Chapter 7

APPLICATIONS OF VECTORS

In Chapter 6, we discussed some of the basic ideas about vectors. In this chapter, we will use vectors in both mathematical and physical situations to calculate quantities that would otherwise be difficult to determine. You will discover how vectors enable calculations in situations involving the velocity at which a plane flies under windy conditions and the force at which two other people must pull to balance the force created by two others in a game of tug-o-war. In addition, we will introduce the concept of vector multiplication and show how vectors can be applied in a variety of contexts.

CHAPTER EXPECTATIONS

In this chapter, you will

- use vectors to model and solve problems arising from real-world applications involving velocity and force, **Sections 7.1, 7.2**
- perform the operation of the dot product on two vectors, **Sections 7.3, 7.4**
- determine properties of the dot product, **Sections 7.3, 7.4**
- determine the scalar and vector projections of a vector, **Section 7.5**
- perform the operation of cross product on two algebraic vectors in three-dimensional space, **Section 7.6**
- determine properties of the cross product, **Section 7.6**
- solve problems involving the dot product and cross product, **Section 7.7**

Review of Prerequisite Skills

In this chapter, you will use vectors in applications involving elementary force and velocity problems. As well, you will be introduced to the study of scalar and vector products. You will find it helpful to be able to

- find the magnitude and the direction of vectors using trigonometry
- plot points and find coordinates of points in two- and three-dimensional systems

Exercise

1. The velocity of an airplane is 800 km/h north. A wind is blowing due east at 100 km/h. Determine the velocity of the airplane relative to the ground.

2. A particle is displaced 5 units to the west and then displaced 12 units in a direction $N45°W$. Find the magnitude and direction of the resultant displacement.

3. Draw the x-axis, y-axis, and z-axis, and plot the following points:

a. $A(0, 1, 0)$
b. $B(-3, 2, 0)$
c. $C(-2, 0, 1)$
d. $D(0, 2, -3)$

4. Express each of the following vectors in component form (a, b, c). Then determine its magnitude.

a. $3\vec{i} - 2\vec{j} + 7\vec{k}$
b. $-9\vec{i} + 3\vec{j} + 14\vec{k}$
c. $\vec{i} + \vec{j}$
d. $2\vec{i} - 9\vec{k}$

5. Describe where the following general points are located.

a. $A(x, y, 0)$
b. $B(x, 0, z)$
c. $C(0, y, z)$

6. Find a single vector that is equivalent to each linear combination.

a. $(-6, 0) + 7(1, -1)$
b. $(4, -1, 3) - (-2, 1, 3)$
c. $2(-1, 1, 3) + 3(-2, 3, -1)$
d. $-\frac{1}{2}(4, -6, 8) + \frac{3}{2}(4, -6, 8)$

7. If $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j}$, determine a single vector that is equivalent to each linear combination.

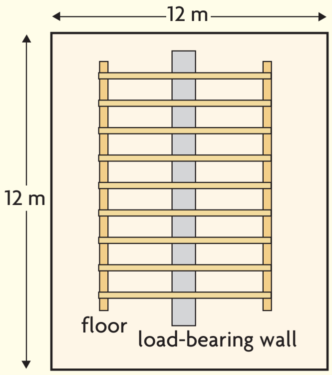
a. $\vec{a} + \vec{b}$
b. $\vec{a} - \vec{b}$
c. $2\vec{a} - 3\vec{b}$

CAREER LINK Investigate

CHAPTER 7: FORCES IN ARCHITECTURE: STRUCTURAL ENGINEERING

Structural engineers are a specific kind of architect: they help in the design of large-scale structures, such as bridges and skyscrapers. The role of structural engineers is to make sure that the structure being built will be stable and not collapse. To do this, they need to calculate all the forces acting on the structure, including the weight of building materials, occupants, and any furniture or items that might be stored in the building. They also need to account for the forces of wind, water, and seismic activity, including hurricanes and earthquakes. To design a safe structure effectively, the composition of forces must be calculated to find the resultant force. The strength and structure of the materials must exert a force greater than the equilibrant force. A simple example of this is a load-bearing wall inside a house. A structural engineer must calculate the total weight of the floor above, which is considered a dead load. Then the engineer has to factor in the probable weight of the occupants and their furniture, which is considered a live load. The load-bearing wall must be built to exert an opposing force that is greater than the force created by the live load.

Type of Load	Load (kg/m^2)
live	90
dead	150



Case Study—Replacing a Load-Bearing Wall with a Steel Support Beam

The table at the left shows the normal loads created by a timber floor and a non-load-bearing wall above. A homeowner wants to make one large room out of two. This will require removing a wall that is bearing the load of the floor above and replacing the wall with a steel support beam. The horizontal and vertical yellow segments represent the framing for the area of the upper floor that is currently being supported by the wall, without help from the walls at the edges of the rooms. Complete the discussion questions to determine what size of beam will be required to bear the load.

DISCUSSION QUESTIONS

1. Find the area of the floor that is currently being supported by the load-bearing wall. Use the information in the table to calculate the live and dead loads for this area.
2. Find the resultant downward force created by the weight of the floor above including an estimate for the expected weight of four occupants and their furniture.
3. Determine the equilibrant force required by a steel support beam that would support the force you calculated in question 2. Explain why, for safety reasons, a beam that supports a greater force is used.

Section 7.1—Vectors as Forces

The concept of **force** is something that everyone is familiar with. When we think of force, we usually think of it associated with effort or muscular exertion. This is experienced when an object is moved from one place to another. Examples of activities that involve forces are pulling a toboggan, lifting a book, shooting a basketball, or pedalling a bicycle. Each of these activities involves the use of muscular action that exerts a force. There are, however, many other examples of force in which muscular action is not present. For example, the attraction of the Moon to Earth, the attraction of a piece of metal to a magnet, the thrust exerted by an engine when gasoline combusts in its cylinders, or the force exerted by shock absorbers in cars to reduce vibration.

Force as a Vector Quantity

Force can be considered something that either pushes or pulls an object. When a large enough force is applied to an object at rest, the object tends to move. When a push or pull is applied to a body that is already in motion, the motion of the body tends to change. Generally speaking, force can be defined as that which changes, or tends to change, the state of rest, or uniform motion of a body.

When describing certain physical quantities, there is little value in describing them with magnitude alone. For example, if we are describing the velocity of wind, it is not very practical to say that the wind has a speed of 30 km/h without specifying the direction of the wind. It makes more sense to say that a wind has a speed of 30 km/h travelling south. Similarly, the description of a force without specifying its magnitude and direction has little practical value. Because force is described by both magnitude and direction, it is a vector. The rules that apply to vectors also apply to forces.

Before we consider situations involving the calculation of force, it is necessary to describe the unit in which force is measured. On Earth, force is defined as the product between the mass of an object and the acceleration due to gravity ($9.8\ \text{m/s}^2$). So a 1 kg mass exerts a downward force of $1\ \text{kg} \times 9.8\ \text{m/s}^2$ or $9.8\ \text{kg} \cdot \text{m/s}^2$. This unit of measure is called a newton and is abbreviated as N. Because of Earth's gravitational field, which acts downward, we say that a 1 kg mass exerts a force of 9.8 N. Thus, the force exerted by a 2 kg mass at Earth's surface is about 19.6 N. A person having a mass of 60 kg would exert approximately 60×9.8, or 588 N, on the surface of Earth. So weight, expressed in newtons, is a force acting with a downward direction.

In problems involving forces, it is often the case that two or more forces act simultaneously on an object. To better understand the effect of these forces, it is useful to be able to find the single force that would produce exactly the same effect as all the forces acting together produced. This single force is called the **resultant**, or **sum**, of all the forces working on the object. It is important that we be able to determine the direction and magnitude of this single force. When we find the resultant of several forces, this resultant may be substituted for the individual forces, and the separate forces need not be considered further. The process of finding the resultant of all the forces acting on an object is called the **composition of forces**.

Resultant and Composition of Forces

The resultant of several forces is the single force that can be used to represent the combined effect of all the forces. The individual forces that make up the resultant are referred to as the components of the resultant.

If several forces are acting on an object, it is often advantageous to find a single force which, when applied to the object, would prevent any further motion that these original forces tended to produce. This single force is called the **equilibrant** because it would keep the object in a state of equilibrium.

Equilibrant of Several Forces

The equilibrant of a number of forces is the single force that opposes the resultant of the forces acting on an object. When the equilibrant is applied to the object, this force maintains the object in a state of equilibrium.

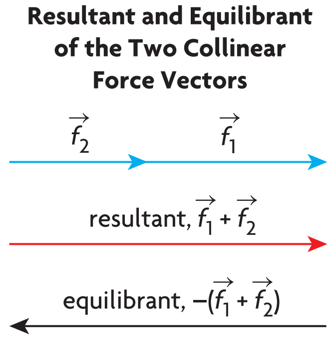
In the first example, we will consider collinear forces and demonstrate how to calculate their resultant and equilibrant. Collinear forces are those forces that act along the same straight line (in the same or opposite direction).

EXAMPLE 1 Representing force using vectors

Two children, James and Fred, are pushing on a rock. James pushes with a force of 80 N in an easterly direction, and Fred pushes with a force of 60 N in the same direction. Determine the resultant and equilibrant of these two forces.

Solution

To visualize the first force, we represent it with a horizontal line segment measuring 8 cm, pointing east. We represent the 60 N force with a line segment of 6 cm, also pointing east. The vectors used to represent forces are proportionate in length to the magnitude of the forces they represent.

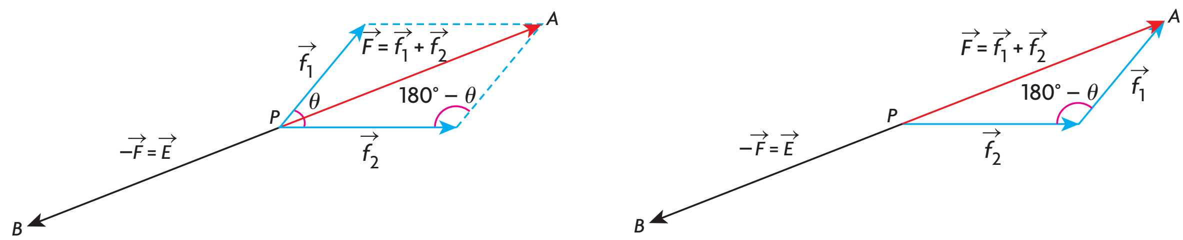


The resultant of these forces, $\vec{f_1} + \vec{f_2}$, is the single vector pointing east with a magnitude of 140 N. The combined effort of James and Fred working together exerts a force on the rock of magnitude 140 N in an easterly direction. The equilibrant of these forces is the vector, $-(\vec{f_1} + \vec{f_2})$, which has a magnitude of 140 N pointing in the opposite direction, west. In general, the resultant and equilibrant are two vectors having the same magnitude but pointing in opposite directions.

It is not typical that forces acting on an object are collinear. In the following diagram, the two noncollinear forces, $\vec{f_1}$ and $\vec{f_2}$, are applied at the point P and could be thought of as two forces applied to an object in an effort to move it.

The natural question is, how do we determine the resultant of these two forces? Since forces are vectors, it follows from our work in the previous chapter that the resultant of two noncollinear forces is represented by either the diagonal of the parallelogram determined by these two vectors when placed tail to tail or the third side of the triangle formed when the vectors are placed head to tail. In the following diagrams, vector $\overrightarrow{PA} = \vec{F}$ is the resultant of $\vec{f_1}$ and $\vec{f_2}$, while the vector $\overrightarrow{PB} = \vec{E}$ is the equilibrant of $\vec{f_1}$ and $\vec{f_2}$.

Resultant and Equilibrant of the Force Vectors $\vec{f_1}$ and $\vec{f_2}$



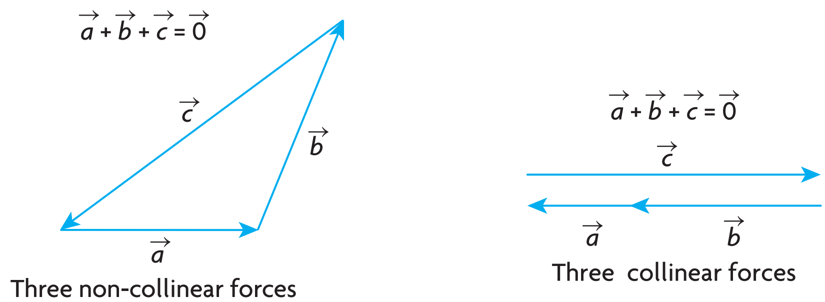
The resultant vector $\vec{F}$ and the equilibrant vector $\vec{E}$ are examples of two vectors that are in a state of equilibrium. When both these forces are applied to an object at point P, the object does not move. Since these vectors have the same magnitude but opposite directions it follows that $\vec{F} + \vec{E} = \vec{F} + (-\vec{F}) = \vec{0}$.

Vectors in a State of Equilibrium

When three noncollinear vectors are in a state of equilibrium, these vectors will always lie in the same plane and form a linear combination. When the three vectors are arranged head to tail, the result is a triangle because the resultant of two of the forces is opposed by the third force. This means that if three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in equilibrium, such that $\vec{c}$ is the equilibrant of $\vec{a}$ and $\vec{b}$, then $-\vec{c} = \vec{a} + \vec{b}$ or $\vec{a} + \vec{b} + \vec{c} = (-\vec{c}) + (\vec{c}) = \vec{0}$.

It is important to note that it also is possible for three vectors to be in equilibrium when the three forces are collinear. As with noncollinear vectors, one of the three forces is balanced by the resultant of the two other forces. In this case, the three forces do not form a triangle in the traditional sense. Instead, the sides of the "triangle" lie along the same straight line.

Forces in Equilibrium



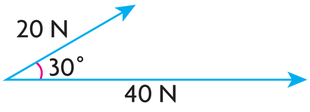
In the following example, the resultant of two noncollinear forces is calculated.

EXAMPLE 2 Connecting the resultant force to vector addition

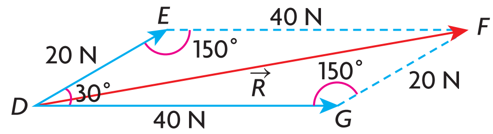
Two forces of 20 N and 40 N act at an angle of 30° to each other. Determine the resultant of these two forces.

Solution

We start the solution to this problem by drawing both a position diagram and a vector diagram. A position diagram indicates the actual position of the given vectors, and a vector diagram takes the information given in the position diagram and puts it in a form that allows for the determination of the resultant vector using either the triangle or parallelogram law. As before, the position diagram is drawn approximately to scale, and the side lengths of the parallelogram are labelled. The resultant of the two given vectors is $\overrightarrow{DF} = \vec{R}$, and the supplement of $\angle EDG$ is $\angle FED$, which measures 150°.



Position diagram



Vector diagram

In $\triangle DEF$, $|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\cos 150°$ (Cosine law)

$$|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\left(\frac{-\sqrt{3}}{2}\right)$$

$$|\vec{R}|^2 = 2000 + 800\sqrt{3}$$

$$|\vec{R}|^2 \doteq 3385.64$$

Therefore, $|\vec{R}| \doteq 58.19$ N. If we let $\vec{E}$ represent the equilibrant, then $|\vec{E}| \doteq 58.19$ N.

Since we are asked to calculate the resultant and equilibrant of the two forces, we must also calculate angles so that we can state each of their relative positions. To do this, we use the sine law.

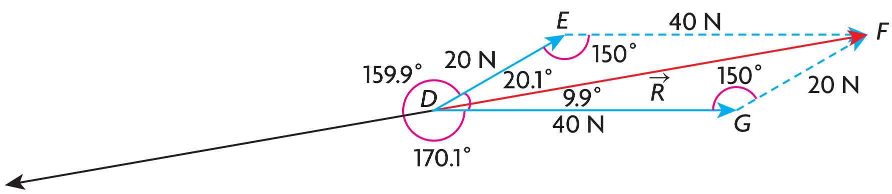
In $\triangle DEF$, $\dfrac{\sin \angle DEF}{|\vec{R}|} = \dfrac{\sin \angle EDF}{|\overrightarrow{EF}|}$ (Sine law)

$$\frac{\sin 150°}{58.19} \doteq \frac{\sin \angle EDF}{40}$$

$$\sin \angle EDF \doteq \frac{40(\sin 150°)}{58.19}$$

$$\sin \angle EDF \doteq 0.3437$$

Thus, $\angle EDF \doteq 20.1°$



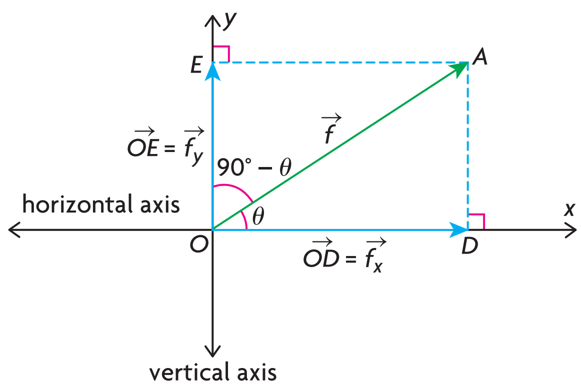
The resultant and equilibrant are forces, each having a magnitude of approximately 58.19 N. The resultant makes an angle of 20.1° with the 20 N force and 9.9° with the 40 N force. The equilibrant makes an angle of 159.9° with the 20 N force and an angle of 170.1° with the 40 N force.

We have shown that if we take any two forces that act at the same point, acting at an angle of θ to each other, the forces may be composed to obtain the resultant of these two forces. Furthermore, the resultant of any two forces is unique because there is only one parallelogram that can be formed with these two forces.

Resolving a Vector into Its Components

In many situations involving forces, we are interested in a process that is the opposite of composition. This process is called **resolution**, which means taking a single force and decomposing it into two components. When we resolve a force into two components, it is possible to do this in an infinite number of ways because there are infinitely many parallelograms having a particular single force as the diagonal. However, the most useful and important way to resolve a force vector occurs when this vector is resolved into two components that are at right angles to each other. These components are usually referred to as the horizontal and vertical components.

In the following diagram, we demonstrate how to resolve the force vector $\vec{f}$ into its horizontal and vertical components.



The vector resolved into components is the vector $\overrightarrow{OA}$, or vector $\vec{f}$. From A, the head of the vector, perpendicular lines are drawn to meet the x-axis and y-axis at points D and E, respectively. The vectors $\overrightarrow{OD}$ and $\overrightarrow{OE}$ are called the horizontal and vertical components of the vector $\overrightarrow{OA}$, where the angle between $\vec{f}$ and the x-axis is labelled θ.

To calculate $|\overrightarrow{OD}|$, we use the cosine ratio in the right triangle OAD.

$$\text{In } \triangle OAD, \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\overrightarrow{OD}|}{|\overrightarrow{OA}|}$$

$$\text{Therefore, } |\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$$

This means that the vector $\overrightarrow{OD}$, the horizontal component of $\overrightarrow{OA}$, has magnitude $|\overrightarrow{OA}|\cos\theta$.

The magnitude of the vertical component of $\overrightarrow{OA}$ is calculated in the same way using $\triangle OEA$.

$$\text{In } \triangle OEA, \cos(90° - \theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\overrightarrow{OE}|}{|\overrightarrow{OA}|}$$

$$\text{Therefore, } |\overrightarrow{OE}| = |\overrightarrow{OA}|\cos(90° - \theta)$$

$$\text{Since } \sin\theta = \cos(90° - \theta),$$

$$|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$$

What we have shown is that $|\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$ and $|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$. If we replace $\overrightarrow{OA}$ with $\vec{f}$, this would imply that $|\vec{f_x}| = |\vec{f}|\cos\theta$ and $|\vec{f_y}| = |\vec{f}|\sin\theta$, where $\vec{f_x}$ and $\vec{f_y}$ represent the horizontal and vertical components of $\vec{f}$, respectively.

Resolution of a Vector into Horizontal and Vertical Components

If the vector $\vec{f}$ is resolved into its respective horizontal and vertical components, $\vec{f_x}$ and $\vec{f_y}$, then $|\vec{f_x}| = |\vec{f}|\cos\theta$ and $|\vec{f_y}| = |\vec{f}|\sin\theta$, where θ is the angle that $\vec{f}$ makes with the x-axis.

