R^3
Vector equation	$(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$
Parametric equations	$x = x_0 + sa_1 + tb_1, y = y_0 + sa_2 + tb_2, z = z_0 + sa_3 + tb_3$
Cartesian equation	$Ax + BY + Cz + D = 0: \vec{n} = (A, B, C)$

You also saw that when lines and planes intersect, angles are formed between them. Both lines and planes have normals, which are vectors that run perpendicular to the respective line or plane. The size of an angle can be determined using the normal vectors and the following formula:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

Sketching the graph of a plane in R^3 can be accomplished by examining the Cartesian equation of the plane. Determine whether the equation contains one, two, or three variables and whether it contains a constant. This information helps to narrow down which case you need to consider to sketch the graph. Once you have determined the specific case (see the In Summary table in Section 8.6), you can determine the appropriate points and lines to help you sketch a representation of the plane.

Review Exercise

1. Determine vector and parametric equations of the plane that contains the points $A(1, 2, -1)$, $B(2, 1, 1)$, and $C(3, 1, 4)$.

2. In question 1, there are a variety of different answers possible, depending on the points and direction vectors chosen. Determine two Cartesian equations for this plane using two different vector equations, and verify that these two equations are identical.

3. a. Determine the vector, parametric, and symmetric equations of the line passing through points $A(-3, 2, 8)$ and $B(4, 3, 9)$.
   b. Determine the vector and parametric equations of the plane containing the points $A(-3, 2, 8)$, $B(4, 3, 9)$, and $C(-2, -1, 3)$.
   c. Explain why a symmetric equation cannot exist for a plane.

4. Determine the vector, parametric, and symmetric equations of the line passing through the point $A(7, 1, -2)$ and perpendicular to the plane with equation $2x - 3y + z - 1 = 0$.

5. Determine the Cartesian equation of each of the following planes:
   a. through the point $P(0, 1, -2)$, with normal $\vec{n} = (-1, 3, 3)$
   b. through the points $(3, 0, 1)$ and $(0, 1, -1)$, and perpendicular to the plane with equation $x - y - z + 1 = 0$
   c. through the points $(1, 2, 1)$ and $(2, 1, 4)$, and parallel to the x-axis

6. Determine the Cartesian equation of the plane that passes through the origin and contains the line $\vec{r} = (3, 7, 1) + t(2, 2, 3)$, $t \in \mathbf{R}$.

7. Find the vector and parametric equations of the plane that is parallel to the yz-plane and contains the point $A(-1, 2, 1)$.

8. Determine the Cartesian equation of the plane that contains the line $\vec{r} = (2, 3, 2) + t(1, 1, 4)$, $t \in \mathbf{R}$, and the point $(4, -3, 2)$.

9. Determine the Cartesian equation of the plane that contains the following lines:
   $L_1: \vec{r} = (4, 4, 5) + t(5, -4, 6)$, $t \in \mathbf{R}$, and
   $L_2: \vec{r} = (4, 4, 5) + s(2, -3, -4)$, $s \in \mathbf{R}$

10. Determine an equation for the line that is perpendicular to the plane $3x - 2y + z = 1$ passing through $(2, 3, -3)$. Give your answer in vector, parametric, and symmetric form.

11. A plane has $3x + 2y - z + 6 = 0$ as its Cartesian equation. Determine the vector and parametric equations of this plane.

12. Determine an equation for the line that has the same x- and z-intercepts as the plane with equation $2x + 5y - z + 7 = 0$. Give your answer in vector, parametric, and symmetric form.

13. Determine the vector, parametric, and Cartesian forms of the equation of the plane containing the lines $L_1 : \vec{r} = (3, -4, 1) + s(1, -3, -5),\ s \in \mathbf{R}$, and $L_2 : \vec{r_2} = (7, -1, 0) + t(2, -6, -10),\ t \in \mathbf{R}$.

14. Sketch each of the following planes:

    a. $\pi_1 : 2x + 3y - 6z - 12 = 0$

    b. $\pi_2 : 2x + 3y - 12 = 0$

    c. $\pi_3 : x - 3z - 6 = 0$

    d. $\pi_4 : y - 2z - 4 = 0$

    e. $\pi_5 : 2x + 3y - 6z = 0$

15. Determine the vector, parametric, and Cartesian equations of each of the following planes:

    a. passing through the points $P(1, -2, 5)$ and $Q(3, 1, 2)$ and parallel to the line with equation $L : \vec{r} = 2t\vec{i} + (4t + 3)\vec{j} + (t + 1)\vec{k},\ t \in \mathbf{R}$

    b. containing the point $A(1, 1, 2)$ and perpendicular to the line joining the points $B(2, 1, -6)$ and $C(-2, 1, 5)$

    c. passing through the points $(4, 1, -1)$ and $(5, -2, 4)$ and parallel to the z-axis

    d. passing through the points $(1, 3, -5)$, $(2, 6, 4)$, and $(3, -3, 3)$

16. Show that $L_1 : \vec{r} = (1, 2, 3) + s(-3, 5, 21) + t(0, 1, 3)$, s, $t \in \mathbf{R}$, and $L_2 : \vec{r} = (1, -1, -6) + u(1, 1, 1) + v(2, 5, 11)$, $u, v \in \mathbf{R}$, are equations for the same plane.

17. The two lines $L_1 : \vec{r} = (-1, 1, 0) + s(2, 1, -1),\ s \in \mathbf{R}$, and $L_2 : \vec{r} = (2, 1, 2) + t(2, 1, -1),\ t \in \mathbf{R}$, are parallel but do not coincide. The point $A(5, 4, -3)$ is on L_1. Determine the coordinates of a point B on L_2 such that $\overrightarrow{AB}$ is perpendicular to L_2.

18. Write a brief description of each plane.

    a. $\pi_1 : 2x - 3y = 6$

    b. $\pi_2 : x - 3z = 6$

    c. $\pi_3 : 2y - z = 6$

19. a. Which of the following points lies on the line $x = 2t,\ y = 3 + t,\ z = 1 + t$?

    $A(2, 4, 2)$ $\quad B(-2, 2, 1)$ $\quad C(4, 5, 2)$ $\quad D(6, 6, 2)$

    b. If the point $(a, b, -3)$ lies on the line, determine the values of a and b.

20. Calculate the acute angle that is formed by the intersection of each pair of lines.

a. $L_1: \frac{x-1}{1} = \frac{y-3}{5}$ and $L_2: \frac{x-2}{2} = \frac{1-y}{3}$

b. $y = 4x + 2$ and $y = -x + 3$

c. $L_1: x = -1 + 3t, y = 1 + 4t, z = -2t$ and
$L_2: x = -1 + 2s, y = 3s, z = -7 + s$

d. $L_1: (x, y, z) = (4, 7, -1) + t(4, 8, -4)$ and
$L_2: (x, y, z) = (1, 5, 4) + s(-1, 2, 3)$

21. Calculate the acute angle that is formed by the intersection of each pair of planes.

a. $2x + 3y - z + 9 = 0$ and $x + 2y + 4 = 0$

b. $x - y - z - 1 = 0$ and $2x + 3y - z + 4 = 0$

22. a. Which of the following lines is parallel to the plane $4x + y - z - 10 = 0$?

i. $\vec{r} = (3, 0, 2) + t(1, -2, 2)$

ii. $x = -3t, y = -5 + 2t, z = -10t$

iii. $\frac{x-1}{4} = \frac{y+6}{-1} = \frac{z}{1}$

b. Do any of these lines lie in the plane in part a.?

23. Does the point (4, 5, 6) lie in the plane
$(x, y, z) = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$?
Support your answer with the appropriate calculations.

24. Determine the parametric equations of the plane that contains the following two parallel lines:

$L_1: (x, y, z) = (2, 4, 1) + t(3, -1, 1)$ and

$L_2: (x, y, z) = (1, 4, 4) + t(-6, 2, -2)$

25. Explain why the vector equation of a plane has two parameters, while the vector equation of a line has only one.

26. Explain why any plane with a vector equation of the form
$(x, y, z) = (a, b, c) + s(d, e, f) + t(a, b, c)$ will always pass through the origin.