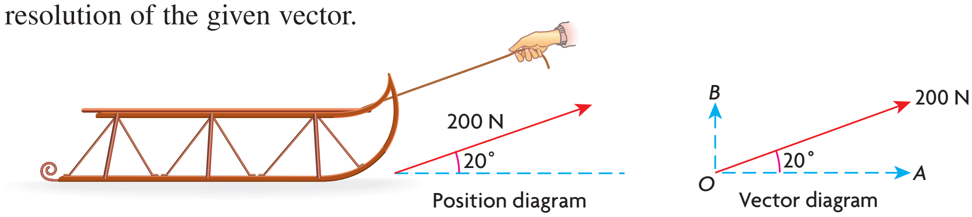
EXAMPLE 3 Connecting forces to the components of a given vector

Kayla pulls on a rope attached to her sleigh with a force of 200 N. If the rope makes an angle of 20° with the horizontal, determine:

a. the force that pulls the sleigh forward
b. the force that tends to lift the sleigh

Solution

In this problem, we are asked to resolve the force vector into its two rectangular components. We start by drawing a position diagram and, beside it, show the resolution of the given vector.



From the diagram, the vector $\overrightarrow{OA}$ is the horizontal component of the given force vector that pulls the sleigh forward. The vector $\overrightarrow{OB}$ is the vertical component of the given force vector that tends to lift the sleigh. To calculate their magnitudes, we directly apply the formulas developed.

$$|\overrightarrow{OA}| = 200(\cos 20°) \qquad \text{and} \qquad |\overrightarrow{OB}| = 200(\sin 20°)$$
$$\doteq 200(0.9397) \qquad\qquad \doteq 200(0.3420)$$
$$\doteq 187.94 \text{ N} \qquad\qquad \doteq 68.40 \text{ N}$$

The sleigh is pulled forward with a force of approximately 187.94 N, and the force that tends to lift it is approximately 68.40 N.

In the following example, we will use two different methods to solve the problem. In the first solution, a triangle of forces will be used. In the second solution, the concept of resolution of forces will be used.

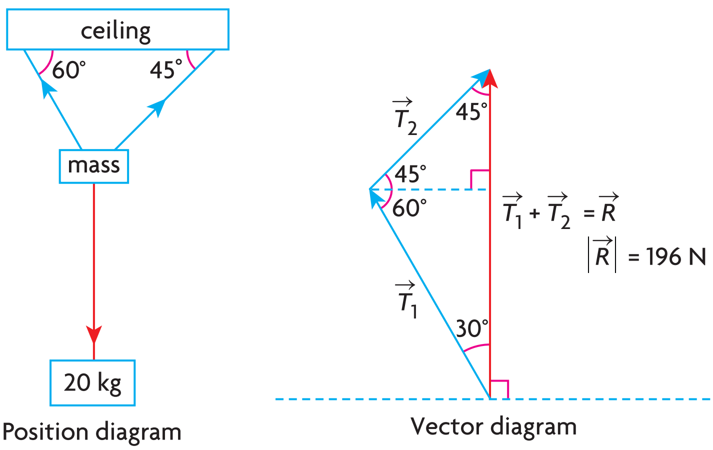
EXAMPLE 4 Selecting a strategy to solve a problem involving several forces

A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 60° and 45° with the ceiling. Determine the tension in each of the ropes.

Solution

Method 1 Triangle of forces

First, recall that the downward force exerted per kilogram is 9.8 N. So the 20 kg mass exerts a downward force of 196 N. Draw a position diagram and a vector diagram. Let the tension vectors for the two pieces of rope be $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$, and let their resultant be $\overrightarrow{R}$. The magnitude of the resultant force created by the tensions in the ropes must equal the magnitude of the downward force on the mass caused by gravity since the system is in a state of equilibrium.



To calculate the required tensions, it is necessary to use the sine law in the vector diagram.

Thus, $\dfrac{|\overrightarrow{T_1}|}{\sin 45°} = \dfrac{|\overrightarrow{T_2}|}{\sin 30°} = \dfrac{196}{\sin 105°}$

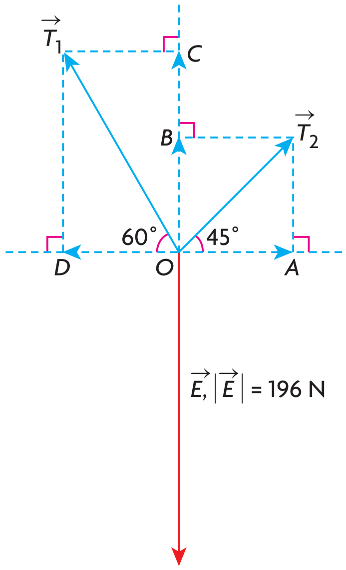
$|\vec{T_1}|\sin 105° = 196(\sin 45°)$ and $|\vec{T_2}|\sin 105° = 196(\sin 30°)$

$|\vec{T_1}| = \dfrac{196(0.7071)}{0.9659} \doteq 143.48$ N and $|\vec{T_2}| = \dfrac{196(0.5)}{0.9659} \doteq 101.46$ N

Therefore, the tensions in the two ropes are approximately 143.48 N and 101.46 N.

Method 2 Resolution of Forces

We start by drawing a diagram showing the tension vectors, $\vec{T_1}$ and $\vec{T_2}$, and the equilibrant, $\vec{E}$. The tension vectors are shown in their resolved form.



For the tension vectors, the magnitudes of their components are calculated.

Horizontal components:

$|\vec{OA}| = \cos 45°|\vec{T_2}| \doteq 0.7071|\vec{T_2}|$ and $|\vec{OB}| \doteq 0.7071|\vec{T_2}|$;

Vertical components:

$|\vec{OC}| = \sin 60°|\vec{T_1}| \doteq 0.8660|\vec{T_1}|$ and $|\vec{OD}| = 0.5|\vec{T_1}|$

For the system to be in equilibrium, the magnitudes of the horizontal and vertical components must balance each other.

Horizontal components: $|\vec{OA}| = |\vec{OD}|$ or $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

Vertical components: $|\vec{OB}| + |\vec{OC}| = |\vec{E}|$ or $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

This gives the following system of two equations in two unknowns.

① $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

② $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

In equation ①, $|\vec{T_1}| \doteq \dfrac{0.7071|\vec{T_2}|}{0.5}$ or $|\vec{T_1}| \doteq 1.4142|\vec{T_2}|$.

If we substitute this into equation ②, we obtain

$$0.7071|\vec{T_2}| + 0.8660[(1.4142)(|\vec{T_2}|)] \doteq 196$$

$$\begin{aligned} 1.9318|\vec{T_2}| &\doteq 196 \\ |\vec{T_2}| &\doteq \frac{196}{1.9318} \\ &\doteq 101.46 \text{ N} \\ \text{Since } |\vec{T_1}| &\doteq 1.4142|\vec{T_2}|, \\ |\vec{T_1}| &\doteq 1.4142(101.46) \\ &\doteq 143.48 \text{ N} \end{aligned}$$

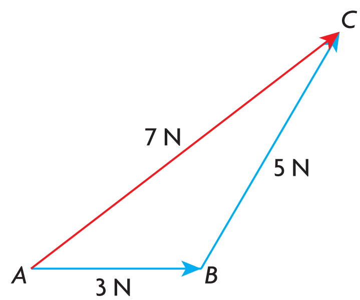
Therefore, the tensions in the ropes are 143.48 N and 101.46 N, as before.

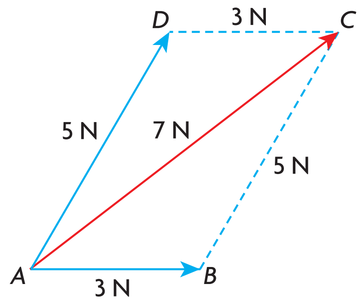
EXAMPLE 5 Reasoning about equilibrium in a system involving three forces

a. Is it possible for three forces of 15 N, 18 N, and 38 N to keep a system in a state of equilibrium?
b. Three forces having magnitudes 3 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces.

Solution

a. For a system to be in equilibrium, it is necessary that a triangle be formed having lengths proportional to 15, 18, and 38. Since $15 + 18 < 38$, a triangle cannot be formed because the triangle inequality states that for a triangle to be formed, the sum of any two sides must be greater than or equal to the third side. Therefore, three forces of 15 N, 18 N, and 38 N cannot keep a system in a state of equilibrium.
b. We start by drawing the triangle of forces and the related parallelogram.





Using $\triangle ABC$,

$$|\vec{AC}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2 - 2|\vec{AB}||\vec{BC}|\cos \angle CBA \qquad \text{(Cosine law)}$$

$$7^2 = 3^2 + 5^2 - 2(3)(5)\cos \angle CBA$$

$$49 = 34 - 30 \cos \angle CBA$$
$$\frac{-1}{2} = \cos \angle CBA$$
$$120° = \cos^{-1}(-0.5) \angle CBA$$

The angle that is required is $\angle DAB$, the supplement of $\angle CBA$. $\angle DAB = 60°$, and the angle between the 3 N and 5 N force is 60°.

IN SUMMARY

Key Ideas

- Problems involving forces can be solved using strategies involving vectors.
- When two or more forces are applied to an object, the net effect of the forces can be represented by the resultant vector determined by adding the vectors that represent each of the forces.
- A system is in a state of equilibrium when the net effect of all the forces acting on an object causes no movement of the object.

Need to Know

- $\vec{F} = \vec{F_1} + \vec{F_2}$ is the resultant of $\vec{F_1}$ and $\vec{F_2}$.
- $-\vec{F} = -(\vec{F_1} + \vec{F_2})$ is the equilibrant of $\vec{F_1}$ and $\vec{F_2}$.
- If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in a state of equilibrium.

Exercise 7.1

PART A

1. a. Name some common household items that have approximate weights of 10 N, 50 N, and 100 N.

   b. What is your weight in newtons?

2. Three forces of 10 N, 20 N, and 30 N are in a state of equilibrium.

   a. Draw a sketch of these three forces.

   b. What is the angle between the equilibrant and each of the smaller forces?

3. Two forces of 10 N and 20 N are acting on an object. How should these forces be arranged to produce the largest possible resultant?

4. Explain in your own words why three forces must lie in the same plane if they are acting on an object in equilibrium.

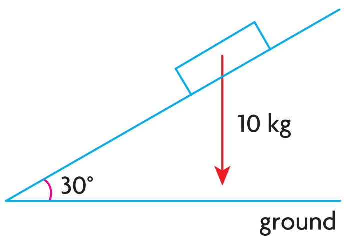
K

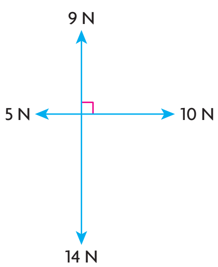
PART B

5. Determine the resultant and equilibrant of each pair of forces acting on an object.
   a. $\vec{f_1}$ has a magnitude of 5 N acting due east, and $\vec{f_2}$ has a magnitude of 12 N acting due north.
   b. $\vec{f_1}$ has a magnitude of 9 N acting due west, and $\vec{f_2}$ has a magnitude of 12 N acting due south.
6. Which of the following sets of forces acting on an object could produce equilibrium?
   a. 2 N, 3 N, 4 N
   b. 9 N, 40 N, 41 N
   c. $\sqrt{5}$ N, 6 N, 9 N
   d. 9 N, 10 N, 19 N
7. Using a vector diagram, explain why it is easier to do chin-ups when your hands are 30 cm apart instead of 90 cm apart. (Assume that the force exerted by your arms is the same in both cases.)
8. A force, $\vec{f_1}$, of magnitude 6 N acts on particle P. A second force, $\vec{f_2}$, of magnitude 8 N acts at 60° to $\vec{f_1}$. Determine the resultant and equilibrant of $\vec{f_1}$ and $\vec{f_2}$.
9. Resolve a force of 10 N into two forces perpendicular to each other, such that one component force makes an angle of 15° with the 10 N force.

A

10. A 10 kg block lies on a smooth ramp that is inclined at 30°. What force, parallel to the ramp, would prevent the block from moving? (Assume that 1 kg exerts a force of 9.8 N.)





11. Three forces, with magnitudes 13 N, 7 N, and 8 N, are in a state of equilibrium.
    a. Draw a sketch of these three forces.
    b. Determine the angle between the two smallest forces.
12. Four forces of magnitude 5 N, 9 N, 10 N, and 14 N are arranged as shown in the diagram at the left. Determine the resultant of these forces.

13. Two forces, $\vec{f_1}$ and $\vec{f_2}$, act at right angles to each other. The magnitude of the resultant of these two forces is 25 N, and $|\vec{f_1}| = 24$ N.

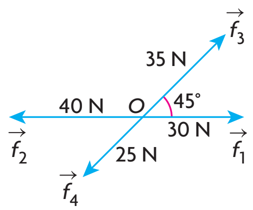
   a. Determine $|\vec{f_2}|$.

   b. Determine the angle between $\vec{f_1}$ and the resultant, and the angle between $\vec{f_1}$ and the equilibrant.

C 14. Three forces, each having a magnitude of 1 N, are arranged to produce equilibrium.

   a. Draw a sketch showing an arrangement of these forces, and demonstrate that the angle between the resultant and each of the other two forces is 60°.

   b. Explain how to determine the angle between the equilibrant and the other two vectors.



T 15. Four forces, $\vec{f_1}$, $\vec{f_2}$, $\vec{f_3}$, and $\vec{f_4}$, are acting on an object and lie in the same plane, as shown. The forces $\vec{f_1}$ and $\vec{f_2}$ act in an opposite direction to each other, with $|\vec{f_1}| = 30$ N and $|\vec{f_2}| = 40$ N. The forces $\vec{f_3}$ and $\vec{f_4}$ also act in opposite directions, with $|\vec{f_3}| = 35$ N and $|\vec{f_4}| = 25$ N. If the angle between $\vec{f_1}$ and $\vec{f_3}$ is 45°, determine the resultant of these four forces.

16. A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 30° and 45° with the ceiling. Determine the tension in each of the ropes.

17. A mass of 5 kg is suspended by two strings, 24 cm and 32 cm long, from two points that are 40 cm apart and at the same level. Determine the tension in each of the strings.

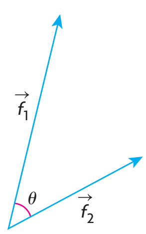
PART C

18. Two tugs are towing a ship. The smaller tug is 15° off the port bow, and the larger tug is 20° off the starboard bow. The larger tug pulls twice as hard as the smaller tug. In what direction will the ship move?

19. Three forces of 5 N, 8 N, and 10 N act from the corner of a rectangular solid along its three edges.

   a. Calculate the magnitude of the equilibrant of these three forces.

   b. Determine the angle that the equilibrant makes with each of the three forces.



20. Two forces, $\vec{f_1}$ and $\vec{f_2}$, make an angle θ with each other when they are placed tail to tail, as shown. Prove that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.

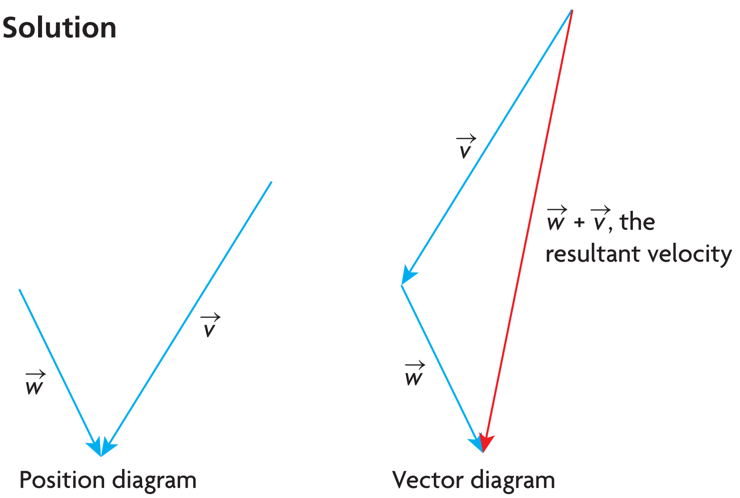
In the previous chapter, we showed that velocity is a vector because it had both magnitude (speed) and direction. In this section, we will demonstrate how two velocities can be combined to determine their resultant velocity.

EXAMPLE 1

Representing velocity with diagrams

An airplane has a velocity of $\vec{v}$ (relative to the air) when it encounters a wind having a velocity of $\vec{w}$ (relative to the ground). Draw a diagram showing the possible positions of the velocities and another diagram showing the resultant velocity.

Solution



The resultant velocity of any two velocities is their sum. In all calculations involving resultant velocities, it is necessary to draw a triangle showing the velocities so there is a clear recognition of the resultant and its relationship to the other two velocities. When the velocity of the airplane is mentioned, it is understood that we are referring to its air speed. When the velocity of the wind is mentioned, we are referring to its velocity relative to a fixed point, the ground. The resultant velocity of the airplane is the velocity of the airplane relative to the ground and is called the ground velocity of the airplane.

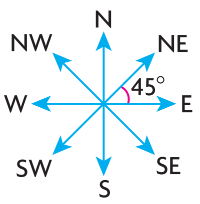
EXAMPLE 2

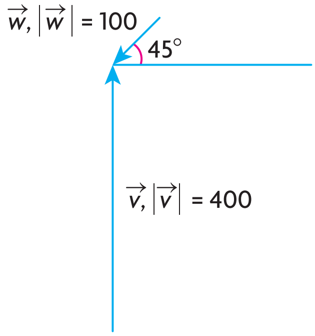
Selecting a vector strategy to determine ground velocity

A plane is heading due north with an air speed of 400 km/h when it is blown off course by a wind of 100 km/h from the northeast. Determine the resultant ground velocity of the airplane.

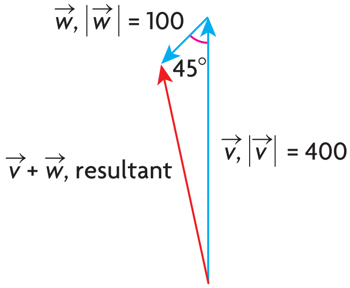
Solution

We start by drawing position and vector diagrams where $\vec{w}$ represents the velocity of the wind and $\vec{v}$ represents the velocity of the airplane in kilometres per hour.





Position diagram



Vector diagram

Use the cosine law to determine the magnitude of the resultant velocity.

$$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta,\ \theta = 45°,\ |\vec{w}| = 100,\ |\vec{v}| = 400$$

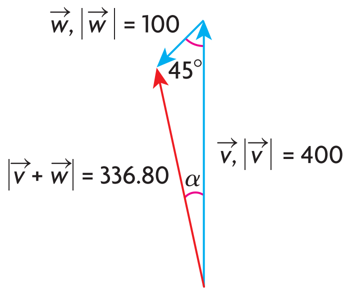
$$|\vec{v} + \vec{w}|^2 = 400^2 + 100^2 - 2(100)(400)\cos 45°$$

$$|\vec{v} + \vec{w}|^2 = 160\,000 + 10\,000 - 80\,000\left(\frac{1}{\sqrt{2}}\right)$$

$$|\vec{v} + \vec{w}|^2 = 170\,000 - \frac{80\,000}{\sqrt{2}}$$

$$|\vec{v} + \vec{w}| \doteq 336.80$$

To state the required velocity, the direction of the resultant vector is needed. Use the sine law to calculate α, the angle between the velocity vector of the plane and the resultant vector.



$$\frac{\sin\alpha}{100} \doteq \frac{\sin 45°}{336.80}$$

$$\sin\alpha \doteq \frac{100\sin 45°}{336.80} \doteq 0.2099$$

$$\alpha \doteq 12.1°$$

Therefore, the resultant velocity is approximately 336.80 km/h, $N12.1°W$ (or $W77.9°N$).

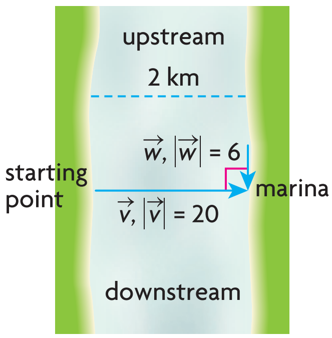
EXAMPLE 3 Using vectors to represent velocities

A river is 2 km wide and flows at 6 km/h. Anna is driving a motorboat, which has a speed of 20 km/h in still water and she heads out from one bank in a direction perpendicular to the current. A marina lies directly across the river from the starting point on the opposite bank.

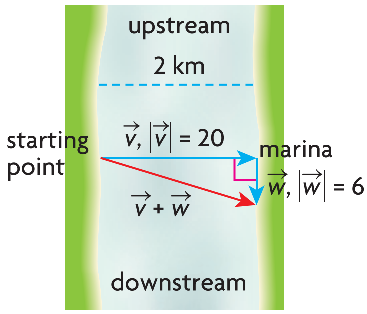
a. How far downstream from the marina will the current push the boat?
b. How long will it take for the boat to cross the river?
c. If Anna decides that she wants to end up directly across the river at the marina, in what direction should she head? What is the resultant velocity of the boat?

Solution

a. As before, we construct a vector and position diagram, where $\vec{w}$ and $\vec{v}$ represent the velocity of the river and the boat, respectively, in kilometres per hour.

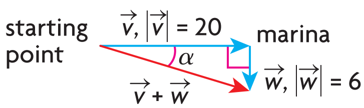


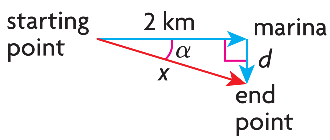
Position diagram



Vector diagram

The distance downstream that the boat lands can be calculated in a variety of ways, but the easiest way is to redraw the velocity triangle from the vector diagram, keeping in mind that the velocity triangle is *similar* to the distance triangle. This is because the distance travelled is directly proportional to the velocity.





Using similar triangles, $\frac{6}{20} = \frac{d}{2}$, $d = 0.6$.

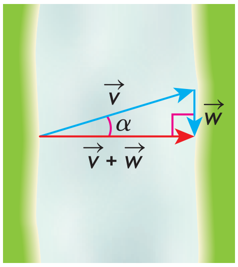
The boat will touch the opposite bank 0.6 km downstream.

b. To calculate the actual distance between the starting and end points, the Pythagorean theorem is used for the distance triangle, with x being the required distance. Thus, $x^2 = 2^2 + (0.6)^2 = 4.36$ and $x \doteq 2.09$, which means that the actual distance the boat travelled was approximately 2.09 km.

To calculate the length of time it took to make the trip, it is necessary to calculate the speed at which this distance was travelled. Again, using similar triangles, $\frac{20}{2} \doteq \frac{|\vec{v} + \vec{w}|}{2.09}$. Solving this proportion, $|\vec{v} + \vec{w}| \doteq 20.9$, so the actual speed of the boat crossing the river was about 20.9 km/h. The actual time taken to cross the river is $t = \frac{d}{v} \doteq \frac{2.09}{20.9} \doteq 0.1$ h, or about 6 min.

Therefore, the boat landed 0.6 km downstream, and it took approximately 6 min to make the crossing.

c. To determine the velocity with which she must travel to reach the marina, we will draw the related vector diagram.



We are given $|\vec{w}| = 6$ and $|\vec{v}| = 20$. To determine the direction in which the boat must travel, let α represent the angle upstream at which the boat heads out.

$$\sin \alpha = \frac{6}{20} \text{ or } \sin^{-1}\left(\frac{6}{20}\right) = \alpha$$

$$\alpha \doteq 17.5°$$

To calculate the magnitude of the resultant velocity, use the Pythagorean theorem.
$|\vec{v}|^2 = |\vec{w}|^2 + |\vec{v} + \vec{w}|^2$ where $|\vec{v}| = 20$ and $|\vec{w}| = 6$

Thus, $20^2 = 6^2 + |\vec{v} + \vec{w}|^2$

$$|\vec{v} + \vec{w}|^2 = 400 - 36$$

$$|\vec{v} + \vec{w}| \doteq 19.08$$

This implies that if Anna wants to travel directly across the river, she will have to travel upstream 17.5° with a speed of approximately 19.08 km/h. The nose of the boat will be headed upstream at 17.5°, but the boat will actually be moving directly across the river at a water speed of 19.08 km/h.

IN SUMMARY

Key Idea

- Problems involving velocities can be solved using strategies involving vectors.

Need to Know

- The velocity of an object is stated relative to a frame of reference. The frame of reference used influences the stated velocity of the object.
- Air speed/water speed is the speed of a plane/boat relative to a person on board. Ground speed is the speed of a plane or boat relative to a person on the ground and includes the effect of wind or current.
- The resultant velocity $\vec{v_r} = \vec{v_1} + \vec{v_2}$.

Exercise 7.2

PART A

1. A woman walks at 4 km/h down the corridor of a train that is travelling at 80 km/h on a straight track.
   a. What is her resultant velocity in relation to the ground if she is walking in the same direction as the train?
   b. If she walks in the opposite direction as the train, what is her resultant velocity?
2. An airplane heading north has an air speed of 600 km/h.
   a. If the airplane encounters a wind from the north at 100 km/h, what is the resultant ground velocity of the plane?
   b. If there is a wind from the south at 100 km/h, what is the resultant ground velocity of the plane?

PART B

3. An airplane has an air speed of 300 km/h and is heading due west. If it encounters a wind blowing south at 50 km/h, what is the resultant ground velocity of the plane?
4. K Adam can swim at the rate of 2 km/h in still water. At what angle to the bank of a river must he head if he wants to swim directly across the river and the current in the river moves at the rate of 1 km/h?
5. A child, sitting in the backseat of a car travelling at 20 m/s, throws a ball at 2 m/s to her brother who is sitting in the front seat.
   a. What is the velocity of the ball relative to the children?
   b. What is the velocity of the ball relative to the road?
6. A boat heads 15° west of north with a water speed of 12 m/s. Determine its resultant velocity, relative to the ground, when it encounters a 5 m/s current from 15° north of east.
7. An airplane is heading due north at 800 km/h when it encounters a wind from the northeast at 100 km/h.
   a. What is the resultant velocity of the airplane?
   b. How far will the plane travel in 1 h?
8. An airplane is headed north with a constant velocity of 450 km/h. The plane encounters a wind from the west at 100 km/h.
   a. In 3 h, how far will the plane travel?
   b. In what direction will the plane travel?

A 9. A small airplane has an air speed of 244 km/h. The pilot wishes to fly to a destination that is 480 km due west from the plane's present location. There is a 44 km/h wind from the south.

a. In what direction should the pilot fly in order to reach the destination?

b. How long will it take to reach the destination?

10. Judy and her friend Helen live on opposite sides of a river that is 1 km wide. Helen lives 2 km downstream from Judy on the opposite side of the river. Judy can swim at a rate of 3 km/h, and the river's current has a speed of 4 km/h. Judy swims from her cottage directly across the river.

a. What is Judy's resultant velocity?

b. How far away from Helen's cottage will Judy be when she reaches the other side?

c. How long will it take Judy to reach the other side?

T 11. An airplane is travelling $N60°E$ with a resultant ground speed of 205 km/h. The nose of the plane is actually pointing east with an airspeed of 212 km/h.

a. What is the wind direction?

b. What is the wind speed?

C 12. Barbara can swim at 4 km/h in still water. She wishes to swim across a river to a point directly opposite from where she is standing. The river is moving at a rate of 5 km/h. Explain, with the use of a diagram, why this is not possible.

PART C

13. Mary leaves a dock, paddling her canoe at 3 m/s. She heads downstream at an angle of 30° to the current, which is flowing at 4 m/s.

a. How far downstream does Mary travel in 10 s?

b. What is the length of time required to cross the river if its width is 150 m?

14. Dave wants to cross a 200 m wide river whose current flows at 5.5 m/s. The marina he wants to visit is located at an angle of S45°W from his starting position. Dave can paddle his canoe at 4 m/s in still water.

a. In which direction should he head to get to the marina?

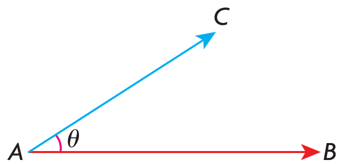
b. How long will the trip take?

15. A steamboat covers the distance between town A and town B (located downstream) in 5 h without making any stops. Moving upstream from B to A, the boat covers the same distance in 7 h (again making no stops). How many hours does it take a raft moving with the speed of the river current to get from A to B?

Section 7.3—The Dot Product of Two Geometric Vectors

In Chapter 6, the concept of multiplying a vector by a scalar was discussed. In this section, we introduce the dot product of two vectors and deal specifically with geometric vectors. When we refer to geometric vectors, we are referring to vectors that do not have a coordinate system associated with them. The dot product for any two vectors is defined as the product of their magnitudes multiplied by the cosine of the angle between the two vectors when the two vectors are placed in a tail-to-tail position.

Dot Product of Two Vectors



$\vec{AC} \cdot \vec{AB} = |\vec{AC}||\vec{AB}|\cos\theta, 0 \le \theta \le 180°$

Observations about the Dot Product

There are some elementary but important observations that can be made about this calculation. First, the result of the dot product is always a scalar. Each of the quantities on the right side of the formula above is a scalar quantity, and so their product must be a scalar. For this reason, the dot product is also known as the **scalar product**. Second, the dot product can be positive, zero, or negative, depending upon the size of the angle between the two vectors.

Sign of the Dot Product

For the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \le \theta \le 180°$:

- for $0 \le \theta < 90°$, $\cos\theta > 0$, so $\vec{a} \cdot \vec{b} > 0$
- for $\theta = 90°$, $\cos\theta = 0$, so $\vec{a} \cdot \vec{b} = 0$
- for $90° < \theta \le 180°$, $\cos\theta < 0$, so $\vec{a} \cdot \vec{b} < 0$

The dot product is only calculated for vectors when the angle θ between the vectors is to 0° to 180°, inclusive. (For convenience in calculating, the angle between the vectors is usually expressed in degrees, but radian measure is also correct.)

Perhaps the most important observation to be made about the dot product is that when two nonzero vectors are perpendicular, their dot product is always 0. This will have many important applications in Chapter 8, when we discuss lines and planes.

EXAMPLE 1 Calculating the dot product of two geometric vectors

Two vectors, $\vec{a}$ and $\vec{b}$, are placed tail to tail and have magnitudes 3 and 5, respectively. There is an angle of 120° between the vectors. Calculate $\vec{a} \cdot \vec{b}$.

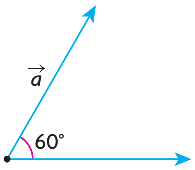
Solution

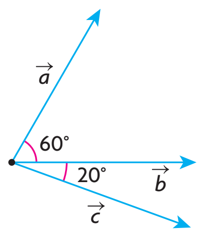
Since $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $\cos 120° = -0.5$,

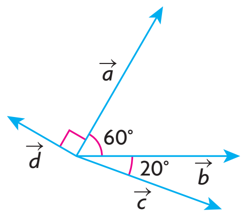
$$\vec{a} \cdot \vec{b} = (3)(5)(-0.5)$$
$$= -7.5$$

Notice that, in this example, it is stated that the vectors are tail to tail when taking the dot product. This is the convention that is always used, since this is the way of defining the angle between any two vectors.

INVESTIGATION







A. Given vectors $\vec{a}$ and $\vec{b}$ where $|\vec{a}| = 5$, $|\vec{b}| = 8$ and the angle between the vectors is 60°, calculate $\vec{a} \cdot \vec{b}$.

B. For the vectors given in part A calculate $\vec{b} \cdot \vec{a}$. What do you notice? Will this relationship always hold regardless of the two vectors used and the measure of the angle between them? Explain.

C. For the vectors given in part A calculate $\vec{a} \cdot \vec{a}$ and $\vec{b} \cdot \vec{b}$. Based on your observations, what can you conclude about $\vec{u} \cdot \vec{u}$ for any vector $\vec{u}$?

D. Using the vectors given in part A and a third vector $\vec{c}$, $|\vec{c}| = 4$, as shown in the diagram, calculate each of the following without rounding:

i. $|\vec{b} + \vec{c}|$

ii. the angle between $\vec{b} + \vec{c}$ and $\vec{a}$.

iii. $\vec{a} \cdot (\vec{b} + \vec{c})$

iv. $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

E. Compare your results from part iii and iv in part D. What property does this demonstrate? Write an equivalent expression for $\vec{c} \cdot (\vec{a} + \vec{b})$ and confirm it using the appropriate calculations with the vectors given in part D.

F. Using the 3 vectors given above, explain why $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$.

G. A fourth vector $\vec{d}$, $|\vec{d}| = 3$, is given as shown in the diagram. Explain why $\vec{a} \cdot \vec{d} = \vec{a} \cdot (-\vec{d}) = (-\vec{a}) \cdot \vec{d} = (-\vec{a}) \cdot (-\vec{d})$

H. Using the vectors given, calculate $\vec{a} \cdot \vec{0}$, $\vec{b} \cdot \vec{0}$ and $\vec{c} \cdot \vec{0}$. What does this imply?

Properties of the Dot Product

It should also be noted that the dot product can be calculated in whichever order we choose. In other words, $\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|\cos\theta = |\vec{q}||\vec{p}|\cos\theta = \vec{q}\cdot\vec{p}$. We can change the order in the multiplication because the quantities in the formula are just scalars (that is, numbers) and the order of multiplication does not affect the final answer. This latter property is known as the *commutative* property for the dot product.

Another property that proves to be quite important for both computation and theoretical purposes is the dot product between a vector $\vec{p}$ and itself. The angle between $\vec{p}$ and itself is 0°, so $\vec{p}\cdot\vec{p} = |\vec{p}||\vec{p}|(1) = |\vec{p}|^2$ since $\cos(0°) = 1$.

EXAMPLE 2 **Calculating the dot product between a vector and itself**

a. If $|\vec{a}| = \sqrt{7}$, calculate $\vec{a}\cdot\vec{a}$.

b. Calculate $\vec{i}\cdot\vec{i}$.

Solution

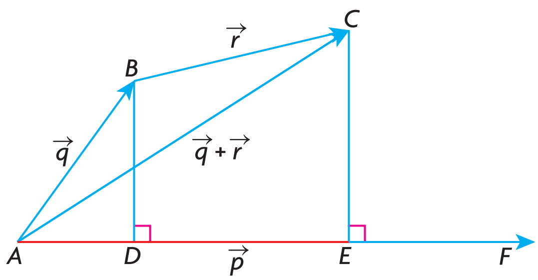
a. This is an application of the property just shown. So, $\vec{a}\cdot\vec{a} = (\sqrt{7})(\sqrt{7}) = 7$.

b. Since we know that $\vec{i}$ is a unit vector (along the positive x-axis), $\vec{i}\cdot\vec{i} = (1)(1) = 1$. In general, for any vector $\vec{x}$ of unit length, $\vec{x}\cdot\vec{x} = |\vec{x}|^2 = 1$. Thus, $\vec{j}\cdot\vec{j} = 1$ and $\vec{k}\cdot\vec{k} = 1$, where $\vec{j}$ and $\vec{k}$ are the unit vectors along the positive y- and z-axes, respectively.

Another important property that the dot product follows is the *distributive* property. In elementary algebra, the distributive property states that $p(q + r) = pq + pr$. We will prove that the distributive property also holds for the dot product. We will prove this geometrically below and algebraically in the next section.

Theorem: For the vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, $\vec{p}(\vec{q} + \vec{r}) = \vec{p}\cdot\vec{q} + \vec{p}\cdot\vec{r}$.

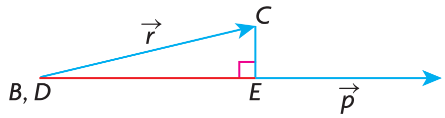
Proof: The vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, are drawn, and the diagram is labelled as shown with $\overrightarrow{AC} = \vec{q} + \vec{r}$. To help visualize the dot products, lines from B and C have been drawn perpendicular to $\vec{p}$ (which is $\overrightarrow{AF}$).



Using the definition of a dot product, we write $\vec{q}\cdot\vec{p} = |\vec{q}||\vec{p}|\cos BAF$.

If we look at the right-angled triangle ABD and use the cosine ratio, we note that $\cos BAD = \frac{AD}{|\vec{q}|}$ or $AD = |\vec{q}|\cos BAD$. The two angles BAD and BAF are identical, and so $AD = |\vec{q}|\cos BAF$.
Rewriting the formula $\vec{q} \cdot \vec{p} = |\vec{q}||\vec{p}|\cos BAF$ as $\vec{q} \cdot \vec{p} = (|\vec{q}|\cos BAF)|\vec{p}|$, and substituting $AD = |\vec{q}|\cos BAF$, we obtain, $\vec{q} \cdot \vec{p} = AD|\vec{p}|$.

We also consider the vectors $\vec{r}$ and $\vec{p}$. We translate the vector $\overrightarrow{BC}$ so that point B is moved to be coincident with D. (The vector $\overrightarrow{BC}$ maintains the same direction and size under this translation.)



Writing the dot product for $\vec{r}$ and $\vec{p}$, we obtain $\vec{r} \cdot \vec{p} = |\vec{r}||\vec{p}|\cos CDE$. If we use trigonometric ratios in the right triangle, $\cos CDE = \frac{DE}{|\vec{r}|}$ or $DE = |\vec{r}|\cos CDE$.
Substituting $DE = |\vec{r}|\cos CDE$ into $\vec{r} \cdot \vec{p} = |\vec{r}||\vec{p}|\cos CDE$, we obtain $\vec{r} \cdot \vec{p} = DE|\vec{p}|$. If we use the formula for the dot product of $\vec{q} + \vec{r}$ and $\vec{p}$, we get the following: $(\vec{q} + \vec{r}) \cdot \vec{p} = |\vec{q} + \vec{r}||\vec{p}|\cos CAE$. Using the same reasoning as before, $\cos CAE = \frac{AE}{|\vec{q} + \vec{r}|}$ and $AE = (\cos CAE)|\vec{q} + \vec{r}|$, and then, by substitution, $(\vec{q} + \vec{r}) \cdot \vec{p} = |\vec{p}|AE$.

Adding the two quantities $\vec{q} \cdot \vec{p}$ and $\vec{r} \cdot \vec{p}$,

$$\begin{aligned}
\vec{q} \cdot \vec{p} + \vec{r} \cdot \vec{p} &= AD|\vec{p}| + DE|\vec{p}| \\
&= |\vec{p}|(AD + DE) && \text{(Factoring)} \\
&= |\vec{p}|AE \\
&= (\vec{q} + \vec{r}) \cdot \vec{p}
\end{aligned}$$

Thus, $\vec{q} \cdot \vec{p} + \vec{r} \cdot \vec{p} = (\vec{q} + \vec{r}) \cdot \vec{p}$, or, written in the more usual way,

$$\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$$

We list some of the properties of the dot product below.
This final property has not been proven, but it comes directly from the definition of the dot product and proves most useful in computation.

Properties of the Dot Product

Commutative Property: $\vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p}$,

Distributive Property: $\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$,

Magnitudes Property: $\vec{p} \cdot \vec{p} = |\vec{p}|^2$,

Associative Property with a scalar K: $(k\vec{p}) \cdot \vec{q} = \vec{p} \cdot (k\vec{q}) = k(\vec{p} \cdot \vec{q})$

EXAMPLE 3 Selecting a strategy to determine the angle between two geometric vectors

If the vectors $\vec{a} + 3\vec{b}$ and $4\vec{a} - \vec{b}$ are perpendicular, and $|\vec{a}| = 2|\vec{b}|$, determine the angle (to the nearest degree) between the nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution

Since the two given vectors are perpendicular, $(\vec{a} + 3\vec{b}) \cdot (4\vec{a} - \vec{b}) = 0$.

Multiplying, $\vec{a} \cdot (4\vec{a} - \vec{b}) + 3\vec{b} \cdot (4\vec{a} - \vec{b}) = 0$

$4\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + 12\vec{b} \cdot \vec{a} - 3\vec{b} \cdot \vec{b} = 0$ (Distributive property)

Simplifying, $4|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 3|\vec{b}|^2 = 0$ (Commutative property)

Since $|\vec{a}| = 2|\vec{b}|$, $|\vec{a}|^2 = (2|\vec{b}|)^2 = 4|\vec{b}|^2$ (Squaring both sides)

Substituting, $4(4|\vec{b}|^2) + 11((2|\vec{b}|)(|\vec{b}|)\cos\theta) - 3|\vec{b}|^2 = 0$

Solving for $\cos\theta$,

$$\cos\theta = \frac{-13|\vec{b}|^2}{22|\vec{b}|^2}$$

$$\cos\theta = \frac{-13}{22}, \ |\vec{b}|^2 \neq 0$$

Thus, $\cos^{-1}\left(\frac{-13}{22}\right) = \theta, \theta \doteq 126.2°$

Therefore, the angle between the two vectors is approximately 126.2°.