27. a. Explain why the three points $(2, 3, -1)$, $(8, 5, -5)$ and $(-1, 2, 1)$ do not determine a plane.

b. Explain why the line $\vec{r} = (4, 9, -3) + t(1, -4, 2)$ and the point $(8, -7, 5)$ do not determine a plane.

28. Find a formula for the scalar equation of a plane in terms of a, b, and c, where a, b, and c are the x-intercept, the y-intercept, and the z-intercept of a plane, respectively. Assume that all intercepts are nonzero.

29. Determine the Cartesian equation of the plane that has normal vector $(6, -5, 12)$ and passes through the point $(5, 8, -3)$.

30. A plane passes through the points $A(1, -3, 2)$, $B(-2, 4, -2)$, and $C(3, 2, 1)$.
    a. Determine a vector equation of the plane.
    b. Determine a set of parametric equations of the plane.
    c. Determine the Cartesian equation of the plane.
    d. Determine if the point $(3, 5, -4)$ lies on the plane.

31. Determine the Cartesian equation of the plane that is parallel to the plane $4x - 2y + 5z - 10 = 0$ and passes through each point below.
    a. $(0, 0, 0)$
    b. $(-1, 5, -1)$
    c. $(2, -2, 2)$

32. Show that the following pairs of lines intersect. Determine the coordinates of the point of intersection and the angles formed by the lines.
    a. $L_1\colon x = 5 + 2t$, $y = -3 + t$ and $L_2\colon x = 23 - 2s$, $y = 6 - s$
    b. $L_1\colon \dfrac{x + 3}{3} = \dfrac{y + 1}{4}$ and $L_2\colon \dfrac{x - 6}{3} = \dfrac{y - 2}{-2}$

33. Determine the vector equation, parametric equations, and, if possible, symmetric equation of the line that passes through the point $P(1, 3, 5)$ and
    a. has direction vector $(-2, -4, -10)$
    b. also passes through the point $Q(-7, 9, 3)$
    c. is parallel to the line that passes through $R(4, 8, -5)$ and $S(-2, -5, 9)$
    d. is parallel to the x-axis
    e. is perpendicular to the line $(x, y, z) = (1, 0, 5) + t(-3, 4, -6)$
    f. is perpendicular to the plane determined by the points $A(4, 2, 1)$, $B(3, -4, 2)$, and $C(-3, 2, 1)$

34. Determine the Cartesian equation of the plane that
    a. contains the point $P(-2, 6, 1)$ and has normal vector $(2, -4, 5)$
    b. contains the point $P(-2, 0, 6)$ and the line $\frac{x - 4}{3} = \frac{y + 2}{-5} = \frac{z - 1}{2}$
    c. contains the point $P(3, 3, 3)$ and is parallel to the xy-plane
    d. contains the point $P(-4, 2, 4)$ and is parallel to the plane $3x + y - 4z + 8 = 0$
    e. is perpendicular to the yz-plane and has y-intercept 4 and z-intercept -2
    f. is perpendicular to the plane $x - 2y + z = 6$ and contains the line $(x, y, z) = (2, -1, -1) + t(3, 1, 2)$

Chapter 8 Test

1. a. Given the points $A(1, 2, 4)$, $B(2, 0, 3)$, and $C(4, 4, 4)$,
   - i. determine the vector and parametric equations of the plane that contains these three points
   - ii. determine the corresponding Cartesian equation of the plane that contains these three points

   b. Does the point with coordinates $\left(1, -1, -\frac{1}{2}\right)$ lie on this plane?

2. The plane π intersects the coordinate axes at $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$.
   a. Write an equation for this plane, expressing it in the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
   b. Determine the coordinates of a normal to this plane.

3. a. Determine a vector equation for the plane containing the origin and the line with equation $\vec{r} = (2, 1, 3) + t(1, 2, 5), t \in \mathbf{R}$.
   b. Determine the corresponding Cartesian equation of this plane.

4. a. Determine a vector equation for the plane that contains the following two lines:
   $L_1 : \vec{r} = (4, -3, 5) + t(2, 0, -3), t \in \mathbf{R}$, and
   $L_2 : \vec{r} = (4, -3, 5) + s(5, 1, -1), s \in \mathbf{R}$
   b. Determine the corresponding Cartesian equation of this plane.

5. a. A line has $\frac{x - 2}{4} = \frac{y - 4}{-2} = z$ as its symmetric equations. Determine the coordinates of the point where this line intersects the yz-plane.
   b. Write a second symmetric equation for this line using the point you found in part a.

6. a. Determine the angle between π_1 and π_2 where the two planes are defined as $\pi_1 : x + y - z = 0$ and $\pi_2 : x - y + z = 0$.
   b. Given the planes $\pi_3 : 2x - y + kz = 5$ and $\pi_4 : kx - 2y + 8z = 9$,
   - i. determine a value of k if these planes are parallel
   - ii. determine a value of k if these planes are perpendicular

   c. Explain why the two given equations that contain the parameter k in part b cannot represent two identical planes.

7. a. Using a set of coordinate axes in R^2, sketch the line $x + 2y = 0$.
   b. Using a set of coordinate axes in R^3, sketch the plane $x + 2y = 0$.
   c. The equation $Ax + By = 0$, $A, B \neq 0$, represents an equation of a plane in R^3. Explain why this plane must always contain the z-axis.

$\cos A = 1$
$\cos B = 1$
$\cos C = -1$
area of triangle $ABC = 0$

Chapter 7 Test, p. 422

1. **a.** $(-4, -1, -3)$
b. $(-4, -1, -3)$
c. 0
d. 0

2. **a.** scalar projection: $\frac{1}{3}$,
vector projection: $\frac{1}{9}(2, -1, -2)$.
b. x-axis: 48.2°; y-axis: 109.5°; z-axis: 131.8°
c. $\sqrt{26}$ or 5.10

3. Both forces have a magnitude of 78.10 N. The resultant makes an angle 33.7° to the 40 N force and 26.3° to the 50 N force. The equilibrant makes an angle 146.3° to the 40 N force and 153.7° to the 50 N force.

4. 1004.99 km/h, N 5.7° W

5. **a.** 96 m downstream
b. 28.7° upstream

6. 3.50 square units.

7. cord at 45°: about 254.0 N; cord at 70°: about 191.1 N

8. **a.** 0
33
$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$
$= \frac{1}{4}(33) - \frac{1}{4}(33) = 0$
So, the equation holds for these vectors.

b.
$$\begin{aligned}|\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{x}) + (\vec{y}\cdot\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$
$$\begin{aligned}|\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot -\vec{y}) + (-\vec{y}\cdot\vec{x}) + (-\vec{y}\cdot -\vec{y})\\ &= (\vec{x}\cdot\vec{x}) - 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$
So, the right side of the equation is
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2 = \frac{1}{4}(4(\vec{x}\cdot\vec{y})) = \vec{x}\cdot\vec{y}$$

Chapter 8

Review of Prerequisite Skills, pp. 424–425

1. **a.** $(2, -9, 6)$
b. $(13, -12, -41)$

2. **a.** yes **c.** yes
b. yes **d.** no

3. yes

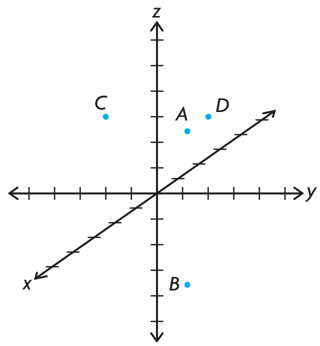
4. $t = 18$

5. **a.** $(3, 1)$
b. $(5, 6)$
c. $(-4, 7, 0)$

6. $\sqrt{2802}$

7. **a.** $(-22, -8, -13)$
b. $(0, 0, -3)$

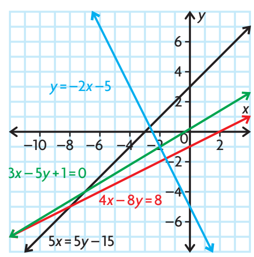
8.