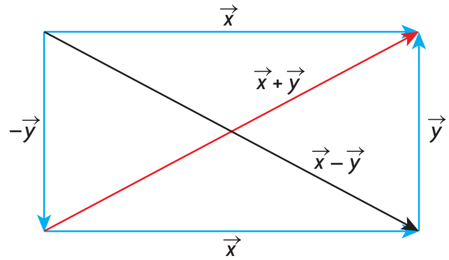
It is often necessary to square the magnitude of a vector expression. This is illustrated in the following example.

EXAMPLE 4 Proving that two vectors are perpendicular using the dot product

If $|\vec{x} + \vec{y}| = |\vec{x} - \vec{y}|$, prove that the nonzero vectors, $\vec{x}$ and $\vec{y}$, are perpendicular.

Solution

Consider the following diagram.



Since $|\vec{x}+\vec{y}| = |\vec{x}-\vec{y}|$, $|\vec{x}+\vec{y}|^2 = |\vec{x}-\vec{y}|^2$ (Squaring both sides)

$|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$ and $|\vec{x}-\vec{y}|^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$ (Magnitudes property)

Therefore, $(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$

$$|\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2 = |\vec{x}|^2 - 2\vec{x}\cdot\vec{y} + |\vec{y}|^2$$ (Multiplying out)

So, $4\vec{x}\cdot\vec{y} = 0$ and $\vec{x}\cdot\vec{y} = 0$

Thus, the two required vectors are shown to be perpendicular. (Geometrically, this means that if diagonals in a parallelogram are equal in length, then the sides must be perpendicular. In actuality, the parallelogram is a rectangle.)

In this section, we dealt with the dot product and its geometric properties. In the next section, we will illustrate these same ideas with algebraic vectors.

IN SUMMARY

Key Idea

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

Need to Know

- If $0° \leq \theta < 90°$, then $\vec{a}\cdot\vec{b} > 0$
- If $\theta = 90°$, then $\vec{a}\cdot\vec{b} = 0$
- If $90° < \theta \leq 180°$, then $\vec{a}\cdot\vec{b} < 0$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{i}\cdot\vec{i} = 1$, $\vec{j}\cdot\vec{j} = 1$, and $\vec{k}\cdot\vec{k} = 1$
- $(k\vec{a})\cdot\vec{b} = \vec{a}\cdot(k\vec{b}) = k(\vec{a}\cdot\vec{b})$

Exercise 7.3

PART A

1. If $\vec{a} \cdot \vec{b} = 0$, why can we not necessarily conclude that the given vectors are perpendicular? (In other words, what restrictions must be placed on the vectors to make this statement true?)

C 2. Explain why the calculation $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not meaningful.

3. A student writes the statement $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$ and then concludes that $\vec{a} = \vec{c}$. Construct a simple numerical example to show that this is not correct if the given vectors are all nonzero.

4. Why is it correct to say that if $\vec{a} = \vec{c}$, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$?

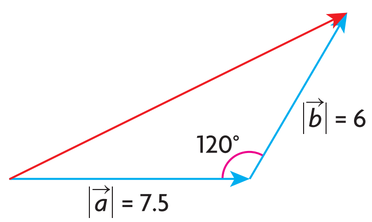
5. If two vectors $\vec{a}$ and $\vec{b}$ are unit vectors pointing in opposite directions, what is the value of $\vec{a} \cdot \vec{b}$?

PART B

6. If θ is the angle (in degrees) between the two given vectors, calculate the dot product of the vectors.
   a. $|\vec{p}| = 4, |\vec{q}| = 8, \theta = 60°$
   b. $|\vec{x}| = 2, |\vec{y}| = 4, \theta = 150°$
   c. $|\vec{a}| = 0, |\vec{b}| = 8, \theta = 100°$
   d. $|\vec{p}| = 1, |\vec{q}| = 1, \theta = 180°$
   e. $|\vec{m}| = 2, |\vec{n}| = 5, \theta = 90°$
   f. $|\vec{u}| = 4, |\vec{v}| = 8, \theta = 145°$

7. Calculate, to the nearest degree, the angle between the given vectors.
   a. $|\vec{x}| = 8, |\vec{y}| = 3, \vec{x} \cdot \vec{y} = 12\sqrt{3}$
   b. $|\vec{m}| = 6, |\vec{n}| = 6, \vec{m} \cdot \vec{n} = 6$
   c. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = 3$
   d. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = -3$
   e. $|\vec{a}| = 7, |\vec{b}| = 3, \vec{a} \cdot \vec{b} = 10.5$
   f. $|\vec{u}| = 10, |\vec{v}| = 10, \vec{u} \cdot \vec{v} = -50$

K 8. For the two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are shown in the diagram below, calculate the dot product.



9. Use the properties of the dot product to simplify each of the following expressions:
   a. $(\vec{a} + 5\vec{b}) \cdot (2\vec{a} - 3\vec{b})$
   b. $3\vec{x} \cdot (\vec{x} - 3\vec{y}) - (\vec{x} - 3\vec{y}) \cdot (-3\vec{x} + \vec{y})$

10. What is the value of the dot product between $\vec{0}$ and any nonzero vector? Explain.

A 11. The vectors $\vec{a} - 5\vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular. If $\vec{a}$ and $\vec{b}$ are unit vectors, then determine $\vec{a} \cdot \vec{b}$.

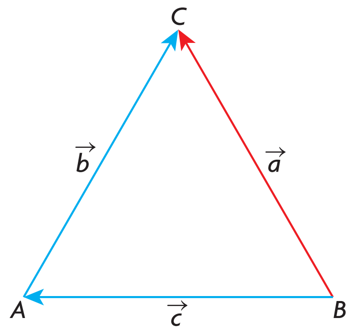
12. If $\vec{a}$ and $\vec{b}$ are any two nonzero vectors, prove each of the following to be true:
   a. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$
   b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

13. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfy the relationship $\vec{a} = \vec{b} + \vec{c}$.
   a. Show that $|\vec{a}|^2 = |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$.
   b. If the vectors $\vec{b}$ and $\vec{c}$ are perpendicular, how does this prove the Pythagorean theorem?

14. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three mutually perpendicular vectors of lengths 1, 2, and 3, respectively. Calculate the value of $(\vec{u} + \vec{v} + \vec{w}) \cdot (\vec{u} + \vec{v} + \vec{w})$.

T 15. Prove the identity $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$.

16. The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are of unit length and form the sides of equilateral triangle ABC such that $\vec{a} - \vec{b} - \vec{c} = \vec{0}$ (as shown). Determine the numerical value of $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b} + \vec{c})$.



PART C

17. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Determine the value of $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ if $|\vec{a}| = 1$, $|\vec{b}| = 2$, and $|\vec{c}| = 3$.

18. The vector $\vec{a}$ is a unit vector, and the vector $\vec{b}$ is any other nonzero vector. If $\vec{c} = (\vec{b} \cdot \vec{a})\vec{a}$ and $\vec{d} = \vec{b} - \vec{c}$, prove that $\vec{d} \cdot \vec{a} = 0$.

Section 7.4—The Dot Product of Algebraic Vectors

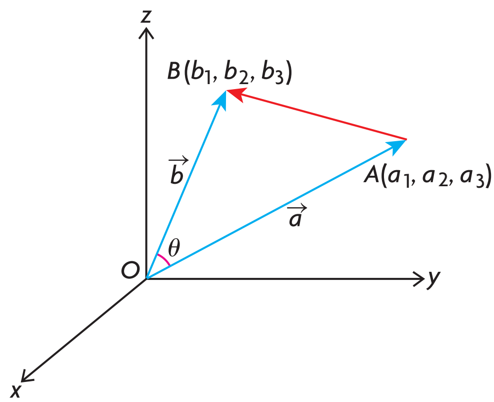
In the previous section, the dot product was discussed in geometric terms. In this section, the dot product will be expressed in terms of algebraic vectors in R^2 and R^3. Recall that a vector expressed as $\vec{a} = (-1, 4, 5)$ is referred to as an algebraic vector. The geometric properties of the dot product developed in the previous section will prove useful in understanding the dot product in algebraic form. The emphasis in this section will be on developing concepts in R^3, but these ideas apply equally well to R^2 or to higher dimensions.

Defining the Dot Product of Algebraic Vectors

Theorem: In R^3, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = a_1b_1 + a_2b_2 + a_3b_3$.

Proof: Draw $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, as shown in the diagram.



In ΔOAB, $|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$ (Cosine law)

So, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$ and

$|\overrightarrow{AB}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

We know that $|\overrightarrow{OA}|^2 = a_1^2 + a_2^2 + a_3^2$ and $|\overrightarrow{OB}|^2 = b_1^2 + b_2^2 + b_3^2$.

It should also be noted that $\vec{a} \cdot \vec{b} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$. (Definition of dot product)

We substitute each of these quantities in the expression for the cosine law.

This gives $(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 =$

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a} \cdot \vec{b}$$

Expanding, we get

$$b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2$$
$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a}\cdot\vec{b} \qquad \text{(Simplify)}$$
$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a}\cdot\vec{b} \qquad \text{(Divide by } -2\text{)}$$
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Observations about the Algebraic Form of the Dot Product

There are some important observations to be made about this expression for the dot product. First and foremost, the quantity on the right-hand side of the expression, $a_1b_1 + a_2b_2 + a_3b_3$, is evaluated by multiplying corresponding components and then adding them. Each of these quantities, a_1b_1, a_2b_2, and a_3b_3, is just a real number, so their sum is a real number. This implies that $\vec{a}\cdot\vec{b}$ is itself just a real number, or a scalar product. Also, since the right side is an expression made up of real numbers, it can be seen that $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\cdot\vec{a}$. This is a restatement of the commutative law for the dot product of two vectors. All the other rules for computation involving dot products can now be proven using the properties of real numbers and the basic definition of a dot product.

In this proof, we have used vectors in R^3 to calculate a formula for $\vec{a}\cdot\vec{b}$. It is important to understand, however, that this procedure could be used in the same way for two vectors, $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, in R^2, to obtain the formula $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$.

EXAMPLE 1 **Proving the distributive property of the dot product in R^3**

Prove that the distributive property holds for dot products in R^3—that is, $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

Solution

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$.

In showing this statement to be true, the right side will be expressed in component form and then rearranged to be the same as the left side.

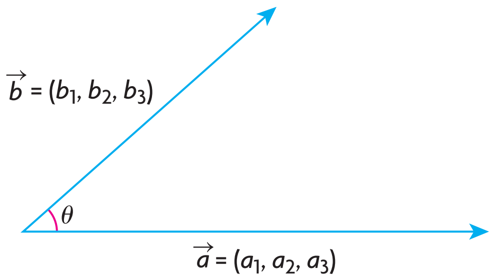
$$\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) + (a_1, a_2, a_3)\cdot(c_1, c_2, c_3)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \qquad \text{(Definition of dot product)}$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \qquad \text{(Rearranging terms)}$$
$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \qquad \text{(Factoring)}$$
$$= \vec{a}\cdot(\vec{b} + \vec{c})$$

This example shows how to prove the distributive property for the dot product in R^3. The value of writing the dot product in component form is that it allows us to combine the geometric form with the algebraic form, and create the ability to do calculations that would otherwise not be possible.

> **Computation of the Dot Product of Algebraic Vectors**
>
> In R^2, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2x_2$, where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$.
>
> In R^3, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2y_2 + x_3y_3$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$.
>
> In both cases q is the angle between $\vec{x}$ and $\vec{y}$.

The dot product expressed in component form has significant advantages over the geometric form from both a computational and theoretical point of view. At the outset, the calculation appears to be somewhat artificial or contrived, but as we move ahead, we will see its applicability to many situations.

A useful application of the dot product is to calculate the angle between two vectors. Solving for $\cos\theta$ in the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ gives the following result.

> **Formula for the Angle between Two Vectors**
>
> 
>
>
> When two vectors are placed tail to tail, as shown, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

EXAMPLE 2 Selecting a strategy to determine the angle between two algebraic vectors

a. Given the vectors $\vec{a} = (-1, 2, 4)$ and $\vec{b} = (3, 4, 3)$, calculate $\vec{a} \cdot \vec{b}$.

b. Calculate, to the nearest degree, the angle between $\vec{a}$ and $\vec{b}$.

Solution

a. $\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4) + (4)(3) = 17$

b. $|\vec{a}|^2 = (-1)^2 + (2)^2 + (4)^2 = 21,\ |\vec{a}| = \sqrt{21}$

$|\vec{b}|^2 = (3)^2 + (4)^2 + (3)^2 = 34,\ |\vec{b}| = \sqrt{34}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{17}{\sqrt{21}\sqrt{34}} \qquad \text{(Substitution)}$$

$$\cos\theta \doteq 0.6362$$

$$\theta \doteq \cos^{-1}(0.6362)$$

$$\theta \doteq 50.5°$$

Therefore, the angle between the two vectors is approximately 50.5°.

In the previous section, we showed that when two nonzero vectors are perpendicular, their dot product equals zero—that is, $\vec{a}\cdot\vec{b} = 0$.

EXAMPLE 3 Using the dot product to solve a problem involving perpendicular vectors

a. For what values of k are the vectors $\vec{a} = (-1, 3, -4)$ and $\vec{b} = (3, k, -2)$ perpendicular?

b. For what values of m are the vectors $\vec{x} = (m, m, -3)$ and $\vec{y} = (m, -3, 6)$ perpendicular?

Solution

a. Since $\vec{a}\cdot\vec{b} = 0$ for perpendicular vectors,

$$-1(3) + 3(k) - 4(-2) = 0$$
$$3k = -5$$
$$k = \frac{-5}{3}$$

In calculations of this type involving the dot product, the calculation should be verified as follows:

$$\begin{aligned}(-1, 3, -4)\cdot\left(3, \tfrac{-5}{3}, -2\right) &= -1(3) + 3\left(\tfrac{-5}{3}\right) - 4(-2)\\ &= -3 - 5 + 8\\ &= 0\end{aligned}$$

This check verifies that the calculation is correct.

b. Using the conditions for perpendicularity of vectors,

$$\begin{aligned}(m, m, -3)\cdot(m, -3, 6) &= 0\\ m^2 - 3m - 18 &= 0\\ (m-6)(m+3) &= 0\\ m = 6 \text{ or } m &= -3\end{aligned}$$

Check:

For $m = 6$, $(6, 6, -3) \cdot (6, -3, 6) = 36 - 18 - 18 = 0$

For $m = -3$, $(-3, -3, -3) \cdot (-3, -3, 6) = 9 + 9 - 18 = 0$

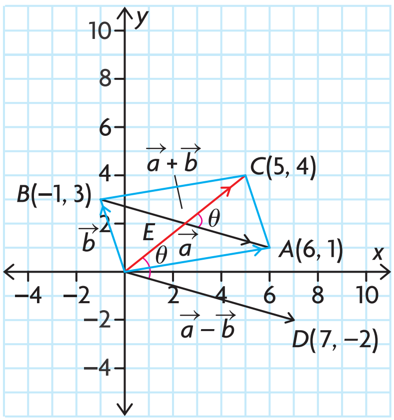
We can combine various operations that we have learned for calculation purposes in R^2 and R^3.

EXAMPLE 4 Using the dot product to solve a problem involving a parallelogram

A parallelogram has its sides determined by $\vec{a} = (6, 1)$ and $\vec{b} = (-1, 3)$. Determine the angle between the diagonals of the parallelogram formed by these vectors.

Solution

The diagonals of the parallelogram are determined by the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, as shown in the diagram. The components of these vectors are $\vec{a} + \vec{b} = (6 + (-1), 1 + 3) = (5, 4)$ and $\vec{a} - \vec{b} = (6 - (-1), 1 - 3) = (7, -2)$, as shown in the diagram



At this point, the dot product is applied directly to find θ, the angle between the vectors $\overrightarrow{OD}$ and $\overrightarrow{OC}$.

$$\text{Therefore, } \cos\theta = \frac{(5, 4) \cdot (7, -2)}{|(5, 4)||(7, -2)|}$$

$$\cos\theta = \frac{27}{\sqrt{41}\sqrt{53}}$$

$$\cos\theta \doteq 0.5792$$

$$\text{Therefore, } \theta \doteq 54.61°$$

The angle between the diagonals is approximately 54.6°. The answer given is 54.6°, but its supplement, 125.4°, is also correct.

One of the most important properties of the dot product is its application to determining a perpendicular vector to two given vectors, which will be demonstrated in the following example.

EXAMPLE 5 Selecting a strategy to determine a vector perpendicular to two given vectors

Find a vector (or vectors) perpendicular to each of the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$.

Solution

Let the required vector be $\vec{x} = (x, y, z)$. Since $\vec{x}$ is perpendicular to each of the two given vectors, $(x, y, z) \cdot (1, 5, -1) = 0$ and $(x, y, z) \cdot (-3, 1, 2) = 0$.

Multiplying gives $x + 5y - z = 0$ and $-3x + y + 2z = 0$, which is a system of two equations in three unknowns.

① $x + 5y - z = 0$

② $-3x + y + 2z = 0$

③ $3x + 15y - 3z = 0$ (Multiplying equation ① by 3)

④ $16y - z = 0$ (Adding equations ② and ③)

$z = 16y$

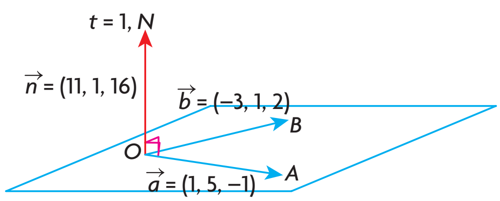
Now, we substitute $z = 16y$ into equation ① to solve for x in terms of y. We obtain $x + 5y - 16y = 0$, or $x = 11y$.

We have solved for x and z by expressing each variable in terms of y. The solution to the system of equations is $(11y, y, 16y)$ or $(11t, t, 16t)$ if we let $y = t$. The substitution of t (called a parameter) for y is not necessarily required for a correct solution and is done more for convenience of notation. This kind of substitution will be used later to great advantage and will be discussed in Chapter 9 at length.

We can find vectors to satisfy the required conditions by replacing t with any real number, $t \neq 0$. Since we can use any real number for t to produce the required vector, this implies that an infinite number of vectors are perpendicular to both $\vec{a}$ and $\vec{b}$. If we use $t = 1$, we obtain $(11, 1, 16)$.

As before, we verify the solution:
$(11, 1, 16) \cdot (1, 5, -1) = 11 + 5 - 16 = 0$ and
$(11, 1, 16) \cdot (-3, 1, 2) = -33 + 1 + 32 = 0$

It is interesting to note that the vector $(11t, t, 16t)$, $t \neq 0$, represents a general vector perpendicular to the plane in which the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$ lie. This is represented in the diagram shown, where $t = 1$.



Determining the components of a vector perpendicular to two nonzero vectors will prove to be important in later applications.

IN SUMMARY

Key Idea

- The dot product is defined as follows for algebraic vectors in R^2 and R^3, respectively:
  - If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2,)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$
  - If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Need to Know

- The properties of the dot product hold for both geometric and algebraic vectors.
- Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are perpendicular if $\vec{a} \cdot \vec{b} = 0$.
- For two nonzero vectors $\vec{a}$ and $\vec{b}$, where θ is the angle between the vectors, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

Exercise 7.4

PART A

1. How many vectors are perpendicular to $\vec{a} = (-1, 1)$? State the components of three such vectors.

2. For each of the following pairs of vectors, calculate the dot product and, on the basis of your result, say whether the angle between the two vectors is acute, obtuse, or 90°.
   a. $\vec{a} = (-2, 1), \vec{b} = (1, 2)$
   b. $\vec{a} = (2, 3, -1), \vec{b} = (4, 3, -17)$
   c. $\vec{a} = (1, -2, 5), \vec{b} = (3, -2, -2)$

3. Give the components of a vector that is perpendicular to each of the following planes:
   a. *xy*-plane
   b. *xz*-plane
   c. *yz*-plane

4. a. From the set of vectors $\left\{(1, 2, -1), (-4, -5, -6), (4, 3, 10), \left(5, -3, \frac{-5}{6}\right)\right\}$, select two pairs of vectors that are perpendicular to each other.
   b. Are any of these vectors collinear? Explain.