9. **a.** $(-7, -3)$
b. $(10, 14)$
c. $(2, -8, 5)$
d. $(-4, 5, 4)$

10. **a.** $(7, 3)$
b. $(-10, -14)$
c. $(-2, 8, -5)$
d. $(4, -5, -4)$

11. **a.** slope: -2; y-intercept: -5
b. slope: $\frac{1}{2}$; y-intercept: -1
c. slope: $\frac{3}{5}$; y-intercept: $\frac{1}{5}$
d. slope: 1; y-intercept: 3



12. Answers may vary. For example:
a. $(8, 14)$
b. $(-15, 12, 9)$
c. $\vec{i} + 3\vec{j} - 2\vec{k}$
d. $-20\vec{i} + 32\vec{j} + 8\vec{k}$

13. **a.** 33
b. -33
c. 77
d. $(-11, -8, 28)$
e. $(11, 8, -28)$
f. $(55, 40, -140)$

14. The dot product of two vectors yields a real number, while the cross product of two vectors gives another vector.

Section 8.1, pp. 433–434

1. Direction vectors for a line are unique only up to scalar multiplication. So, since each of the given vectors is just a scalar multiple of $\left(\frac{1}{3}, \frac{1}{6}\right)$, each is an acceptable direction vector for the line.

2. **a.** Answers may vary. For example, $(-2, 7)$, $(1, 5)$, and $(4, 3)$.
b. $t = -5$
If $t = -5$, then $x = -14$ and $y = 15$. So $P(-14, 15)$ is a point on the line.

3. Answers may vary. For example:
a. direction vector: $(2, 1)$; point: $(3, 4)$
b. direction vector: $(2, -7)$; point: $(1, 3)$
c. direction vector: $(0, 2)$; point: $(4, 1)$
d. direction vector: $(-5, 0)$; point: $(0, 6)$

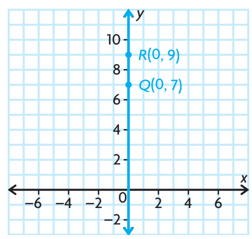
4. Answers may vary. For example:
$\vec{r} = (2, 1) + t(-5, 4), t \in \mathbf{R}$
$\vec{q} = (-3, 5) + s(5, -4), s \in \mathbf{R}$

5. **a.** $R(-9, 18)$ is a point on the line. When $t = 7$, $x = -9$ and $y = 18$.
b. Answers may vary. For example:
$\vec{r} = (-9, 18) + t(-1, 2), t \in \mathbf{R}$
c. Answers may vary. For example:
$\vec{r} = (-2, 4) + t(-1, 2), t \in \mathbf{R}$

6. Answers may vary. For example:
a. $(-3, -4)$, $(0, 0)$, and $(3, 4)$
b. $\vec{r} = t(1, 1), t \in \mathbf{R}$
c. This describes the same line as part a.

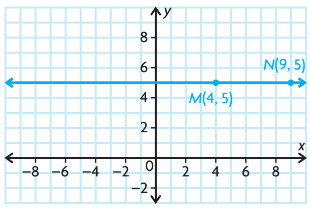
7. One can multiply a direction vector by a constant to keep the same line, but multiplying the point yields a different line.

8. **a.**



b. $\vec{r} = (0, 7) + t(0, 2), t \in \mathbf{R}$; $x = 0$, $y = 7 + 2t, t \in \mathbf{R}$

9. a.



b. $\vec{r} = (4, 5) + t(5, 0), t \in \mathbf{R}$;
$x = 4 + 5t, y = 5, t \in \mathbf{R}$

10. a. $\vec{r} = (2, 0) + t(5, -3), t \in \mathbf{R}$

b. $(0, -1.2)$

11. 9

12. First, all the relevant vectors are found.

$\overrightarrow{AB} = (-2, 3)$

$\overrightarrow{AC} = (-4, 6)$

$\overrightarrow{AD} = (-6, 9)$

a. $\overrightarrow{AC} = (-4, 6) = 2(-2, 3) = 2\overrightarrow{AB}$

b. $\overrightarrow{AD} = (-6, 9) = 3(-2, 3) = 3\overrightarrow{AB}$

c. $\overrightarrow{AC} = (-4, 6) = \frac{2}{3}(-6, 9) = \frac{2}{3}\overrightarrow{AD}$

13. a. $A(5, 12)$; $B(-12, -5)$

b. $\sqrt{578}$ or about 24.04

14. In the parametric form, the second equation becomes $x = 1 + 6t$, $y = 6 + 4t, t \in \mathbf{R}$. If t is solved for in this equation, we obtain $t = \frac{x - 1}{6}$ and $t = \frac{y - 6}{4}$. Setting these two expressions equal to each other, the line is described by $\frac{x - 1}{6} = \frac{y - 6}{4}$, or by simplifying, $y - 6 = \frac{2}{3}x - \frac{2}{3}$. So, the second equation describes a line with a slope of $\frac{2}{3}$. If y is solved for in the first expression, we see that $y = \frac{2}{3}x + 5$. $(1, 6)$ is on the second line but not the first. Hence, both equations are lines with slope of $\frac{2}{3}$ and must be parallel.

Section 8.2, pp. 443–444

1. a. $\vec{m} = (6, -5)$

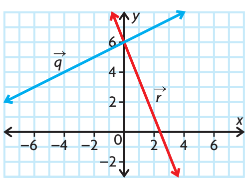
b. $\vec{n} = (5, 6)$

c. $(0, 9)$

d. $\vec{r} = (7, 9) + t(6, -5), t \in \mathbf{R}$;
$x = 7 + 6t, y = 9 - 5t, t \in \mathbf{R}$

e. $\vec{r} = (-2, 1) + t(5, 6), t \in \mathbf{R}$;
$x = -2 + 5t, y = 1 + 6t, t \in \mathbf{R}$

2. a.–b.



c. It produces a different line.

3. a. $\vec{r} = (0, -6) + t(8, 7), t \in \mathbf{R}$;
$x = 8t, y = -6 + 7t, t \in \mathbf{R}$

b. $\vec{r} = (0, 5) + t(2, 3), t \in \mathbf{R}$;
$x = 2t, y = 5 + 3t, t \in \mathbf{R}$

c. $\vec{r} = (0, -1) + t(1, 0), t \in \mathbf{R}$;
$x = t, y = -1, t \in \mathbf{R}$

d. $\vec{r} = (4, 0) + t(0, 1), t \in \mathbf{R}$;
$x = 4, y = t, t \in \mathbf{R}$

4. If the two lines have direction vectors that are collinear and share a point in common, then the two lines are coincident. In this example, both have $(3, 2)$ as a parallel direction vector and both have $(-4, 0)$ as a point on the line. Hence, the two lines are coincident.

5. a. The normal vectors for the lines are $(2, -3)$ and $(4, -6)$, which are collinear. Since in two dimensions, any two direction vectors perpendicular to $(2, -3)$ are collinear, the lines have collinear direction vectors. Hence, the lines are parallel.

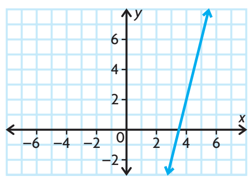
b. $k = 12$

6. $4x + 5y - 21 = 0$

7. $x + y - 2 = 0$

8. $2x + y - 16 = 0$

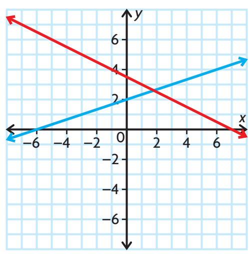
9. a.



b. $4x - y - 14 = 0$

10. a. 82° **c.** 63° **e.** 54°
b. 42° **d.** 37° **f.** 63°

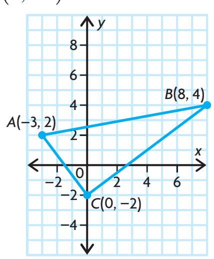
11. a.



b. acute: 45°, obtuse: 135°

12. a. $(0, -2)$

b.



c. $\overrightarrow{CA} = (-3 - 0, 2 - (-2))$
$= (-3, 4)$

$\overrightarrow{CB} = (8 - 0, 4 - (-2))$
$= (8, 6)$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-3)(8) + (4)(6)$
$= -24 + 24$
$= 0$

Since the dot product of the vectors is 0, the vectors are perpendicular, and $\angle ACB = 90°$.

13. The sum of the interior angles of a quadrilateral is 360°. The normals make 90° angles with their respective lines at A and C. The angle of the quadrilateral at B is $180° - \theta$. Let x represent the measure of the interior angle of the quadrilateral at O.

$$90° + 90° + 180° - \theta + x = 360°$$
$$360° - \theta + x = 360°$$
$$x = \theta$$

Therefore, the angle between the normals is the same as the angle between the lines.

14. $2 \pm \sqrt{3}$

Section 8.3, pp. 449–450

1. a. $(-3, 1, 8)$

b. $(1, -1, 3)$

c. $(-2, 1, 3)$

d. $(-2, -3, 1)$

e. $(3, -2, -1)$

f. $\left(\frac{1}{3}, -\frac{3}{4}, \frac{2}{5}\right)$

2. a. $(-1, 1, 9)$

b. $(2, 1, -1)$

c. $(3, -4, -1)$

d. $(-1, 0, 2)$

e. $(0, 0, 2)$

f. $(2, -1, 2)$

3. a. $\vec{r} = (-1, 2, 4) + t(4, -5, 1), t \in \mathbf{R}$;
$\vec{q} = (3, -3, 5) + s(-4, 5, -1), s \in \mathbf{R}$

b. $x = -1 + 4t, y = 2 - 5t,$
$z = 4 + t, t \in \mathbf{R}$; $x = 3 - 4s,$
$y = -3 + 5s, z = 5 - s, s \in \mathbf{R}$

4. a. $\vec{r} = (-1, 5, -4) + t(1, 0, 0), t \in \mathbf{R}$

b. $x = -1 + t, y = 5, z = -4, t \in \mathbf{R}$

c. Since two of the coordinates in the direction vector are zero, a symmetric equation cannot exist.

5. a. $\vec{r} = (-1, 2, 1) + t(3, -2, 1), t \in \mathbf{R}$;
$x = -1 + 3t, y = 2 - 2t,$
$z = 1 + t, t \in \mathbf{R}$;
$\frac{x + 1}{3} = \frac{y - 2}{-2} = \frac{z - 1}{1}$

b. $\vec{r} = (-1, 1, 0) + t(0, 1, 1), t \in \mathbf{R}$;
$x = -1, y = 1 + t, z = t, t \in \mathbf{R}$;
$\frac{y-1}{1} = \frac{z}{1}, x = -1$

c. $\vec{r} = (-2, 3, 0) + t(0, 1, 1), t \in \mathbf{R}$;
$x = -2, y = 3 + t, z = t, t \in \mathbf{R}$;
$\frac{y-3}{1} = \frac{z}{1}, x = -2$

d. $\vec{r} = (-1, 0, 0) + t(0, 1, 0), t \in \mathbf{R}$;
$x = -1, y = t, z = 0, t \in \mathbf{R}$;
Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.

e. $\vec{r} = t(-4, 3, 0), t \in \mathbf{R}$;
$x = -4t, y = 3t, z = 0, t \in \mathbf{R}$;
$\frac{x}{-4} = \frac{y}{3}, z = 0$

f. $\vec{r} = (1, 2, 4) + t(0, 0, 1), t \in \mathbf{R}$;
$x = 1, y = 2, z = 4 + t, t \in \mathbf{R}$;
Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.

6. **a.** $x = -6 + t, y = 10 - t, z = 7 + t, t \in \mathbf{R}$;
$x = -7 + s, y = 11 - s, z = 5, s \in \mathbf{R}$
b. about $35.3°$

7. The directional vector of the first line is $(8, 2, -2) = -2(-4, -1, 1)$. So, $(-4, -1, 1)$ is a directional vector for the first line as well. Since $(-4, -1, 1)$ is also the directional vector of the second line, the lines are the same if the lines share a point. $(1, 1, 3)$ is a point on the second line. Since
$1 = \frac{1+7}{8} = \frac{1+1}{2} = \frac{3-5}{-2}$, $(1, 1, 3)$ is a point on the first line as well. Hence, the lines are the same.

8. **a.** The line that passes through $(0, 0, 3)$ with a directional vector of $(-3, 1, -6)$ is given by the parametric equation is $x = 3t, y = t, z = 3 - 6t, t \in \mathbf{R}$. So, the y-coordinate is equal to -2 only when $t = -2$. At $t = -2, x = -3(-2) = 6$ and $z = 3 - 6(-2) = 15$. So, $A(6, -2, 15)$ is a point on the line. So, the y-coordinate is equal to 5 only when $t = 5$. At $t = 5$, $x = -3(5) = -15$ and $z = 3 - 6(5) = -27$. So, $B(-15, 5, -27)$ is a point on the line.
b. $x = -3t, y = t, z = 3 - 6t, -2 \le t \le 5$

9. -1

10. **a.** $(8, 4, -3), (0, -8, 13), (4, -2, 5)$
b. $(-9, 3, 15), (1, 1, 3), (-4, 2, 9)$
c. $(-4, 3, -4), (2, 1, 4), (-1, 2, 0)$
d. $(-4, -1, -2), (-4, 5, 8)\ (-4, 2, 3)$

11. **a.** $x = 4 - 4t, y = -2 - 6t, z = 5 + 8t, t \in \mathbf{R}$;
$\frac{x-4}{-4} = \frac{y+2}{-6} = \frac{z-5}{8}$
b. $\vec{r} = (-4, 2, 9) + s(5, -1, -6), s \in \mathbf{R}$; $\frac{x+4}{5} = \frac{y-2}{-1} = \frac{z-9}{-6}$
c. $\vec{r} = (-1, 2, 0) + t(3, -1, 4), t \in \mathbf{R}$;
$x = -1 + 3t, y = 2 - t, z = 4t, t \in \mathbf{R}$
d. $\vec{r} = (-4, 2, 3) + t(0, 3, 5), t \in \mathbf{R}$;
$x = -4, y = 2 + 3t, z = 3 + 5t, t \in \mathbf{R}$

12. $x = 2 - 34t, y = -5 + 25t, z = 13t, t \in \mathbf{R}$

13. $(-2, -1, 2), (2, 1, 2)$.