5. In Example 5, a vector was found that was perpendicular to two nonzero vectors.
   a. Explain why it would not be possible to do this in R^2 if we selected the two vectors $\vec{a} = (1, -2)$ and $\vec{b} = (1, 1)$.
   b. Explain, in general, why it is not possible to do this if we select any two vectors in R^2.

PART B

K

6. Determine the angle, to the nearest degree, between each of the following pairs of vectors:
   a. $\vec{a} = (5, 3)$ and $\vec{b} = (-1, -2)$
   b. $\vec{a} = (-1, 4)$ and $\vec{b} = (6, -2)$
   c. $\vec{a} = (2, 2, 1)$ and $\vec{b} = (2, 1, -2)$
   d. $\vec{a} = (2, 3, -6)$ and $\vec{b} = (-5, 0, 12)$

7. Determine k, given two vectors and the angle between them.
   a. $\vec{a} = (-1, 2, -3), \vec{b} = (-6k, -1, k), \theta = 90°$
   b. $\vec{a} = (1, 1), \vec{b} = (0, k), \theta = 45°$

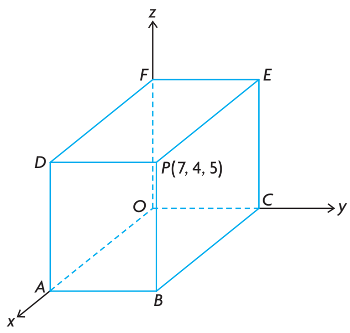
8. In R^2, a square is determined by the vectors $\vec{i}$ and $\vec{j}$.
   a. Sketch the square.
   b. Determine vector components for the two diagonals.
   c. Verify that the angle between the diagonals is 90°.

9. Determine the angle, to the nearest degree, between each pair of vectors.
   a. $\vec{a} = (1 - \sqrt{2}, \sqrt{2}, -1)$ and $\vec{b} = (1, 1)$
   b. $\vec{a} = (\sqrt{2} - 1, \sqrt{2} + 1, \sqrt{2})$ and $\vec{b} = (1, 1, 1)$

C

10. a. For the vectors $\vec{a} = (2, p, 8)$ and $\vec{b} = (q, 4, 12)$, determine values of p and q so that the vectors are
       i. collinear
       ii. perpendicular
    b. Are the values of p and q unique? Explain why or why not.

11. $\triangle ABC$ has vertices at $A(2, 5)$, $B(4, 11)$, and $C(-1, 6)$. Determine the angles in this triangle.



12. A rectangular box measuring 4 by 5 by 7 is shown in the diagram at the left.
    a. Determine the coordinates of each of the missing vertices.
    b. Determine the angle, to the nearest degree, between $\overrightarrow{AE}$ and $\overrightarrow{BF}$.
13. a. Given the vectors $\vec{p} = (-1, 3, 0)$ and $\vec{q} = (1, -5, 2)$, determine the components of a vector perpendicular to each of these vectors.
    b. Given the vectors $\vec{m} = (1, 3, -4)$ and $\vec{n} = (-1, -2, 3)$, determine the components of a vector perpendicular to each of these vectors.
14. Find the value of p if the vectors $\vec{r} = (p, p, 1)$ and $\vec{s} = (p, -2, -3)$ are perpendicular to each other.
15. a. Determine the algebraic condition such that the vectors $\vec{c} = (-3, p, -1)$ and $\vec{d} = (1, -4, q)$ are perpendicular to each other.
    b. If $q = -3$, what is the corresponding value of p?

A

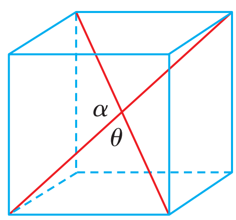
16. Given the vectors $\vec{r} = (1, 2, -1)$ and $\vec{s} = (-2, -4, 2)$, determine the components of two vectors perpendicular to each of these vectors. Explain your answer.
17. The vectors $\vec{x} = (-4, p, -2)$ and $\vec{y} = (-2, 3, 6)$ are such that $\cos^{-1}\left(\frac{4}{21}\right) = \theta$, where θ is the angle between $\vec{x}$ and $\vec{y}$. Determine the value(s) of p.

PART C

18. The diagonals of a parallelogram are determined by the vectors $\vec{a} = (3, 3, 0)$ and $\vec{b} = (-1, 1, -2)$.
    a. Show that this parallelogram is a rhombus.
    b. Determine vectors representing its sides and then determine the length of these sides.
    c. Determine the angles in this rhombus.

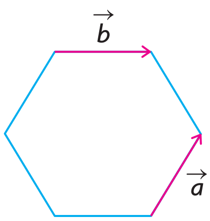
T

19. The rectangle $ABCD$ has vertices at $A(-1, 2, 3)$, $B(2, 6, -9)$, and $D(3, q, 8)$.
    a. Determine the coordinates of the vertex C.
    b. Determine the angle between the two diagonals of this rectangle.
20. A cube measures 1 by 1 by 1. A line is drawn from one vertex to a diagonally opposite vertex through the centre of the cube. This is called a body diagonal for the cube. Determine the angles between the body diagonals of the cube.



Mid-Chapter Review

1. a. If $|\vec{a}| = 3$ and $|\vec{b}| = 2$, and the angle between these two vectors is 60°, determine $\vec{a} \cdot \vec{b}$.
   b. Determine the numerical value of $(3\vec{a} + 2\vec{b}) \cdot (4\vec{a} - 3\vec{b})$.
2. A mass of 15 kg is suspended by two cords from a ceiling. The cords have lengths of 15 cm and 20 cm, and the distance between the points where they are attached on the ceiling is 25 cm. Determine the tension in each of the two cords.
3. In a square that has side lengths of 10 cm, what is the dot product of the vectors representing the diagonals?
4. An airplane is travelling at 500 km/h due south when it encounters a wind from $W45°N$ at 100 km/h.
   a. What is the resultant velocity of the airplane?
   b. How long will it take for the airplane to travel 1000 km?
5. A 15 kg block lies on a smooth ramp that is inclined at 40° to the ground.
   a. Determine the force that this block exerts in a direction perpendicular to the ramp.
   b. What is the force, parallel to the inclined plane, needed to prevent the block from slipping?
6. A regular hexagon, with sides of 3 cm, is shown below. Determine $\vec{a} \cdot \vec{b}$.



7. Given the vectors $\vec{a} = (4, -5, 20)$ and $\vec{b} = (1, 2, 2)$, determine the following:
   a. $\vec{a} \cdot \vec{b}$
   b. the cosine of the angle between the two vectors
8. Given the vectors $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}, \vec{b} = 2\vec{i} - 3\vec{j} + 4\vec{k}$, and $\vec{c} = 3\vec{i} - \vec{j} - \vec{k}$, determine the following:
   a. $\vec{a} \cdot \vec{b}$
   b. $\vec{b} \cdot \vec{c}$
   c. $\vec{b} + \vec{c}$
   d. $\vec{a} \cdot (\vec{b} + \vec{c})$
   e. $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$
   f. $(2\vec{a} - 3\vec{b}) \cdot (2\vec{a} + \vec{c})$

9. Given the vectors $\vec{p} = x\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{q} = 3x\vec{i} + 10x\vec{j} + \vec{k}$, determine the following:
   a. the value(s) of x that make these vectors perpendicular
   b. the values(s) of x that make these vectors parallel
10. If $\vec{x} = \vec{i} - 2\vec{j} - \vec{k}$ and $\vec{y} = \vec{i} - \vec{j} - \vec{k}$, determine the value of each of the following:
   a. $3\vec{x} - 2\vec{y}$
   b. $3\vec{x} \cdot 2\vec{y}$
   c. $|\vec{x} - 2\vec{y}|$
   d. $(2\vec{x} - 3\vec{y}) \cdot (\vec{x} + 4\vec{y})$
   e. $2\vec{x} \cdot \vec{y} - 5\vec{y} \cdot \vec{x}$
11. Three forces of 3 N, 4 N, and 5 N act on an object so that the object is in equilibrium. Determine the angle between the largest and smallest forces.
12. A force of 3 N and a force of 4 N act on an object. If these two forces make an angle of 60° to each other, find the resultant and equilibrant of these two forces.
13. The sides of a parallelogram are determined by the vectors $\vec{m} = (2, -3, 5)$ and $\vec{n} = (-1, 7, 5)$. Determine
   a. the larger angle between the diagonals of this parallelogram
   b. the smaller angle between the sides
14. Martina is planning to fly to a town 1000 km due north of her present location. There is a 45 km/h wind blowing from $N30°E$.
   a. If her plane travels at 500 km/h, what direction should the pilot head to reach the destination?
   b. How long will the trip take?
15. Determine the coordinates of a unit vector that is perpendicular to $\vec{a} = (-1, 2, 5)$ and $\vec{b} = (1, 3, 5)$.
16. Clarence leaves a dock, paddling a canoe at 3 m/s. He heads downstream at an angle of 45° to the current, which is flowing at 4 m/s.
   a. How far downstream does he travel in 10 s?
   b. What is the length of time required to cross the river if it is 180 m wide?
17. a. Under what conditions does $(\vec{x} + \vec{y}) \cdot (\vec{x} - \vec{y}) = 0$?
   b. Give a geometrical interpretation of the vectors $\vec{a}$, $\vec{b}$, $\vec{a} + \vec{b}$, and $\vec{a} - \vec{b}$.
18. A lawn roller with a mass of 60 kg is being pulled with a force of 350 N. If the handle of the roller makes an angle of 40° with the ground, what horizontal component of the force is causing the roller to move forward?

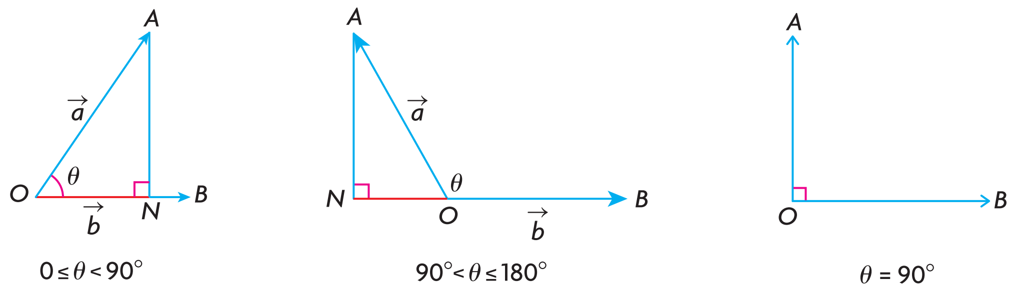
Section 7.5—Scalar and Vector Projections

In the last two sections, the concept of the dot product was discussed, first in geometric form and then in algebraic form. In this section, the dot product will be used along with the concept of **projections**. These concepts are closely related, and each has real significance from both a practical and theoretical point of view.

When two vectors, $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$, are placed tail to tail, and θ is the angle between the vectors, $0° \leq \theta \leq 180°$, the **scalar projection** of $\vec{a}$ on $\vec{b}$ is ON, as shown in the following diagram. The scalar projection can be determined using right triangle trigonometry and can be applied to either geometric or algebraic vectors equally well.

Scalar Projection of $\vec{a}$ on $\vec{b}$

The scalar projection of vector $\vec{a}$ onto $\vec{b}$ is ON, where $ON = |\vec{a}|\cos\theta$.



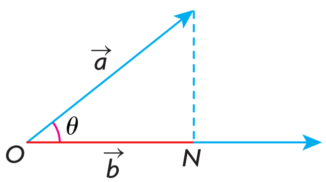
Observations about the Scalar Projection

A number of observations should be made about scalar projections. The scalar projection of $\vec{a}$ on $\vec{b}$ is obtained by drawing a line from the head of vector $\vec{a}$ perpendicular to $\overrightarrow{OB}$, or an extension of $\overrightarrow{OB}$. If the point where this line meets the vector is labelled N, then the scalar projection $\vec{a}$ on $\vec{b}$ is ON. Since ON is a real number, or scalar, and also a projection, it is called a scalar projection. If the angle between two given vectors is such that $0° \leq \theta < 90°$, then the scalar projection is positive; otherwise, it is negative for $90° < \theta \leq 180°$ and 0 if $\theta = 90°$.

The sign of scalar projections should not be surprising, since it corresponds exactly to the sign convention for dot products that we saw in the previous two sections. An important point is that the scalar projection between perpendicular vectors is always 0 because the angle between the vectors is 90° and cos 90° = 0. Another important point is that it is not possible to take the scalar projection of the vector $\vec{a}$ on $\vec{0}$. This would result in a statement involving division by 0, which is meaningless.

Another observation should be made about scalar projections that is not immediately obvious from the given definition. The scalar projection of vector $\vec{a}$ on vector $\vec{b}$ is in general not equal to the scalar projection of vector $\vec{b}$ on vector $\vec{a}$, which can be seen from the following.

Calculating the scalar projection of $\vec{a}$ on $\vec{b}$:



Projection of $\vec{a}$ onto $\vec{b}$

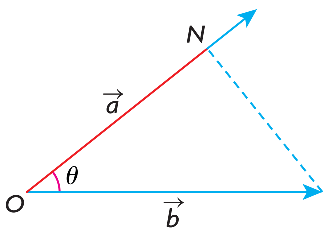
When calculating this projection, what is needed is to solve for $|\vec{a}|\cos\theta$ in the dot product formula.

We know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Rewrite this formula as $\vec{a}\cdot\vec{b} = \left(|\vec{a}|\cos\theta\right)|\vec{b}|$.

Solving for $|\vec{a}|\cos\theta$ gives $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Calculating the scalar projection of $\vec{b}$ on $\vec{a}$:

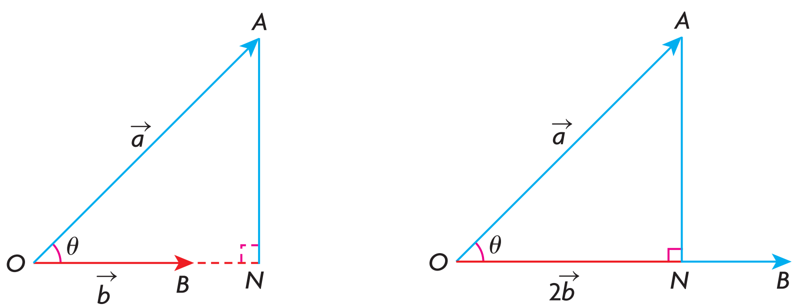


Projection of $\vec{b}$ onto $\vec{a}$

To find the scalar projection of $\vec{b}$ on $\vec{a}$, it is necessary to solve for $|\vec{b}|\cos\theta$ in the dot product formula. This is done in exactly the same way as above, and we find that $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

From this, we can see that, in general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. It is correct to say, however, that these scalar projections are equal if $|\vec{a}| = |\vec{b}|$.

Another observation to make about scalar projections is that the scalar projection of $\vec{a}$ on $\vec{b}$ is independent of the length of $\vec{b}$. This is demonstrated in the following diagram:



From the diagram, we can see that the scalar projection of vector $\vec{a}$ on vector $\vec{b}$ equals ON. If we take the scalar projection of $\vec{a}$ on $2\vec{b}$, this results in the exact same line segment ON.

EXAMPLE 1 Reasoning about the characteristics of the scalar projection

a. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is identical to the scalar projection of $\vec{a}$ on $2\vec{b}$.

b. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is not the same as $\vec{a}$ on $-2\vec{b}$.

Solution

a. The scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $2\vec{b}$ is $\dfrac{\vec{a}\cdot 2\vec{b}}{|2\vec{b}|}$. If we use the properties of the dot product and the fact that $|2\vec{b}| = 2|\vec{b}|$, this quantity can be written as

$\dfrac{2(\vec{a}\cdot\vec{b})}{2|\vec{b}|}$, and then simplified to $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

From this, we see that what was shown geometrically is verified algebraically.

b. As before, the scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $-2\vec{b}$ is $\dfrac{\vec{a}\cdot(-2\vec{b})}{|-2\vec{b}|}$. Using the same approach as above and recognizing that $|-2\vec{b}| = 2|\vec{b}|$, this can be rewritten as

$\dfrac{\vec{a}\cdot(-2\vec{b})}{|-2\vec{b}|} = \dfrac{-2\vec{a}\cdot\vec{b}}{2|\vec{b}|} = \dfrac{-(\vec{a}\cdot\vec{b})}{|\vec{b}|}$.

In this case, the direction of the vector $-2\vec{b}$ changes the scalar projection to the opposite sign from the projection of $\vec{a}$ on $\vec{b}$.

The following example shows how to calculate scalar projections involving algebraic vectors. All the properties applying to geometric vectors also apply to algebraic vectors.

EXAMPLE 2 Selecting a strategy to calculate the scalar projection involving algebraic vectors

For the vectors $\vec{a} = (-3, 4, 5\sqrt{3})$ and $\vec{b} = (-2, 2, -1)$, calculate each of the following scalar projections:

a. $\vec{a}$ on $\vec{b}$ b. $\vec{b}$ on $\vec{a}$

Solution

a. The required scalar projection is $|\vec{a}|\cos\theta$ and, since $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, as before, $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

We start by calculating $\vec{a}\cdot\vec{b}$.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= -3(-2) + 4(2) + 5\sqrt{3}(-1)\\ &= 14 - 5\sqrt{3}\\ &\doteq 5.34\end{aligned}$$

Since $|\vec{b}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$,

$$|\vec{a}|\cos\theta \doteq \frac{5.34}{3} \doteq 1.78$$

The scalar projection of $\vec{a}$ on $\vec{b}$ is approximately 1.78.

b. In this case, the required scalar projection is $|\vec{b}|\cos\theta$. Solving as in the solution to part a. $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Since $|\vec{a}| = (-3)^2 + (4)^2 + (5\sqrt{3})^2 = 10$,

$$|\vec{b}|\cos\theta \doteq \frac{5.34}{10} \doteq 0.53$$

The scalar projection of $\vec{b}$ on $\vec{a}$ is approximately 0.53.

Calculating Scalar Projections

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

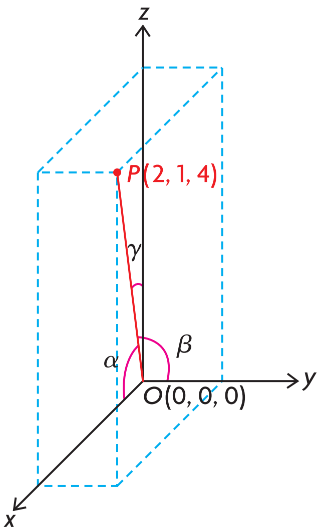
The scalar projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

In general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

Scalar projections are sometimes used to calculate the angle that a position vector $\overrightarrow{OP}$ makes with each of the positive coordinate axes. This concept is illustrated in the next example.

EXAMPLE 3 Selecting a strategy to determine the direction angles of a vector in R^3

Determine the angle that the vector $\overrightarrow{OP} = (2, 1, 4)$ makes with each of the coordinate axes.



Solution

To calculate the required **direction angles**, it is necessary to project $\overrightarrow{OP}$ on each of the coordinate axes. To carry out the calculation, we use the standard basis vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$ so that $\overrightarrow{OP}$ can be projected along the x-axis, y-axis, and z-axis, respectively. We define α as the angle between $\overrightarrow{OP}$ and the positive x-axis, β as the angle between $\overrightarrow{OP}$ and the positive y-axis, and γ as the angle between $\overrightarrow{OP}$ and the positive z-axis.

Calculating α*:*

To calculate the angle that $\overrightarrow{OP}$ makes with the x-axis, we start by writing $\overrightarrow{OP} \cdot \vec{i} = |\overrightarrow{OP}||\vec{i}| \cos \alpha$, which implies $\cos \alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$.

Since $|\overrightarrow{OP}| = \sqrt{21}$ and $|\vec{i}| = 1$, we substitute to find

$$\cos \alpha = \frac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$$

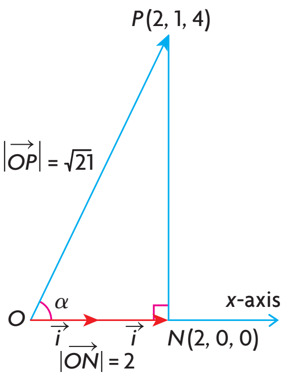
$$\cos \alpha = \frac{(2, 1, 4) \cdot (1, 0, 0)}{\sqrt{21}(1)}$$

$$\cos \alpha = \frac{2}{\sqrt{21}}$$

Thus, $\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{21}}\right)$ and $\alpha \doteq 64.1°$.