14. $P_1(2, 3, -2)$ and $P_2(4, -3, -4)$

15. about $17°$

Mid-Chapter Review, pp. 451–452

1. **a.** $(-7, -2), (-5, 1), (-3, 4)$
b. $(-1, 5), (2, 3), (5, 1)$
c. $\left(-1, \frac{11}{5}\right), \left(0, \frac{8}{5}\right), (1, 1)$
d. $(-2, -4, 4), (4, 0, 6), (1, -2, 5)$

2. **a.** $\left(\frac{18}{5}, 0\right)$; $(0, 6)$
b. $\left(-\frac{14}{3}, 0\right)$; $(0, -3)$

3. approximately $86.8°$

4. x-axis: about $51°$; y-axis: about $39°$

5. $5x - 7y - 41 = 0$

6. $\frac{x}{3} = \frac{y}{-4} = \frac{z-2}{4}$

7. $x = 1 + t, y = 2 - 9t, z = 5 + t, t \in \mathbf{R}$

8. approximately $79.3°$, $137.7°$, and $49.7°$

9. $y = -4, \frac{x-3}{1} = \frac{z-6}{\sqrt{3}}$

10. x-axis: $x = t, y = 0, z = 0, t \in \mathbf{R}$;
y-axis: $x = 0, y = t, z = 0, t \in \mathbf{R}$;
z-axis: $x = 0, y = 0, z = t, t \in \mathbf{R}$

11. **a.** -7
b. $\frac{1}{19}$

12. 17.2 units, 12

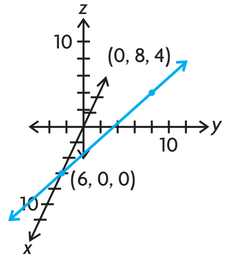
13. **a.** $\vec{r} = (0, 6) + t(4, -3), t \in \mathbf{R}$
b. $x = 4t, y = 6 - 3t, t \in \mathbf{R}$
c. about $36.9°$
d. $\vec{r} = t(3, 4), t \in \mathbf{R}$

14. $x + 6y - 32 = 0$;
$\vec{r} = (-4, 6) + t(12, -2), t \in \mathbf{R}$;
$x = -4 + 12t, y = 6 - 2t, t \in \mathbf{R}$

15. $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

16. **a.** $x = -5 + 3t, y = 10 - 2t, t \in \mathbf{R}$
b. $x = 1 + t, y = -1 + t, t \in \mathbf{R}$
c. $x = 0, y = t, t \in \mathbf{R}$

17. **a.** yz-plane at $(0, 8, 4)$; xz-plane at $(6, 0, 0)$; xy-plane at $(6, 0, 0)$
b. x-axis at $(6, 0, 0)$
c.



18. **a.** $\vec{r} = (1, -2, 8) + t(-5, -2, 1), t \in \mathbf{R}$; $x = 1 - 5t, y = -2 - 2t, z = 8 + t, t \in \mathbf{R}$;
$\frac{x-1}{-5} = \frac{y+2}{-2} = \frac{z-8}{1}$
b. $\vec{r} = (3, 6, 9) + t(2, 4, 6), t \in \mathbf{R}$;
$x = 3 + 2t, y = 6 + 4t, z = 9 + 6t, t \in \mathbf{R}$;
$\frac{x-3}{2} = \frac{y-6}{4} = \frac{z-9}{6}$
c. $\vec{r} = (0, 0, 6) + t(-1, 5, 1), t \in \mathbf{R}$;
$x = -t, y = 5t, z = 6 + t, t \in \mathbf{R}$;
$\frac{x}{-1} = \frac{y}{5} = \frac{z-6}{1}$
d. $\vec{r} = (2, 0, 0) + t(0, 0, -2), t \in \mathbf{R}$;
$x = 2, y = 0, z = -2t, t \in \mathbf{R}$; There is no symmetric equation for this line.

19. $\vec{r} = t(5, -5, -1), t \in \mathbf{R}$

20. $x = t, y = -8 - 13t, z = 1, t \in \mathbf{R}$

21. $(1, 3, -5), -3(1, 3, -5)$.

22. Since $\frac{7-4}{3} = \frac{-1+2}{1} = \frac{8-6}{2} = 1$, the point $(7, -1, 8)$ lies on the line.

Section 8.4 pp. 459–460

1. **a.** plane; **b.** line; **c.** line; **d.** plane;

2. **a.** $(4, -24, 9)$
b. $(1, -2, 5)$
c. $\vec{r} = (2, 1, 3) + s(4, -24, 9) + t(1, -2, 5), t, s \in \mathbf{R}$

3. **a.** $(0, 0, -1)$
b. $(2, -3, -3)$ and $(0, 5, -2)$
c. $(-2, -17, 10)$
d. $m = 0$ and $n = 3$
e. For the point $B(0, 15, -8)$, the first two parametric equations are the same, yielding $m = 0$ and $n = 3$; however, the third equation would then give:
$-8 = -1 - 3m - 2n$
$-8 = -1 - 3(0) - 2(3)$
$-8 = -7$
which is not true. So, there can be no solution.

4. a. $\vec{r} = (-2, 3, 1) + t(0, 0, 1) + s(3, -3, 0), t, s \in \mathbf{R}$
   b. $\vec{r} = (-2, 3, -2) + t(0, 0, 1) + s(3, -3, -1), t, s \in \mathbf{R}$
5. a. $\vec{r} = (1, 0, -1) + s(2, 3, -4) + t(4, 6, -8), t, s \in \mathbf{R}$, does not represent a plane because the direction vectors are the same. We can rewrite the second direction vector as $(2)(2, 3, -4)$. And so we can rewrite the equation as:
   $\vec{r} = (1, 0, -1) + s(2, 3, -4) + 2t(2, 3, -4)$
   $= (1, 0, -1) + (s + 2t)(2, 3, -4)$
   $= (1, 0, -1) + n(2, 3, 4), n \in \mathbf{R}$
   This is an equation of a line in $\mathbf{R}^3$.
6. a. $\vec{r} = (-1, 2, 7) + t(4, 1, 0) + s(3, 4, -1), t, s \in \mathbf{R}$;
   $x = -1 + 4t + 3s$,
   $y = 2 + t + 4s$,
   $z = 7 - s, t, s \in \mathbf{R}$
   b. $\vec{r} = (1, 0, 0) + t(-1, 1, 0) + s(-1, 0, 1), t, s \in \mathbf{R}$;
   $x = 1 - t - s$,
   $y = t$,
   $z = s, t, s \in \mathbf{R}$
   c. $\vec{r} = (1, 1, 0) + t(3, 4, -6) + s(7, 1, 2), t, s \in \mathbf{R}$;
   $x = 1 + 3t + 7s$,
   $y = 1 + 4t + s$,
   $z = -6t + 2s, t, s \in \mathbf{R}$
7. a. $s = 1$ and $t = 1$
   b. $(0, 5, -4) = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2)$ gives the following parametric equations:
   $0 = 2 + 4s + t \Rightarrow t = 2 + 4s$
   $5 = 2s + t$
   $5 = 2s + (2 + 4s)$
   $3 = 6s$
   $\frac{1}{2} = s$
   $t = 2 + 4\left(\frac{1}{4}\right)$
   $t = 2 + 2 = 4$
   The third equation then says:
   $-4 = 1 - s + 2t$
   $-4 = 1 - \frac{1}{2} + 2(4)$
   $-4 = \frac{17}{2}$, which is a false statement. So, the point $A(0, 5, -4)$ is not on the plane.
8. a. $\vec{l} = (-3, 5, 6) + s(-1, 1, 2), s \in \mathbf{R}$;
   $\vec{p} = (-3, 5, 6) + t(2, 1, -3), t \in \mathbf{R}$
   b. $(-3, 5, 6)$
9. $(0, 0, 5)$
10. $\vec{r} = (2, 1, 3) + s(4, 1, 5) + t(3, -1, 2), t, s \in \mathbf{R}$
11. $\vec{r} = m(2, -1, 7) + n(-2, 2, 3)$, $m, n \in \mathbf{R}$
12. a. $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$, $(-1, 1, 0)$
   b. $\vec{r} = s(1, 0, 0) + t(0, 1, 0), t, s \in \mathbf{R}$;
   $x = s$,
   $y = t$,
   $z = 0, t, s \in \mathbf{R}$
13. a. $\vec{r} = s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$
   b. $\vec{r} = (-2, 2, 3) + s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$
   c. The planes are parallel since they have the same direction vectors.
14. $(-4, 7, 1) - (-3, 2, 4) = (-1, 5, -3)$,
   $\frac{27}{13}(-3, 2, 4) - \frac{17}{13}(-4, 7, 1)$
   $= (-1, -5, 7)$
15. $\vec{r} = (0, 3, 0) + t(0, 3, 2), t \in \mathbf{R}$
16. The fact that the plane $\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of the given lines is easily seen when letting $s = 0$ and $t = 0$, respectively.

Section 8.5 pp. 468–469

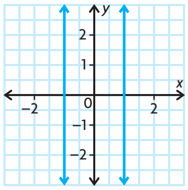
1. a. $\vec{n} = (A, B, C) = (1, -7, -18)$
   b. In the Cartesian equation:
   $Ax + By + Cz + D = 0$
   If $D = 0$, the plane passes through the origin.
   c. $(0, 0, 0)$, $(11, -1, 1)$, $(-11, 1, -1)$
2. a. $\vec{n} = (A, B, C) = (2, -5, 0)$
   b. In the Cartesian equation: $D = 0$. So, the plane passes through the origin.
   c. $(0, 0, 0)$, $(5, 2, 0)$ $(5, 2, 1)$
3. a. $\vec{n} = (A, B, C) = (1, 0, 0)$
   b. In the Cartesian equation: $D = 0$. So, the plane passes through the origin.
   c. $(0, 0, 0)$, $(0, 1, 0)$ $(0, 0, 1)$
4. a. $x + 5y - 7z = 0$
   b. $-8x + 12y + 7z = 0$
5. *Method 1:* Let $A(x, y, z)$ be a point on the plane. Then, $\overrightarrow{PA} = (x + 3, y - 3, z - 5)$ is a vector on the plane.
   $\vec{n} \cdot \overrightarrow{PA} = 0$
   $(x + 3) + 7(y - 3) + 5(z - 5) = 0$
   $x + 7y + 5z - 43 = 0.$
   Method 2: $\vec{n} = (1, 7, 5)$ so the Cartesian equation is
   $x + 7y + 5z + D = 0$
   We know the point $(-3, 3, 5)$ is on the plane and must satisfy the equation, so
   $(-3) + 7(3) + 5(5) + D = 0$
   $43 + D = 0$
   $D = -43$
   This also gives the equation:
   $x + 7y + 5z - 43 = 0.$
6. a. $7x + 19y - 3z - 28 = 0$
   b. $7x + 19y - 3z - 28 = 0$
   c. There is only one simplified Cartesian equation that satisfies the given information, so the equations must be the same.
7. $7x + 17y - 13z - 24 = 0$
8. $20x + 9y + 7z - 47 = 0$
9. a. $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$
   b. $\left(\frac{4}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$
   c. $\left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$
10. $21x - 15y - z - 1 = 0$
11. $2x - 4y - z + 6 = 0$
12. a. First determine their normal vectors, $\vec{n_1}$ and $\vec{n_2}$. Then the angle between the two planes can be determined from the formula:
   $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$
   b. 30°
13. a. 53.3°
   b. $2x - 3y - z + 5 = 0$
14. a. 8
   b. $-\frac{5}{2}$
   c. No, the planes cannot ever be coincident. If they were, then they would also be parallel, so $k = 8$, and we would have the two equations:
   $4x + 8y - 2z + 1 = 0.$
   $2x + 4y - z + 4 = 0 \Rightarrow$
   $4x + 8y - 2z + 8 = 0$. Here all of the coefficients are equal except for the D-values, which means that they don't coincide.
15. $3x + 5y - z - 18 = 0$
16. $-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y + \sqrt{3}z = 0$
17. $8x - 2y - 16z - 5 = 0$

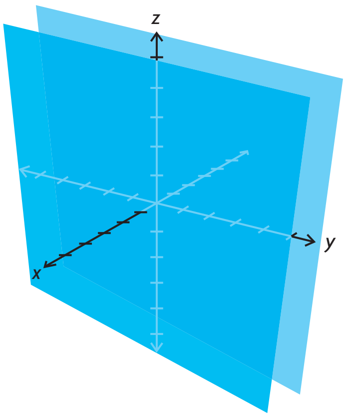
Section 8.6, pp. 476–477

1. a. A plane parallel to the yz-axis, but two units away, in the negative x direction.
   b. A plane parallel to the xz-axis, but three units away, in the positive y direction.
   c. A plane parallel to the xy-axis, but 4 units away, in the positive z direction.

2. $(-2, 3, 4)$

3. P must lie on plane π_1 since the point has an x-coordinate of 5, and doesn't have a y-coordinate of 6.

4. 



5. a. i. x-intercept $= 9$,
y-intercept $= 6$,
no z-intercept

ii. x-intercept $= 40$,
y-intercept $= -30$,
z-intercept $= 24$

iii. no x-intercept,
y-intercept $= 3$,
z-intercept $= -39$

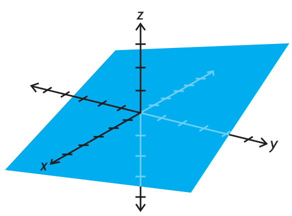
b. i. $(0, 0, 1)$, $(3, -2, 0)$

ii. $(4, 3, 0)$, $(5, 0, -3)$

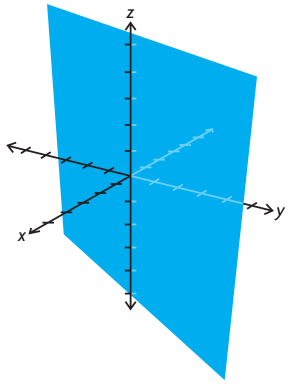
iii. $(1, 0, 0)$, $(0, 1, 13)$

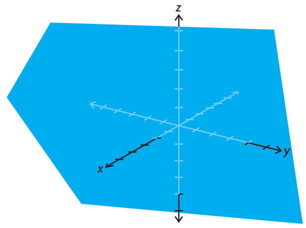
6. a. i. $(0, 0, 0)$, $(1, 2, 0)$, $(0, 5, 1)$

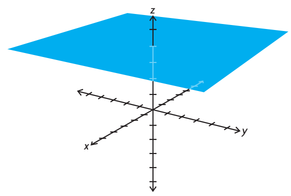
ii. $2x - y = 0$

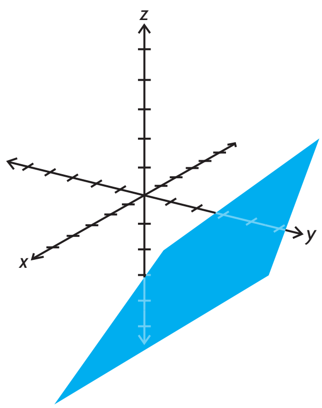
b. 

7. yz-plane, xz-plane, xy-plane

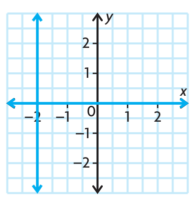
8. a. 

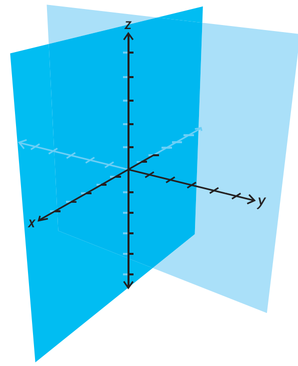
b. 

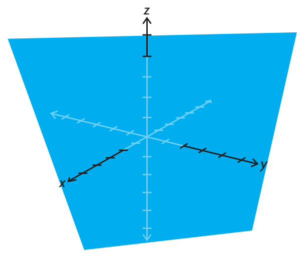
c. 

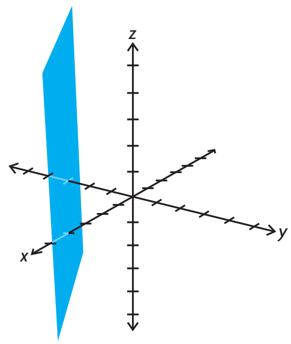
d. 

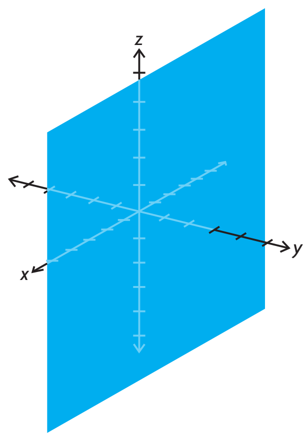
9. a. $y(x + 2) = 0$

b. 

c. 