Therefore, the angle that $\overrightarrow{OP}$ makes with the x-axis is approximately $64.1°$. In its simplest terms, the cosine of the required angle α is the scalar projection of $\overrightarrow{OP}$ on $\vec{i}$, divided by $|\overrightarrow{OP}|$—that is, $\cos \alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}|} = \dfrac{2}{\sqrt{21}}$.

This angle is illustrated in the following diagram:



Calculating β and γ*:*
If we use the same procedure, we can also calculate β and γ, the angles that $\overrightarrow{OP}$ makes with the y-axis and z-axis, respectively.

Thus, $\cos\beta = \frac{(2, 1, 4) \cdot (0, 1, 0)}{\sqrt{21}} = \frac{1}{\sqrt{21}}$

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right), \beta \doteq 77.4^\circ$$

Similarly, $\cos\gamma = \frac{4}{\sqrt{21}}, \gamma \doteq 29.2^\circ$

Therefore, $\overrightarrow{OP}$ makes angles of 64.1°, 77.4°, and 29.2° with the positive x-axis, y-axis and z-axis, respectively.

In our example, specific numbers were used, but the calculation is identical if we consider $\overrightarrow{OP} = (a, b, c)$ and develop a formula for the required direction angles. The cosines of the angles are referred to as the **direction cosines** of α, β, and γ.

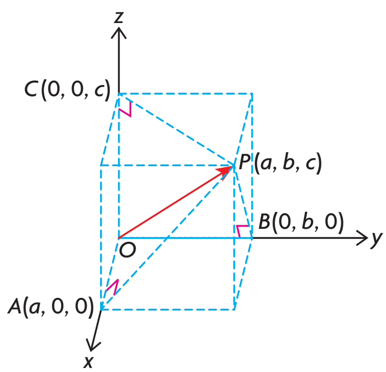
Direction Cosines for $\overrightarrow{OP} = (a, b, c)$

If α, β, and γ are the angles that $\overrightarrow{OP}$ makes with the positive x-axis, y-axis, and z-axis, respectively, then

$$\cos\alpha = \frac{(a, b, c) \cdot (1, 0, 0)}{|\overrightarrow{OP}|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } \cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

These angles can be visualized by constructing a rectangular box and drawing in the appropriate projections. If we are calculating α, the angle that $\overrightarrow{OP}$ makes with the positive x-axis, the projection of $\overrightarrow{OP}$ on the x-axis is just a, and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$, so $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$.
We calculate $\cos\beta$ and $\cos\gamma$ in the same way.



EXAMPLE 4 Calculating a specific direction angle

For the vector $\vec{OP} = (-2\sqrt{2}, 4, -5)$, determine the direction cosine and the corresponding angle that this vector makes with the positive z-axis.

Solution

We can use the formula to calculate γ.

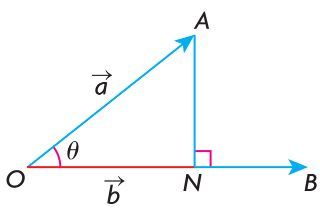
$$\cos\gamma = \frac{-5}{\sqrt{(-2\sqrt{2})^2 + (4)^2 + (-5)^2}} = \frac{-5}{\sqrt{49}} = \frac{-5}{7} \doteq -0.7143$$

and $\gamma \doteq 135.6°$

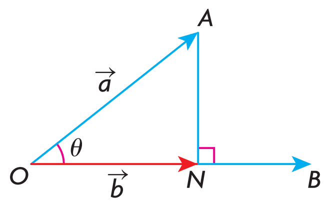
Examining Vector Projections

Thus far, we have calculated scalar projections of a vector onto a vector. This computation can be modified slightly to find the corresponding vector projection of a vector on a vector.

The calculation of the vector projection of $\vec{a}$ on $\vec{b}$ is just the corresponding scalar projection of $\vec{a}$ on $\vec{b}$ multiplied by $\frac{\vec{b}}{|\vec{b}|}$. The expression $\frac{\vec{b}}{|\vec{b}|}$ is a unit vector pointing in the direction of $\vec{b}$.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

Vector Projection of $\vec{a}$ on $\vec{b}$

vector projection of $\vec{a}$ on $\vec{b}$

= (scalar projection of $\vec{a}$ on $\vec{b}$) (unit vector in the direction of $\vec{b}$)

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b},\ \vec{b} \neq \vec{0}$$

EXAMPLE 5 Connecting a scalar projection to its corresponding vector projection

Find the vector projection of $\overrightarrow{OA} = (4, 3)$ on $\overrightarrow{OB} = (4, -1)$.

Solution

The formula for the scalar projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ is $\dfrac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|}$.

$$ON = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{(4, 3) \cdot (4, -1)}{\sqrt{(4)^2 + (-1)^2}}$$
$$= \frac{13}{\sqrt{17}}$$

The vector projection, $\overrightarrow{ON}$, is found by multiplying ON by the unit vector $\dfrac{\overrightarrow{OB}}{|\overrightarrow{OB}|}$.

Since $|\overrightarrow{OB}| = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$,

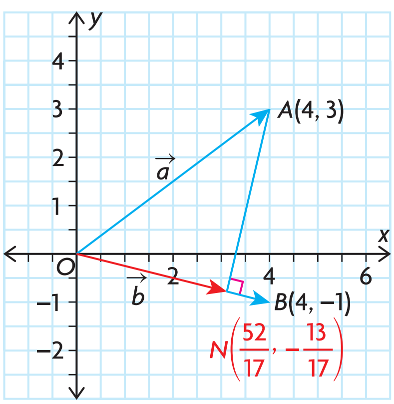
$$\frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{1}{\sqrt{17}}(4, -1)$$

The required vector projection is

$\overrightarrow{ON} = (ON)$(a unit vector in the same direction as $\overrightarrow{OB}$)

$$\overrightarrow{ON} = \frac{13}{\sqrt{17}}\left(\frac{1}{\sqrt{17}}(4, -1)\right)$$
$$= \frac{13}{17}(4, -1)$$
$$= \left(\frac{52}{17}, -\frac{13}{17}\right)$$

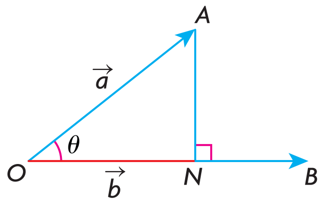
The vector projection $\overrightarrow{ON}$ is shown in red in the following diagram:



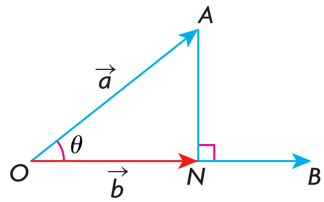
IN SUMMARY

Key Idea

- A projection of one vector onto another can be either a scalar or a vector. The difference is the vector projection has a direction.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

Need to Know

- The scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = |\vec{a}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$
- The direction cosines for $\overrightarrow{OP} = (a, b, c)$ are

  $\cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $\cos\beta = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$,

  $\cos\gamma = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$, where α, β, and γ are the direction angles between the position vector $\overrightarrow{OP}$ and the positive x-axis, y-axis and z-axis, respectively.

Exercise 7.5

PART A

1. a. The vector $\vec{a} = (2, 3)$ is projected onto the x-axis. What is the scalar projection? What is the vector projection?

   b. What are the scalar and vector projections when $\vec{a}$ is projected onto the y-axis?

2. Explain why it is not possible to obtain either a scalar projection or a vector projection when a nonzero vector $\vec{x}$ is projected on $\vec{0}$.

3. Consider two nonzero vectors, $\vec{a}$ and $\vec{b}$, that are perpendicular to each other. Explain why the scalar and vector projections of $\vec{a}$ on $\vec{b}$ must be 0 and $\vec{0}$, respectively. What are the scalar and vector projections of $\vec{b}$ on $\vec{a}$?

4. Draw two vectors, $\vec{p}$ and $\vec{q}$. Draw the scalar and vector projections of $\vec{p}$ on $\vec{q}$. Show, using your diagram, that these projections are not necessarily the same as the scalar and vector projections of $\vec{q}$ on $\vec{p}$.

5. Using the formulas in this section, determine the scalar and vector projections of $\overrightarrow{OP} = (-1, 2, -5)$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$. Explain how you could have arrived at the same answer without having to use the formulas.

PART B

6. a. For the vectors $\vec{p} = (3, 6, -22)$ and $\vec{q} = (-4, 5, -20)$, determine the scalar and vector projections of $\vec{p}$ on $\vec{q}$.
   b. Determine the direction angles for $\vec{p}$.

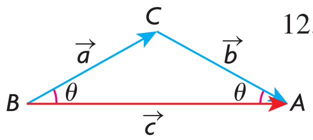
K 7. For each of the following, determine the scalar and vector projections of $\vec{x}$ on $\vec{y}$.
   a. $\vec{x} = (1, 1), \vec{y} = (1, -1)$
   b. $\vec{x} = (2, 2\sqrt{3}), \vec{y} = (1, 0)$
   c. $\vec{x} = (2, 5), \vec{y} = (-5, 12)$

8. a. Determine the scalar and vector projections of $\vec{a} = (-1, 2, 4)$ on each of the three axes.
   b. What are the scalar and vector projections of $m(-1, 2, 4)$ on each of the three axes?

C 9. a. Given the vector $\vec{a}$, show with a diagram that the vector projection of $\vec{a}$ on $\vec{a}$ is $\vec{a}$ and that the scalar projection of $\vec{a}$ on $\vec{a}$ is $|\vec{a}|$.
   b. Using the formulas for scalar and vector projections, explain why the results in part a. are correct if we use $\theta = 0°$ for the angle between the two vectors.

10. a. Using a diagram, show that the vector projection of $-\vec{a}$ on $\vec{a}$ is $-\vec{a}$.
   b. Using the formula for determining scalar projections, show that the result in part a. is true.

A 11. a. Find the scalar and vector projections of $\overrightarrow{AB}$ along each of the axes if A has coordinates $(1, 2, 2)$ and B has coordinates $(-1, 3, 4)$.
   b. What angle does $\overrightarrow{AB}$ make with the y-axis?



12. In the diagram shown, $\triangle ABC$ is an isosceles triangle where $|\vec{a}| = |\vec{b}|$.
    a. Draw the scalar projection of $\vec{a}$ on $\vec{c}$.
    b. Relocate $\vec{b}$, and draw the scalar projection of $\vec{b}$ on $\vec{c}$.
    c. Explain why the scalar projection of $\vec{a}$ on $\vec{c}$ is the same as the scalar projection of $\vec{b}$ on $\vec{c}$.
    d. Does the vector projection of $\vec{a}$ on $\vec{c}$ equal the vector projection of $\vec{b}$ on $\vec{c}$?
13. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 10$ and $|\vec{b}| = 12$, and the angle between them is 135°.
    a. Show that the scalar projection of $\vec{a}$ on $\vec{b}$ does not equal the scalar projection of $\vec{b}$ on $\vec{a}$.
    b. Draw diagrams to illustrate the corresponding vector projections associated with part a.
14. You are given the vector $\overrightarrow{OD} = (-1, 2, 2)$ and the three points, $A(-2, 1, 4)$, $B(1, 3, 3)$, and $C(-6, 7, 5)$.
    a. Calculate the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$.
    b. Verify computationally that the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$ added to the scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ equals the scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$.
    c. Explain why this same result is also true for the corresponding vector projections.

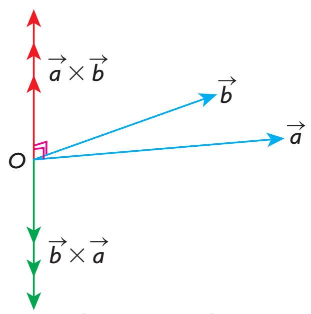
T

15. a. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
    b. Determine the coordinates of a vector $\overrightarrow{OP}$ that makes an angle of 30° with the y-axis, 60° with the z-axis, and 90° with the x-axis.
    c. In Example 3, it was shown that, in general, the direction angles do not always add to 180°—that is, $\alpha + \beta + \gamma \neq 180°$. Under what conditions, however, must the direction angles always add to 180°?

PART C

16. A vector in R^3 makes equal angles with the coordinate axes. Determine the size of each of these angles if the angles are
    a. acute
    b. obtuse
17. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.
18. Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not collinear. The sum of the direction angles of each vector is 180°. Draw diagrams to illustrate possible positions of points A and B.

Section 7.6—The Cross Product of Two Vectors



The Cross Product (Vector Product)

In the previous three sections, the dot product along with some of its applications was discussed. In this section, a second product called the **cross product**, denoted as $\vec{a} \times \vec{b}$, is introduced. The cross product is sometimes referred to as a **vector product** because, when it is calculated, the result is a vector and not a scalar. As we shall see, the cross product can be used in physical applications but also in the understanding of the geometry of R^3.

If we are given two vectors, $\vec{a}$ and $\vec{b}$, and wish to calculate their cross product, what we are trying to find is a particular vector that is perpendicular to each of the two given vectors. As will be observed, if we consider two nonzero, noncollinear vectors, there is an infinite number of vectors perpendicular to the two vectors. If we want to determine the cross product of these two vectors, we choose just one of these perpendicular vectors as our answer. Finding the cross product of two vectors is shown in the following example.

EXAMPLE 1

Calculating the cross product of two vectors

Given the vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 3, -1)$, determine $\vec{a} \times \vec{b}$.

Solution

When calculating $\vec{a} \times \vec{b}$, we are determining a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. We start by letting this vector be $\vec{v} = (x, y, z)$.

Since $\vec{a} \cdot \vec{v} = 0$, $(1, 1, 0) \cdot (x, y, z) = 0$ and $x + y = 0$.

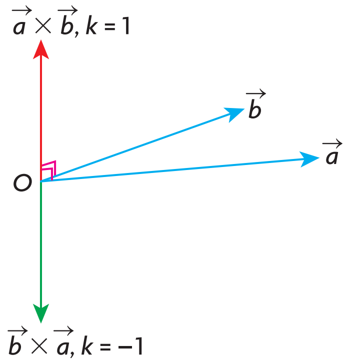
In the same way, since $\vec{b} \cdot \vec{v} = 0$, $(1, 3, -1) \cdot (x, y, z) = 0$ and $x + 3y - z = 0$.

Finding of the cross product of $\vec{a}$ and $\vec{b}$ requires solving a system of two equations in three variables which, under normal circumstances, has an infinite number of solutions. We will eliminate a variable and use substitution to find a solution to this system.

① $x + y = 0$

② $x + 3y - z = 0$

Subtracting eliminates x, $-2y + z = 0$, or $z = 2y$. If we substitute $z = 2y$ in equation ②, it will be possible to express x in terms of y. Doing so gives $x + 3y - (2y) = 0$ or $x = -y$. Since x and z can both be expressed in terms of y, we write the solution as $(-y, y, 2y) = y(-1, 1, 2)$. The solution to this system



is any vector of the form $y(-1, 1, 2)$. It can be left in this form, but is usually written as $k(-1, 1, 2)$, $k \in \mathbf{R}$, where k is a parameter representing any real value. The parameter k indicates that there is an infinite number of solutions and that each of them is a scalar multiple of $(-1, 1, 2)$. In this case, the cross product is defined to be the vector where $k = 1$—that is, $(-1, 1, 2)$. The choosing of $k = 1$ simplifies computation and makes sense mathematically, as we will see in the next section. It should also be noted that the cross product is a vector and, as stated previously, is sometimes called a vector product.

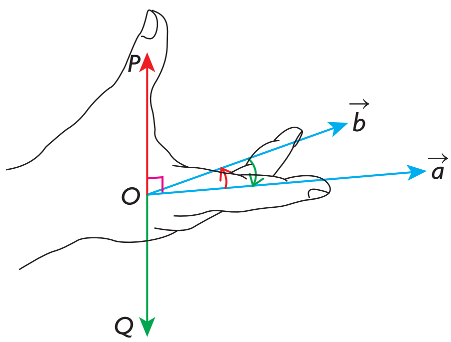
Deriving a Formula for the Cross Product

What is necessary, to be more efficient in calculating $\vec{a} \times \vec{b}$, is a formula.

The vector $\vec{a} \times \vec{b}$ is a vector that is perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$. An infinite number of vectors satisfy this condition, all of which are scalar multiples of each other, but the cross product is one that is chosen in the simplest possible way, as will be seen when the formula is derived below. Another important point to understand about the cross product is that it exists only in R^3. It is not possible to take two noncollinear vectors in R^2 and construct a third vector perpendicular to the two vectors, because this vector would be outside the given plane.

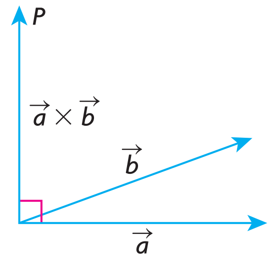
It is also difficult, at times, to tell whether we are calculating $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$. The formula, properly applied, will do the job without difficulty. There are times when it is helpful to be able to identify the cross product without using a formula. From the diagram below, $\vec{a} \times \vec{b}$ is pictured as a vector perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, and, when looking down the axis from P on $\vec{a} \times \vec{b}$, $\vec{a}$ would have to be rotated *counterclockwise* in order to be collinear with $\vec{b}$. In other words, $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

The vector $\vec{b} \times \vec{a}$, the opposite to $\vec{a} \times \vec{b}$, is again perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, but, when looking from Q down the axis formed by $\vec{b} \times \vec{a}$, $\vec{a}$ would have to be rotated *clockwise* in order to be collinear with $\vec{b}$.



Definition of a Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 (3-space) is the vector that is perpendicular to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.
The vector $\vec{b} \times \vec{a}$ is the opposite of $\vec{a} \times \vec{b}$ and points in the opposite direction.



To develop a formula for $\vec{a} \times \vec{b}$, we follow a procedure similar to that followed in Example 1. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, and let $\vec{v} = (x, y, z)$ be the vector that is perpendicular to $\vec{a}$ and $\vec{b}$.

So, ① $\vec{a} \cdot \vec{v} = (a_1, a_2, a_3) \cdot (x, y, z) = a_1x + a_2y + a_3z = 0$
and ② $\vec{b} \cdot \vec{v} = (b_1, b_2, b_3) \cdot (x, y, z) = b_1x + b_2y + b_3z = 0$

As before, we have a system of two equations in three unknowns, which we know from before has an infinite number of solutions. To solve this system of equations, we will multiply the first equation by b_1 and the second equation by a_1 and then subtract.

①$\times\, b_1 \rightarrow$ ③ $b_1a_1x + b_1a_2y + b_1a_3z = 0$
②$\times\, a_1 \rightarrow$ ④ $a_1b_1x + a_1b_2y + a_1b_3z = 0$

Subtracting ③ and ④ eliminates x. Move the z-terms to the right.

$(b_1a_2 - a_1b_2)y = (a_1b_3 - b_1a_3)z$

Multiplying each side by -1 and rearranging gives the desired result:

$(a_1b_2 - b_1a_2)y = (b_1a_3 - a_1b_3)z$

Now

$$\frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1}$$

If we carry out an identical procedure and eliminate z from the system of equations, we have the following:

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3}$$

If we combine the two statements and set them equal to a constant k, we have

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1} = k$$

Note that we can make these fractions equal to k because every proportion can be made equal to a constant k. (For example, if $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = k$, then k could be either $\frac{1}{2}$ or $\frac{10}{20}$ or any nonzero multiple of the form $\frac{1n}{2n}$.) This expression gives us a general form for a vector that is perpendicular to $\vec{a}$ and $\vec{b}$. The cross product, $\vec{a} \times \vec{b}$, is defined to occur when $k = 1$, and $\vec{b} \times \vec{a}$ occurs when $k = -1$.

Formula for Calculating the Cross Product of Algebraic Vectors

$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, $k \in \mathbf{R}$.

If $k = 1$, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

If $k = -1$, then $\vec{b} \times \vec{a} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$

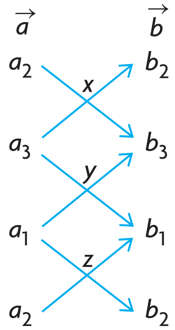
It is not easy to remember this formula for calculating the cross product of two vectors, so we develop a procedure, or a way of writing them, so that the memory work is removed from the calculation.