10. a. 

b. 

c. 

11. a. $\frac{x}{3} + \frac{y}{4} + \frac{z}{6} = 1$

b. $\frac{x}{5} - \frac{z}{7} = 1$

c. $\frac{z}{8} = 1$

Review Exercise, pp. 480–483

1. Answers may vary. For example,
$x = 1 + s + t,$
$y = 2 - s,$
$z = -1 + 2s + 3t$

2. $3x + y - z - 6 = 0$
$\overrightarrow{AC} = (2, -1, 5) = \vec{c}$
$\overrightarrow{BC} = (1, 0, 3) = \vec{b}$
$\vec{r} = (1, 2, -1) + s(2, -1, 5) + t(1, 0, 3), s, t \in \mathbf{R}$
$\vec{b} \times \vec{c} = (1, 0, 3) \times (2, -1, 5) = (3, 1, -1)$
$Ax + By + Cz + D = 0$
$(3)x + (1)y + (-1)z + D = 0$
$3(1) + (2) - 1(-1) + D = 0$
$D = -6$
$3x + y - z - 6 = 0$
Both Cartesian equations are the same regardless of which vectors are used.

3. **a.** Answers may vary. For example,
$\vec{r} = (4, 3, 9) + t(7, 1, 1), t \in \mathbf{R};$
$x = 4 + 7t, y = 3 + t, z = 9 + t, t \in \mathbf{R};$
$\frac{x-4}{7} = \frac{y-3}{1} = \frac{z-9}{1}$
b. Answers may vary. For example,
$\vec{r} = (4, 3, 9) + t(7, 1, 1) + s(3, 2, 3), t, s \in \mathbf{R};$
$x = 4 + 7t + 3s, y = 3 + t + 2s, z = 9 + t + 3s, t, s \in \mathbf{R}$
c. There are two parameters.

4. $\vec{r} = (7, 1, -2) + t(2, -3, 1), t \in \mathbf{R};$
$x = 7 + 2t, y = 1 - 3t, z = -2 + t;$
$\frac{x-7}{2} = \frac{y-1}{-3} = \frac{z+2}{1}$

5. **a.** $x - 3y - 3z - 3 = 0$
b. $3x + 5y - 2z - 7 = 0$
c. $3y + z - 7 = 0$

6. $19x - 7y - 8z = 0$

7. $\vec{r} = (-1, 2, 1) + t(0, 1, 0) + s(0, 0, 1)\ t, s \in \mathbf{R};$
$x = -1, y = 2 + t, z = 1 + s$

8. $3x + y - z - 7 = 0$

9. $34x + 32y - 7z - 229 = 0$

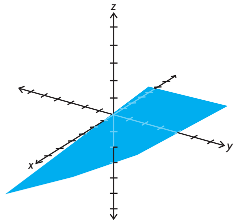
10. Answers may vary. For example,
$\vec{r} = (2, 3, -3) + s(3, -2, 1), s \in \mathbf{R};$
$x = 2 + 3s, y = 3 - 2s, z = -3 + s;$
$\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z+3}{1}$

11. Answers may vary. For example,
$\vec{r} = (0, 0, 6) + s(1, 0, 3) + t(3, -5, -1), s, t \in \mathbf{R};$
$x = s + 3t, y = -5t, z = 6 + 3s - t$

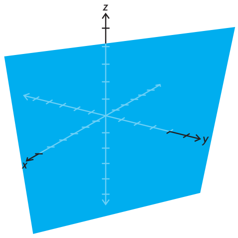
12. Answers may vary. For example,
$\vec{r} = (0, 0, 7) + t(1, 0, 2), t \in \mathbf{R};$
$x = t, y = 0, z = 7 + 2t;$

13. $\vec{r} = (3, -4, 1) + s(1, -3, -5) + t(4, 3, -1), s, t \in \mathbf{R};$
$x = 3 + s + 4t,$
$y = -4 - 3s + 3t$
$z = 1 - 5s - t, s, t \in \mathbf{R};$
$18x - 19y + 15z - 145 = 0$

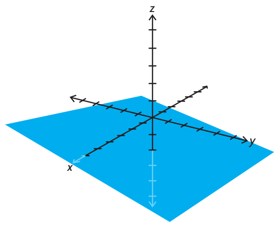
14. **a.**



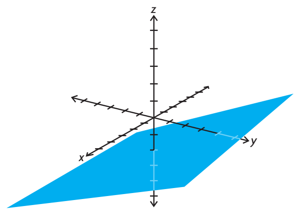
b.



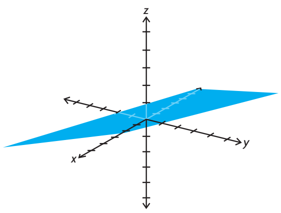
c.



d.



e.



15. **a.** Answers may vary. For example,
$\vec{r} = (3, 1, 2) + t(2, 4, 1) + s(2, 3, -3), t, s \in \mathbf{R};$
$x = 3 + 2t + 2s,$
$y = 1 + 4t + 3s, z = 2 + t - 3s;$
$15x - 8y + 2z - 41 = 0$
b. Answers may vary. For example,
$\overrightarrow{BC} = (-4, 0, 11)$
$D = -18$
$-4x + 11z - 18 = 0$
c. Answers may vary. For example,
$\vec{r} = (4, 1, -1) + t(1, -3, 5) + s(0, 0, 1), t, s \in \mathbf{R};$
$x = 4 + t, y = 1 - 3t,$
$z = -1 + 5t + s;$
$3x + y - 13 = 0$
d. Answers may vary. For example,
$\vec{r} = (1, 3, -5) + t(1, 3, 9) + s(1, -9, -1), t, s \in \mathbf{R};$
$x = 1 + t + s, y = 3 + 3t - 9s,$
$z = -5 + 9t - s;$
$78x + 10y - 12z - 168 = 0$

16. They are in the same plane because both planes have the same normal vectors and Cartesian equations.
$L_1: \vec{r} = (1, 2, 3) + s(-3, 5, 21) + t(0, 1, 3), s, t \in \mathbf{R}$
$L_2: \vec{r} = (1, -1, -6) + u(1, 1, 1) + v(2, 5, 11), u, v \in \mathbf{R}$
$(-3, 5, 21) \times (0, 1, 3) = (-6, 9, -3) = (2, -3, 1)$
$(1, 1, 1) \times (2, 5, 11) = (6, -9, 3) = (2, -3, 1)$
$Ax + By + Cz + D = 0$
$2x - 3y + z + D = 0$
$2(1) - 3(2) + (3) + D = 0$
$D = 1$
$2(1) - 3(-1) + (-6) + D = 0$
$D = 1$
$2x - 3y + z + 1 = 0$

17. $\left(\frac{20}{3}, \frac{10}{3}, -\frac{1}{3}\right)$

18. **a.** The plane is parallel to the z-axis through the points $(3, 0, 0)$ and $(0, -2, 0)$.
b. The plane is parallel to the y-axis through the points $(6, 0, 0)$ and $(0, 0, -2)$.
c. The plane is parallel to the x-axis through the points $(0, 3, 0)$ and $(0, 0, -6)$.

19. **a.** A
b. $a = -8, b = -1$

20. **a.** 45.0° **c.** 37.4°
b. 59.0° **d.** 90°

21. **a.** 44.2°
b. 90°

22. **a.** **i.** no **ii.** yes **iii.** no
b. **i.** yes **ii.** no **iii.** no

23. $(x, y, z) = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$
$(x, y, z) = (4, 1, 6) + 4(3, -2, 1) + 2(-6, 6, -1)$
$(x, y, z) = (4, 5, 8) \neq (4, 5, 6)$

24. $x = 1 + s + 3t, y = 4 - t,$
$z = 4 - 3s + t, s, t \in \mathbf{R}$

25. A plane has two parameters, because a plane goes in two different directions, unlike a line that goes only in one direction.

26. This equation will always pass through the origin, because you can always set $s = 0$ and $t = -1$ to obtain $(0, 0, 0)$.

27. **a.** They do not form a plane, because these three points are collinear.
$\vec{r} = (-1, 2, 1) + t(3, 1, -2)$
b. They do not form a plane, because the point lies on the line.
$\vec{r} = (4, 9, -3) + t(1, -4, 2)$
$\vec{r} = (4, 9, -3) + 4(1, -4, 2)$
$= (8, -7, 5)$

28. $bcx + acy + abz - abc = 0$

29. $6x - 5y + 12z + 46 = 0$

30. **a., b.** $\vec{r} = (1, -3, 2) + t(-3, 7, -4) + s(5, -2, 3)\ t, s \in \mathbf{R}$;
$x = 1 - 3t + 5s$,
$y = -3 + 7t - 2s$,
$z = 2 - 4t + 3s$
c. $13x - 11y - 29z + 12 = 0$
d. no

31. **a.** $4x - 2y + 5z = 0$
b. $4x - 2y + 5z + 19 = 0$
c. $4x - 2y + 5z - 22 = 0$

32. **a.** These lines are coincident. The angle between them is 0°.
b. $\left(\frac{3}{2}, 5\right)$, 86.82°

33. **a.** $\vec{r} = (1, 3, 5) + t(-2, -4, -10)$, $t \in \mathbf{R}$;
$x = 1 - 2t, y = 3 - 4t$,
$z = 5 - 10t$;
$\frac{x - 1}{-2} = \frac{y - 3}{-4} = \frac{z - 5}{-10}$
b. $\vec{r} = (1, 3, 5) + t(-8, 6, -2), t \in \mathbf{R}$;
$x = 1 - 8t, y = 3 + 6t$,
$z = 5 - 2t$;
$\frac{x - 1}{-8} = \frac{x - 3}{6} = \frac{x - 5}{-2}$
c. $\vec{r} = (1, 3, 5) + t(-6, -13, 14)$, $t \in \mathbf{R}$;
$x = 1 - 6t, y = 3 - 13t$,
$z = 5 + 14t$;
$\frac{x - 1}{-6} = \frac{x - 3}{-13} = \frac{x - 5}{14}$
d. $\vec{r} = (1, 3, 5) + t(1, 0, 0), t \in \mathbf{R}$;
$x = 1 + t, y = 3, z = 5$
e. $a = 0, b = 6, c = 4$;
$\vec{r} = (1, 3, 5) + t(0, 6, 4), t \in \mathbf{R}$
f. $\vec{r} = (1, 3, 5) + t(0, 1, 6)$;
$x = 1, y = 3 + t, z = 5 + 6t$

34. **a.** $2x - 4y + 5z + 23 = 0$
b. $29x + 27y + 24z - 86 = 0$
c. $z - 3 = 0$
d. $3x + y - 4z + 26 = 0$
e. $y - 2z - 4 = 0$
f. $-5x + y + 7z + 18 = 0$

Chapter 8 Test, p. 484

1. **a.** **i.** $\vec{r} = (1, 2, 4) + s(1, -2, -1) + t(3, 2, 0), s, t \in \mathbf{R}$;
$x = 1 + s + 3t$,
$y = 2 - 2s + 2t, z = 4 - s$,
$s, t \in \mathbf{R}$
ii. $2x - 3y + 8z - 28 = 0$
b. no

2. **a.** $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$
b. $(6, 4, 3)$

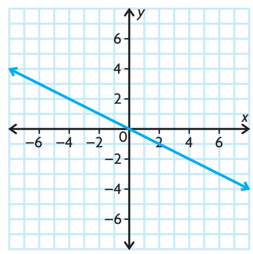
3. **a.** $\vec{r} = s(2, 1, 3) + t(1, 2, 5), s, t \in \mathbf{R}$
b. $-x - 7y + 3z = 0$

4. **a.** $\vec{r} = (4, -3, 5) + s(2, 0, -3) + t(5, 1, -1), s, t \in \mathbf{R}$
b. $3x - 13y + 2z - 61 = 0$

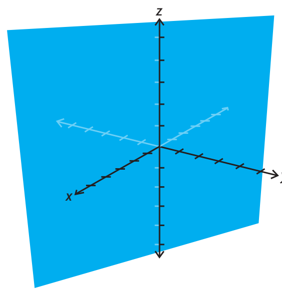
5. **a.** $\left(0, 5, -\frac{1}{2}\right)$
b. $\frac{x}{4} = \frac{y - 5}{-2} = \frac{1}{2}$

6. **a.** about 70.5°
b. **i.** 4
ii. $-\frac{1}{5}$
c. The y-intercepts are different and the planes are parallel.

7. **a.**



b.



c. The equation for the plane can be written as $Ax + By + 0z = 0$. For any real number t,
$A(0) + B(0) + 0(t) = 0$, so $(0, 0, t)$ is on the plane. Since this is true for all real numbers, the z-axis is on the plane.

Chapter 9

Review of Prerequisite Skills, p. 487

1. **a.** yes **b.** no **c.** yes **d.** no

2. Answers may vary. For example:
a. $\vec{r} = (2, 5) + t(5, -2), t \in \mathbf{R}$;
$x = 2 + 5t, y = 5 - 2t, t \in \mathbf{R}$
b. $\vec{r} = (-3, 7) + t(7, -14), t \in \mathbf{R}$;
$x = -3 + 7t, y = 7 - 14t, t \in \mathbf{R}$
c. $\vec{r} = (-1, 0) + t(-2, -11), t \in \mathbf{R}$;
$x = -1 + -2t, y = -11t, t \in \mathbf{R}$
d. $\vec{r} = (1, 3, 5) + t(5, -10, -5), t \in \mathbf{R}$;
$x = 1 + 5t, y = 3 - 10t, z = 5 - 5t$, $t \in \mathbf{R}$
e. $\vec{r} = (2, 0, -1) + t(-3, 5, 3), t \in \mathbf{R}$;
$x = 2 - 3t, y = 5t, z = -1 + 3t$, $t \in \mathbf{R}$
f. $\vec{r} = (2, 5, -1) + t(10, -10, -6)$, $t \in \mathbf{R}$;
$x = 2 + 10t, y = 5 - 10t, z = -1 - 6t, t \in \mathbf{R}$

3. **a.** $2x + 6y - z - 17 = 0$
b. $y = 0$
c. $4x - 3y - 15 = 0$
d. $6x - 5y + 3z = 0$
e. $11x - 6y - 38 = 0$
f. $x + y - z - 6 = 0$

4. $5x + 11y + 2z - 21 = 0$

5. L_1 is not parallel to the plane. L_1 is on the plane.
L_2 is parallel to the plane.
L_3 is not parallel to the plane.

6. **a.** $x - y - z - 2 = 0$
b. $x + 6y - 10z - 30 = 0$

7. $\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0), s, t \in \mathbf{R}$

8. $3y + z = 13$

Section 9.1, pp. 496–498

1. **a.** $\pi: x - 2y - 3z = 6$,
$\vec{r} = (1, 2, -3) + s(5, 1, 1)\ s \in \mathbf{R}$
b. This line lies on the plane.

2. **a.** A line and a plane can intersect in three ways: (1) The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
(2) The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
(3) The line and the plane have an infinite number of intersections. This occurs when the line is

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Chapter 8 Test, p. 484

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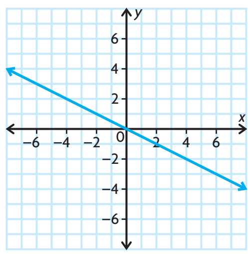

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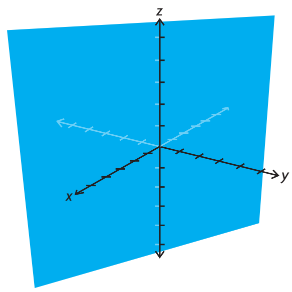

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Chapter 9

Review of Prerequisite Skills, p. 487

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