Method of Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

1. List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, and a_2 below each other as shown.
2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with b_2 and then writing b_3, b_1, and b_2 in exactly the same way as the components of $\vec{a}$.
3. The required formula is now a matter of following the arrows and doing the calculation. To find the x component, for example, we take the down product a_2b_3 and subtract the up product a_3b_2 from it to get $a_2b_3 - a_3b_2$.

(*continued*)

The other components are calculated in exactly the same way, and the formula for each component is listed below.



$x = a_2b_3 - a_3b_2$

$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

INVESTIGATION

A. Given two vectors $\vec{a} = (2, 4, 6)$ and $\vec{b} = (-1, 2, -5)$ calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

B. How are the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ related? Write an expression that relates $\vec{a} \times \vec{b}$ with $\vec{b} \times \vec{a}$.

C. Will the expression you wrote in part B be true for any pair of vectors in R^3? Explain.

D. Using the two vectors given in part A and a third vector $\vec{c} = (4, 3, -1)$ calculate:

i. $\vec{a} \times (\vec{b} + \vec{c})$ ii. $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

E. Compare your results from i and ii in part D. What property does this demonstrate? Write an equivalent expression for $\vec{b} \times (\vec{a} + \vec{c})$ and confirm it using the appropriate calculations.

F. Choose any 3 vectors in R^3 and demonstrate that the property you identified in part E holds for your vectors.

G. Using the three vectors given calculate:

i. $(\vec{a} \times \vec{b}) \times \vec{c}$ ii. $\vec{a} \times (\vec{b} \times \vec{c})$

H. Compare your results from i and ii in part G. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

I. Choose any vector that is collinear with $\vec{a}$ (that is, any vector of the form $k\vec{a}$ $k \in \mathbf{R}$). Calculate $\vec{a} \times k\vec{a}$. Repeat using a different value for k. What can you conclude?

EXAMPLE 2 Calculating cross products

If $\vec{p} = (-1, 3, 2)$ and $\vec{q} = (2, -5, 6)$, calculate $\vec{p} \times \vec{q}$ and $\vec{q} \times \vec{p}$.

Solution

Let $\vec{p} \times \vec{q} = (x, y, z)$. The vectors $\vec{p}$ and $\vec{q}$ are listed in column form, with $\vec{p}$ on the left and $\vec{q}$ on the right, starting from the second component and working down.

$$
\begin{array}{ccc}
\vec{p} & & \vec{q} \\
3 & & -5 \\
 & x & \\
2 & & 6 \\
 & y & \\
-1 & & 2 \\
 & z & \\
3 & & -5
\end{array}
\qquad
\begin{array}{l}
x = 3(6) - (2)(-5) = 28 \\[2em]
y = 2(2) - (-1)(6) = 10 \\[2em]
z = -1(-5) - 3(2) = -1
\end{array}
$$

$\vec{p} \times \vec{q} = (28, 10, -1)$

As already mentioned, $\vec{q} \times \vec{p}$ is the opposite of $\vec{p} \times \vec{q}$, so $\vec{q} \times \vec{p} = -1(28, 10, -1) = (-28, -10, 1)$. It is not actually necessary to calculate $\vec{q} \times \vec{p}$. All that is required is to calculate $\vec{p} \times \vec{q}$ and take the opposite vector to get $\vec{q} \times \vec{p}$.

After completing the calculation of the cross product, the answer should be verified to see if it is perpendicular to the given vectors using the dot product.

Check:

$(28, 10, -1) \cdot (-1, 3, 2) = -28 + 30 - 2 = 0$ and

$(28, 10, -1) \cdot (2, -5, 6) = 56 - 50 - 6 = 0$

There are a number of important properties of cross products that are worth noting. Some of these properties will be verified in the exercises.

Properties of the Cross Product

Let $\vec{p}$, $\vec{q}$, and $\vec{r}$ be three vectors in R^3, and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$,

Distributive law for vector multiplication: $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$,

Scalar law for vector multiplication: $k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$, The first property is one that we have seen in this section and is the first instance we have seen where the commutative property for multiplication has failed. Normally, we expect that the order of multiplication does not affect the product. In this case, changing the order of multiplication does change the result. The other two listed results are results that produce exactly what would be expected, and they will be used in this set of exercises and beyond.

IN SUMMARY

Key Idea

- The cross product $\vec{a} \times \vec{b}$, between two vectors $\vec{a}$ and $\vec{b}$, results in a third vector that is perpendicular to the plane in which the given vectors lie.

Need to Know

- $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

Exercise 7.6

PART A

1. The two vectors $\vec{a}$ and $\vec{b}$ are vectors in R^3, and $\vec{a} \times \vec{b}$ is calculated.
   a. Using a diagram, explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$.
   b. Draw the parallelogram determined by $\vec{a}$ and $\vec{b}$, and then draw the vector $\vec{a} + \vec{b}$. Give a simple explanation of why $(\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$.
   c. Why is it true that $(\vec{a} - \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$? Explain.
2. For vectors in R^3, explain why the calculation $(\vec{a} \cdot \vec{b})(\vec{a} \times \vec{b}) = 0$ is meaningless. (Consider whether or not it is possible for the left side to be a scalar.)

PART B

3. For each of the following calculations, say which are possible for vectors in R^3 and which are meaningless. Give a brief explanation for each.
   a. $\vec{a} \cdot (\vec{b} \times \vec{c})$
   b. $(\vec{a} \cdot \vec{b}) \times \vec{c}$
   c. $(\vec{a} \times \vec{b}) \cdot (\vec{c} + \vec{d})$
   d. $(\vec{a} \cdot \vec{b})(\vec{c} \times \vec{d})$
   e. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$
   f. $\vec{a} \times \vec{b} + \vec{c}$

K 4. Calculate the cross product for each of the following pairs of vectors, and verify your answer by using the dot product.

a. $(2, -3, 5)$ and $(0, -1, 4)$
b. $(2, -1, 3)$ and $(3, -1, 2)$
c. $(5, -1, 1)$ and $(2, 4, 7)$
d. $(1, 2, 9)$ and $(-2, 3, 4)$
e. $(-2, 3, 3)$ and $(1, -1, 0)$
f. $(5, 1, 6)$ and $(-1, 2, 4)$

5. If $(-1, 3, 5) \times (0, a, 1) = (-2, 1, -1)$, determine a.

6. a. Calculate the vector product for $\vec{a} = (0, 1, 1)$ and $\vec{b} = (0, 5, 1)$.
b. Explain geometrically why it makes sense for vectors of the form $(0, b, c)$ and $(0, d, e)$ to have a cross product of the form $(a, 0, 0)$.

7. a. For the vectors $(1, 2, 1)$ and $(2, 4, 2)$, show that their vector product is $\vec{0}$.
b. In general, show that the vector product of two collinear vectors, (a, b, c) and (ka, kb, kc), is always $\vec{0}$.

8. In the discussion, it was stated that $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$ for vectors in R^3. Verify that this rule is true for the following vectors.
a. $\vec{p} = (1, -2, 4)$, $\vec{q} = (1, 2, 7)$, and $\vec{r} = (-1, 1, 0)$
b. $\vec{p} = (4, 1, 2)$, $\vec{q} = (3, 1, -1)$, and $\vec{r} = (0, 1, 2)$

A 9. Verify each of the following:
a. $\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$
b. $\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$
c. $\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$

C 10. Show algebraically that $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot \vec{a} = 0$. What is the meaning of this result?

11. You are given the vectors $\vec{a} = (2, 0, 0)$, $\vec{b} = (0, 3, 0)$, $\vec{c} = (2, 3, 0)$, and $\vec{d} = (4, 3, 0)$.
a. Calculate $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$.
b. Calculate $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.
c. Without doing any calculations (that is, by visualizing the four vectors and using properties of cross products), say why $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d}) = \vec{0}$.

PART C

12. Show that the cross product is not associative by finding vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ such that $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

T 13. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$ is true.

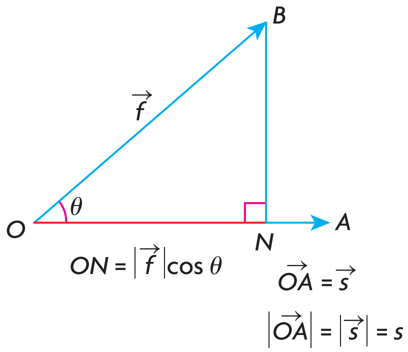
Section 7.7—Applications of the Dot Product and Cross Product

In the previous four sections, the dot product and cross product were discussed in some detail. In this section, some physical and mathematical applications of these concepts will be introduced to give a sense of their usefulness in both physical and mathematical situations.

Physical Application of the Dot Product

When a force is acting on an object so that the object is moved from one point to another, we say that the force has done work. Work is defined as the product of the distance an object has been displaced and the component of the force along the line of displacement.

In the following diagram, $\overrightarrow{OB}$ represents a constant force, $\vec{f}$, acting on an object at O so that this force moves the object from O to A. We will call the distance that the object is displaced s, which is a scalar, where we are assuming that $\vec{s} = \overrightarrow{OA}$ and $s = |\overrightarrow{OA}|$. The scalar projection of $\vec{f}$ on $\overrightarrow{OA}$ equals ON, or $|\vec{f}|\cos\theta$, which is the same calculation for the scalar projection that was done earlier. (This is called the scalar component of $\overrightarrow{OB}$ on $\overrightarrow{OA}$.) The work, W, done by $\vec{f}$ in moving the object is calculated as $W = (|\vec{f}|\cos\theta)(|\overrightarrow{OA}|) = (ON)(s) = \vec{f}\cdot\vec{s}$. As explained before, the force $\vec{f}$ is measured in newtons (N), the displacement is measured in metres (m), and the unit for work is newton-metres, or joules (J). When a 1 N force moves an object 1 m, the amount of work done is 1 J.



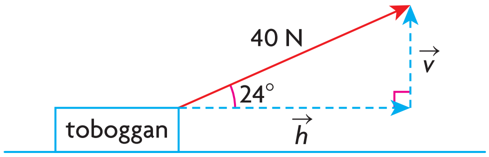
Formula for the Calculation of Work

$W = \vec{f}\cdot\vec{s}$, where $\vec{f}$ is the force acting on an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is the work done, measured in joules (J).

EXAMPLE 1 Using the dot product to calculate work

Marianna is pulling her daughter in a toboggan and is exerting a force of 40 N, acting at 24° to the ground. If Marianna pulls the child a distance of 100 m, how much work was done?

Solution



To solve this problem, the 40 N force has been resolved into its vertical and horizontal components. The horizontal component $\vec{h}$ tends to move the toboggan forward, while the vertical component $\vec{v}$ is the force that tends to lift the toboggan.

From the diagram, $|\vec{h}| = 40 \cos 24°$

$$\doteq 40(0.9135)$$

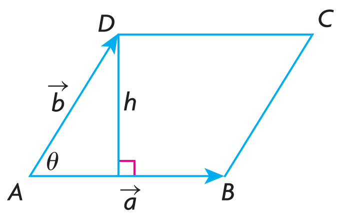
$$\doteq 36.54 \text{ N}$$

The amount of work done is $W \doteq (36.54)(100) \doteq 3654$ J. Therefore, the work done by Marianna is approximately 3654 J.

Geometric Application of the Cross Product

The cross product of two vectors is interesting because calculations involving the cross product can be applied in a number of different ways, giving us results that are important from both a mathematical and physical perspective.

The cross product of two vectors, $\vec{a}$ and $\vec{b}$, can be used to calculate the area of a parallelogram. For any parallelogram, $ABCD$, it is possible to develop a formula for its area, where $\vec{a}$ and $\vec{b}$ are vectors determining its sides and h is its height.



$$\begin{aligned} \text{Area} &= |\vec{a}|h \\ &= |\vec{a}|(|\vec{b}|\sin\theta) \\ &= |\vec{a}||\vec{b}|\sin\theta \end{aligned}$$

where $\sin\theta = \dfrac{h}{|\vec{b}|}$ or $h = |\vec{b}|\sin\theta$

It can be proven that this formula for the area is equal to $|\vec{a} \times \vec{b}|$. That is, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.

Theorem: For two vectors, $\vec{a}$ and $\vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between the two vectors.

Proof: The formula for the cross product is

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Therefore, $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

The right-hand side is expanded and then factored to give

$$|\vec{a} \times \vec{b}|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

This formula can be simplified by making the following substitutions:

$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$, $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$, and
$|\vec{a}||\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

Thus, $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (Factor)

$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$ (Substitution)

$|\vec{a} \times \vec{b}| = \pm |\vec{a}||\vec{b}|\sin\theta$

But since $\sin\theta \geq 0$ for $0^\circ \leq \theta \leq 180^\circ$,

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

This gives us the required formula for the area of a parallelogram, which is equivalent to the magnitude of the cross product between the vectors that define the parallelogram.

EXAMPLE 2 Solving area problems using the cross product

a. Determine the area of the parallelogram determined by the vectors $\vec{p} = (-1, 5, 6)$ and $\vec{q} = (2, 3, -1)$.

b. Determine the area of the triangle formed by the points $A(-1, 2, 1)$, $B(-1, 0, 0)$, and $C(3, -1, 4)$.

Solution

a. The cross product is

$$\begin{aligned}\vec{p} \times \vec{q} &= (5(-1) - 3(6), 6(2) - (-1)(-1), -1(3) - 2(5))\\ &= (-23, 11, -13)\end{aligned}$$

The required area is determined by $|\vec{p} \times \vec{q}|$.

$$\sqrt{(-23)^2 + 11^2 + (-13)^2} = \sqrt{529 + 121 + 169} = \sqrt{819}$$
$$\doteq 28.62 \text{ square units}$$

b. We start by constructing position vectors equal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus,

$$\overrightarrow{AB} = (-1 - (-1), 0 - 2, 0 - 1) = (0, -2, -1) \text{ and } \overrightarrow{AC} = (4, -3, 3)$$

Calculating,

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (-2(3) - (-1)(-3), -1(4) - 0(3), 0(-3) - (-2)(4)) \\ &= (-9, -4, 8)\end{aligned}$$

And

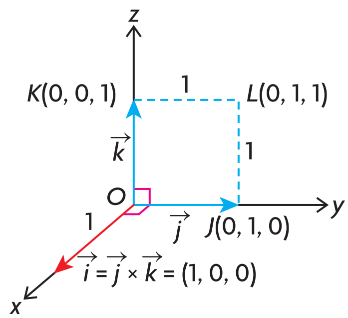
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-9)^2 + (-4)^2 + (8)^2} = \sqrt{161}$$

Therefore, the area of $\triangle ABC$ is one half of the area of the parallelogram formed by vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which is $\frac{1}{2}\sqrt{161} \doteq 6.34$ square units.

This connection between the magnitude of the cross product and area allows us further insight into relationships in R^3. This calculation makes a direct and precise connection between the length of the cross product and the area of the parallelogram formed by two vectors. These two vectors can be anywhere in 3-space, not necessarily in the plane. As well, it also allows us to determine, in particular cases, the cross product of two vectors without having to carry out any computation.

EXAMPLE 3 Reasoning about a cross product involving the standard unit vectors

Without calculating, explain why the cross product of $\vec{j}$ and $\vec{k}$ is $\vec{i}$—that is, $\vec{j} \times \vec{k} = \vec{i}$.



Solution

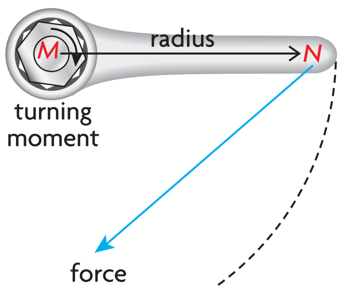
As shown in the diagram, the area of square $OJLK$ is 1. The cross product, $\vec{j} \times \vec{k}$, is a vector perpendicular to the plane determined by $\vec{j}$ and $\vec{k}$, and must therefore lie along either the positive or negative x-axis. Using the definition of the cross product, and knowing that these vectors form a right-handed system, the only possibility is that the cross product must then lie along the *positive* x-axis. The length of the cross product must equal the area of the square $OJLK$, which is 1. So, the required cross product is $\vec{i}$ since $|\vec{i}| = 1$.

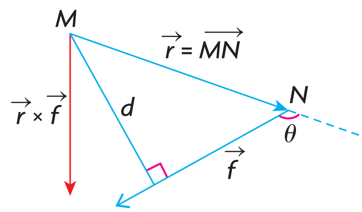
Using the same kind of reasoning, it is interesting to note that $\vec{k} \times \vec{j}$ is $-\vec{i}$, which could be determined by using the definition of a right-handed system and verified by calculation.

Physical Application of the Cross Product

The cross product can also be used in the consideration of forces that involve rotation, or turning about a point or an axis. The rotational or turning effect of a force is something that is commonly experienced in everyday life. A typical example might be the tightening or loosening of a nut using a wrench. A second example is the application of force to a bicycle pedal to make the crank arm rotate. The simple act of opening a door by pushing or pulling on it is a third example of how force can be used to create a turning effect. In each of these cases, there is rotation about either a point or an axis.

In the following situation, a bolt with a right-hand thread is being screwed into a piece of wood by a wrench, as shown. A force $\vec{f}$ is applied to the wrench at point N and is rotating about point M. The vector $\vec{r} = \overrightarrow{MN}$ is the position vector of N with respect to M—that is, it defines the position of N relative to M.





The torque, or the turning effect, of the force $\vec{f}$ about the point M is defined to be the vector $\vec{r} \times \vec{f}$. This vector is perpendicular to the plane formed by the vectors $\vec{r}$ and $\vec{f}$, and gives the direction of the axis through M about which the force tends to twist. In this situation, the vector representing the cross product is directed down as the bolt tightens into the wood and would normally be directed along the axis of the bolt. The magnitude of the torque depends upon two factors: the exerted force, and the distance between the line of the exerted force and the point of rotation, M. The exerted force is $\vec{f}$, and the distance between M and the line of the exerted force is d. The magnitude of the torque is the product of the magnitude of the force (that is, $|\vec{f}|$) and the distance d. Since $d = |\vec{r}|\sin\theta$, the magnitude of the torque $\vec{f}$ about M is $(|\vec{r}|\sin\theta)(|\vec{f}|) = |\vec{r} \times \vec{f}|$. The magnitude of the torque measures the twisting effect of the applied force.

The force $\vec{f}$ is measured in newtons, and the distance d is measured in metres, so the unit of magnitude for torque is (newton)(metres), or joules (J), which is the same unit that work is measured in.

EXAMPLE 4 Using the cross product to calculate torque

A 20 N force is applied at the end of a wrench that is 40 cm in length. The force is applied at an angle of 60° to the wrench. Calculate the magnitude of the torque about the point of rotation M.

Solution

$$|\vec{r} \times \vec{f}| = (|\vec{r}|\sin\theta)|\vec{f}| = (0.40)(20)\frac{\sqrt{3}}{2} \doteq 6.93 \text{ J}$$

One of the implications of calculating the magnitude of torque, $|\vec{r} \times \vec{f}| = |\vec{r}||\vec{f}|\sin\theta$, is that it is maximized when $\sin\theta = 1$ and when the force is applied as far as possible from the turning point—that is, $|\vec{r}|$ is as large as possible. To get the best effect when tightening a bolt, this implies that force should be applied at right angles to the wrench and as far down the handle of the wrench as possible from the turning point.

IN SUMMARY

Key Idea

- Both the dot and cross products have useful applications in geometry and physics.

Need to Know

- $W = \vec{F} \cdot \vec{s}$, where $\vec{F}$ is the force applied to an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is work, measured in joules (J).
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$
- Area of a parallelogram, with sides $\vec{a}$ and $\vec{b}$, equals $|\vec{a} \times \vec{b}|$
- Area of a triangle, with sides $\vec{a}$ and $\vec{b}$, equals $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- Torque equals $\vec{r} \times \vec{f} = |\vec{r}||\vec{f}|\sin\theta$.
- $|\vec{r} \times \vec{f}|$, the magnitude of the torque, measures the overall twisting effect of applied force.

Exercise 7.7

PART A

C 1. A door is opened by pushing inward. Explain, in terms of torque, why this is most easily accomplished when pushing at right angles to the door as far as possible from the hinge side of the door.

2. a. Calculate $|\vec{a} \times \vec{b}|$, where $\vec{a} = (1, 2, 1)$ and $\vec{b} = (2, 4, 2)$.
   b. If $\vec{a}$ and $\vec{b}$ represent the sides of a parallelogram, explain why your answer for part a. makes sense, in terms of the formula for the area of a parallelogram.

PART B

3. Calculate the amount of work done in each situation.
   a. A stove is slid 3 m across the floor against a frictional force of 150 N.
   b. A 40 kg rock falls 40 m down a slope at an angle of 50° to the vertical.
   c. A wagon is pulled a distance of 250 m by a force of 140 N applied at an angle of 20° to the road.
   d. A lawnmower is pushed 500 m by a force of 100 N applied at an angle of 45° to the horizontal.
4. Determine each of the following by using the method shown in Example 3:
   a. $\vec{i} \times \vec{j}$ b. $-\vec{i} \times \vec{j}$ c. $\vec{i} \times \vec{k}$ d. $-\vec{i} \times \vec{k}$

K 5. Calculate the area of the parallelogram formed by the following pairs of vectors:
   a. $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 0, 1)$ b. $\vec{a} = (1, -2, 3)$ and $\vec{b} = (1, 2, 4)$
6. The area of the parallelogram formed by the vectors $\vec{p} = (a, 1, -1)$ and $\vec{q} = (1, 1, 2)$ is $\sqrt{35}$. Determine the value(s) of a for which this is true.
7. In R^3, points $A(-2, 1, 3)$, $B(1, 0, 1)$, and $C(2, 3, 2)$ form the vertices of $\triangle ABC$.
   a. By constructing position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, determine the area of the triangle.
   b. By constructing position vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$, determine the area of the triangle.
   c. What conclusion can be drawn?

A 8. A 10 N force is applied at the end of a wrench that is 14 cm long. The force makes an angle of 45° with the wrench. Determine the magnitude of the torque of this force about the other end of the wrench.

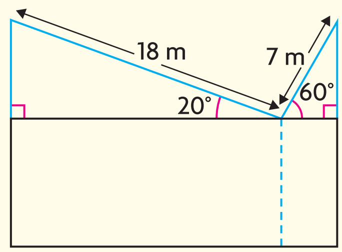
T 9. Parallelogram $OBCA$ has its sides determined by $\overrightarrow{OA} = \vec{a} = (4, 2, 4)$ and $\overrightarrow{OB} = \vec{b} = (3, 1, 4)$. Its fourth vertex is point C. A line is drawn from B perpendicular to side AC of the parallelogram to intersect AC at N. Determine the length of BN.

PART C

10. For the vectors $\vec{p} = (1, -2, 3)$, $\vec{q} = (2, 1, 3)$, and $\vec{r} = (1, 1, 0)$, show the following to be true.
   a. The vector $(\vec{p} \times \vec{q}) \times \vec{r}$ can be written as a linear combination of $\vec{p}$ and $\vec{q}$.
   b. $(\vec{p} \times \vec{q}) \times \vec{r} = (\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p}$

CHAPTER 7: STRUCTURAL ENGINEERING

A structural engineer is designing a special roof for a building. The roof is designed to catch rainwater and hold solar panels to collect sunlight for electricity. Each angled part of the roof exerts a downward force of 50 kg/m^2, including the loads of the panels and rainwater. The building will need a load-bearing wall at the point where each angled roof meets.



a. Calculate the force of the longer angled roof at the point where the roofs meet.

b. Calculate the force of the shorter angled roof at the point where the roofs meet.

c. Calculate the resultant force that the load-bearing wall must counteract to support the roof.

d. Use the given lengths and angles to calculate the width of the building.

e. If the point where the two roofs meet is moved 2 m to the left, calculate the angles that the sloped roofs will make with the horizontal and the length of each roof. Assume that only the point where the roofs meet can be adjusted and that the height of each roof will not change.

f. Repeat parts a. to c., using the new angles you calculated in part e.

g. Make a conjecture about the angles that the two roofs must make with the horizontal (assuming again that the heights are the same but the point where the roofs meet can be adjusted) to minimize the downward force that the load-bearing wall will have to counteract.

h. Calculate the downward force for the angles you conjectured in part g. Then perform the calculations for other angles to test your conjecture.

Key Concepts Review

In Chapter 7, you were introduced to applications of geometric vectors involving force and velocity. You were also introduced to the dot product and cross product between two vectors and should be familiar with the differences in their formulas and applications. Consider the following summary of key concepts:

- When two or more forces are applied to an object, the net effect of the forces can be represented by the resultant vector determined by adding the vectors that represent the forces.

- A system is in a state of equilibrium when the net effect of all the forces acting on an object causes no movement of the object. If there are three forces, this implies that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

- The velocity of a moving object can be influenced by external forces, such as wind and the current of a river. The resultant velocity is determined by adding the vectors that represent the object in motion and the effect of the external force: $\vec{v}_r = \vec{v}_{object} + \vec{v}_{external\ force}$

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

- The dot product between two algebraic vectors $\vec{a}$ and $\vec{b}$ is:

  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$ in R^2 $\qquad$ $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ in R^3

- If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$.

- The cross product $\vec{a} \times \vec{b}$ between two vectors $\vec{a}$ and $\vec{b}$ results in a third vector that is perpendicular to the plane in which the given vectors lie:

  $$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \text{ and } |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta.$$

  Geometrically, $|\vec{a} \times \vec{b}|$ is equivalent to the area of the parallelogram formed by vectors $\vec{a}$ and $\vec{b}$.

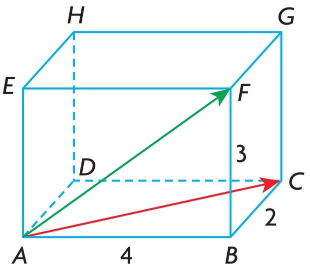
- Work is an application of the dot product, while torque is an application of the cross product.
  - $W = \vec{F} \cdot \vec{s}$, where $\vec{F}$ is the force applied to an object measured in newtons (N), $\vec{s}$ is the objects displacement measured in meters(m), and W is work measured in Joules (J).
  - Torque $= \vec{r} \times \vec{f} = |\vec{r}||\vec{f}|\sin\theta$, where $\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation, $\vec{f}$ is the applied force and θ is the angle between the force and the lever arm.

Review Exercise

1. Given that $\vec{a} = (-1, 2, 1)$, $\vec{b} = (-1, 0, 1)$, and $\vec{c} = (-5, 4, 5)$, determine each of the following:
   a. $\vec{a} \times \vec{b}$
   b. $\vec{b} \times \vec{c}$
   c. $|\vec{a} \times \vec{b}| \times |\vec{b} \times \vec{c}|$
   d. Why is it possible to conclude that the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar?
2. Given that $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors, $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} - 2\vec{k}$, determine each of the following:
   a. $|\vec{a}|$
   b. $|\vec{b}|$
   c. $|\vec{a} - \vec{b}|$
   d. $|\vec{a} + \vec{b}|$
   e. $\vec{a} \cdot \vec{b}$
   f. $\vec{a} \cdot (\vec{a} - 2\vec{b})$
3. a. For what value(s) of a are the vectors $\vec{x} = (3, a, 9)$ and $\vec{y} = (a, 12, 18)$ collinear?
   b. For what value(s) of a are these vectors perpendicular?
4. Determine the angle between the vectors $\vec{x} = (4, 5, 20)$ and $\vec{y} = (-3, 6, 22)$.
5. A parallelogram has its sides determined by $\overrightarrow{OA} = (5, 1)$ and $\overrightarrow{OB} = (-1, 4)$.
   a. Draw a sketch of the parallelogram.
   b. Determine the angle between the two diagonals of this parallelogram.
6. An object of mass 10 kg is suspended by two pieces of rope that make an angle of 30° and 45° with the horizontal. Determine the tension in each of the two pieces of rope.
7. An airplane has a speed of 300 km/h and is headed due west. A wind is blowing from the south at 50 km/h. Determine the resultant velocity of the airplane.
8. The diagonals of a parallelogram are determined by the vectors $\vec{x} = (3, -3, 5)$ and $\vec{y} = (-1, 7, 5)$.
   a. Construct x, y, and z coordinate axes and draw the two given vectors. In addition, draw the parallelogram formed by these vectors.
   b. Determine the area of the parallelogram.
9. Determine the components of a unit vector perpendicular to $(0, 3, -5)$ and to $(2, 3, 1)$.
10. A triangle has vertices $A(2, 3, 7)$, $B(0, -3, 4)$, and $C(5, 2, -4)$.
    a. Determine the largest angle in the triangle.
    b. Determine the area of $\triangle ABC$.

11. A mass of 10 kg is suspended by two pieces of string, 30 cm and 40 cm long, from two points that are 50 cm apart and at the same level. Find the tension in each piece of string.

12. A particle is acted upon by the following four forces: 25 N pulling east, 30 N pulling west, 54 N pulling north, and 42 N pulling south.
    a. Draw a diagram showing these four forces.
    b. Calculate the resultant and equilibrant of these forces.



13. A rectangular box is drawn as shown in the diagram at the left. The lengths of the edges of the box are $AB = 4$, $BC = 2$, and $BF = 3$.
    a. Select an appropriate origin, and then determine coordinates for the other vertices.
    b. Determine the angle between $\overrightarrow{AF}$ and $\overrightarrow{AC}$.
    c. Determine the scalar projection of $\overrightarrow{AF}$ on $\overrightarrow{AC}$.

14. If $\vec{a}$ and $\vec{b}$ are unit vectors, and $|\vec{a} + \vec{b}| = \sqrt{3}$, determine $(2\vec{a} - 5\vec{b}) \cdot (\vec{b} + 3\vec{a})$.

15. Kayla wishes to swim from one side of a river, which has a current speed of 2 km/h, to a point on the other side directly opposite from her starting point. She can swim at a speed of 3 km/h in still water.
    a. At what angle to the bank should Kayla swim if she wishes to swim directly across?
    b. If the river has a width of 300 m, how long will it take for her to cross the river?
    c. If Kayla's speed and the river's speed had been reversed, explain why it would not have been possible for her to swim across the river.

16. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
    a. Determine the coordinates of vectors representing the diagonals.
    b. Determine the angle between the sides of the parallelogram.

17. You are given the vectors $\vec{p} = (2, -2, -3)$ and $\vec{q} = (a, b, 6)$.
    a. Determine values of a and b if $\vec{q}$ is collinear with $\vec{p}$.
    b. Determine an algebraic condition for $\vec{p}$ and $\vec{q}$ to be perpendicular.
    c. Using the answer from part b., determine the components of a unit vector that is perpendicular to $\vec{p}$.

18. For the vectors $\vec{m} = (\sqrt{3}, -2, -3)$ and $\vec{n} = (2, \sqrt{3}, -1)$, determine the following:

a. the angle between these two vectors, to the nearest degree

b. the scalar projection of $\vec{n}$ on $\vec{m}$

c. the vector projection of $\vec{n}$ on $\vec{m}$

d. the angle that $\vec{m}$ makes with the z-axis

19. A number of unit vectors, each of which is perpendicular to the other vectors in the set, is said to form a *special* set. Determine which of the following sets are special.

a. $(1, 0, 0), (0, 0, -1), (0, 1, 0)$

b. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (0, 0, -1)$

20. If $\vec{p} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{q} = 2\vec{i} - \vec{j} + \vec{k}$, and $\vec{r} = \vec{j} - 2\vec{k}$, determine each of the following:

a. $\vec{p} \times \vec{q}$

b. $(\vec{p} - \vec{q}) \times (\vec{p} + \vec{q})$

c. $(\vec{p} \times \vec{r}) \cdot \vec{r}$

d. $(\vec{p} \times \vec{q}) \times \vec{r}$

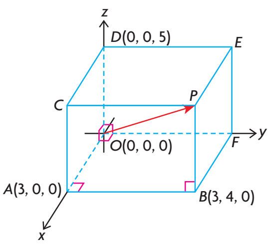
21. Two forces of equal magnitude act on an object so that the angle between their directions is 60°. If their resultant has a magnitude of 20 N, find the magnitude of the equal forces.

22. Determine the components of a vector that is perpendicular to the vectors $\vec{a} = (3, 2, -1)$ and $\vec{b} = (5, 0, 1)$.

23. If $|\vec{x}| = 2$ and $|\vec{y}| = 5$, determine the dot product between $\vec{x} - 2\vec{y}$ and $\vec{x} + 3\vec{y}$ if the angle between $\vec{x}$ and $\vec{y}$ is 60°.

24. The magnitude of the scalar projection of $(1, m, 0)$ on $(2, 2, 1)$ is 4. Determine the value of m.

25. Determine the angle that the vector $\vec{a} = (12, -3, 4)$ makes with the y-axis.



26. A rectangular solid measuring 3 by 4 by 5 is placed on a coordinate axis as shown in the diagram at the left.

a. Determine the coordinates of points C and F.

b. Determine $\overrightarrow{CF}$.

c. Determine the angle between the vectors $\overrightarrow{CF}$ and $\overrightarrow{OP}$.

27. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, where the angle between the two given vectors is $50°$. Determine each of the following:

a. $|\vec{d} + \vec{e}|$ b. $|\vec{d} - \vec{e}|$ c. $|\vec{e} - \vec{d}|$

28. Find the scalar and vector projections of $\vec{i} + \vec{j}$ on each of the following vectors:

a. $\vec{i}$ b. $\vec{j}$ c. $\vec{k} + \vec{j}$

29. a. Determine which of the following are unit vectors:

$\vec{a} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$, $\vec{b} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, $\vec{c} = \left(\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$, and $\vec{d} = (-1, 1, 1)$

b. Which one of vectors $\vec{a}$, $\vec{b}$, or $\vec{c}$ is perpendicular to vector $\vec{d}$? Explain.

30. A 25 N force is applied at the end of a 60 cm wrench. If the force makes a $30°$ angle with the wrench, calculate the magnitude of the torque.

31. a. Verify that the vectors $\vec{a} = (2, 5, -1)$ and $\vec{b} = (3, -1, 1)$ are perpendicular.

b. Find the direction cosines for each vector.

c. If $\overrightarrow{m_1} = (\cos \alpha_a, \cos \beta_a, \cos \gamma_a)$, the direction cosines for $\vec{a}$, and if $\overrightarrow{m_2} = (\cos \alpha_b, \cos \beta_b, \cos \gamma_b)$, the direction cosines for $\overrightarrow{b_2}$, verify that $\overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0$.

32. The diagonals of quadrilateral *ABCD* are $3\vec{i} + 3\vec{j} + 10\vec{k}$ and $-\vec{i} + 9\vec{j} - 6\vec{k}$. Show that quadrilateral *ABCD* is a rectangle.

33. The vector $\vec{v}$ makes an angle of $30°$ with the x-axis and equal angles with both the y-axis and z-axis.

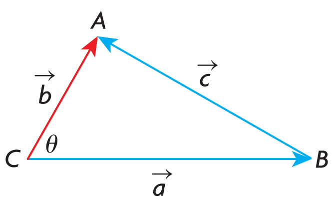
a. Determine the direction cosines for $\vec{v}$.

b. Determine the angle that $\vec{v}$ makes with the z-axis.

34. The vectors $\vec{a}$ and $\vec{b}$ are unit vectors that make an angle of $60°$ with each other. If $\vec{a} - 3\vec{b}$ and $m\vec{a} + \vec{b}$ are perpendicular, determine the value of m.

35. If $\vec{a} = (0, 4, -6)$ and $\vec{b} = (-1, -5, -2)$, verify that

$\vec{a} \cdot \vec{b} = \frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$.



36. Use the fact that $|\vec{c}|^2 = \vec{c} \cdot \vec{c}$ to prove the cosine law for the triangle shown in the diagram with sides $\vec{a}$, $\vec{b}$, and $\vec{c}$.

37. Find the lengths of the sides, the cosines of the angles, and the area of the triangle whose vertices are $A(1, -2, 1)$, $B(3, -2, 5)$, and $C(2, -2, 3)$.

Chapter 7 Test

1. Given the vectors $\vec{a} = (-1, 1, 1)$, $\vec{b} = (2, 1, -3)$, and $\vec{c} = (5, 1, -7)$, calculate the value of each of the following:

   a. $\vec{a} \times \vec{b}$ b. $\vec{b} \times \vec{c}$ c. $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ d. $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{b} \times \vec{c}\right)$

2. Given the vectors $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - 2\vec{k}$, determine the following:

   a. the scalar projection and vector projection of $\vec{a}$ on $\vec{b}$

   b. the angle that $\vec{b}$ makes with each of the coordinate axes

   c. the area of the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$

3. Two forces of 40 N and 50 N act at an angle of 60° to each other. Determine the resultant and equilibrant of these forces.

4. An airplane is heading due north at 1000 km/h when it encounters a wind from the east at 100 km/h. Determine the resultant velocity of the airplane.

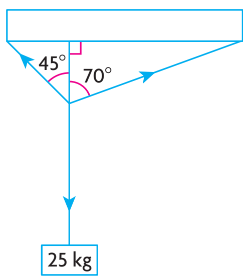
5. A canoeist wishes to cross a 200 m river to get to a campsite directly across from the starting point. The canoeist can paddle at 2.5 m/s in still water, and the current has a speed of 1.2 m/s.

   a. How far downstream would the canoeist land if headed directly across the river?

   b. In what direction should the canoeist head in order to arrive directly across from the starting point?

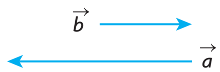
6. Calculate the area of a triangle with vertices $A(-1, 3, 5)$, $B(2, 1, 3)$, and $C(-1, 1, 4)$.

7. A 25 kg mass is suspended from a ceiling by two cords. The cords make angles of 45° and 70° with a perpendicular drawn to the ceiling, as shown. Determine the tension in each cord.



8. a. Using the vectors $\vec{x} = (3, 3, 1)$ and $\vec{y} = (-1, 2, -3)$, verify that the following formula is true: $\vec{x} \cdot \vec{y} = \frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$

   b. Prove that this formula is true for any two vectors.

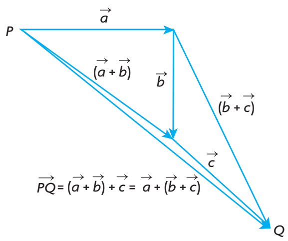
24. 

25. **a.** $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$

26. **Case 1** If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.
Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition.
We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. **a.** $(8, 4, 8)$
b. 12
c. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

3. $\sqrt{19}$

4. **a.** $\vec{x} = 2\vec{b} - 3\vec{a}, \vec{y} = 3\vec{b} - 5\vec{a}$
b. $a = 1, b = 5, c = -11$

5. **a.** $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.
b. $p = -2, q = 3$

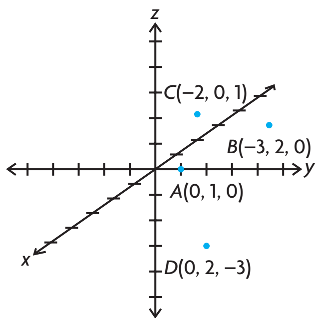
6. **a.** $(1, 12, -29) = -2(3, 1, 4) + 7(1, 2, -3)$
b. $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

7. $\sqrt{13}, \theta \doteq 3.61$; 73.9° relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$
$\overrightarrow{DE} = \vec{b} - \vec{a}$
Also,
$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$
$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$
Thus,
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$

Chapter 7

Review of Prerequisite Skills, p. 350

1. $v \doteq 806$ km/h N 7.1° E
2. 15.93 units W 32.2° N
3. 

4. **a.** $(3, -2, 7)$; $l \doteq 7.87$
b. $(-9, 3, 14)$; $l \doteq 16.91$
c. $(1, 1, 0)$; $l \doteq 1.41$
d. $(2, 0, -9)$; $l \doteq 9.22$

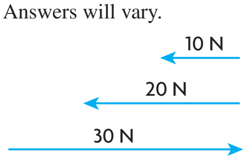
5. **a.** $(x, y, 0)$
b. $(x, 0, z)$
c. $(0, y, z)$

6. **a.** $\vec{i} - 7\vec{j}$
b. $6\vec{i} - 2\vec{j}$
c. $-8\vec{i} + 11\vec{j} + 3\vec{k}$
d. $4\vec{i} - 6\vec{j} + 8\vec{k}$

7. **a.** $\vec{i} + 3\vec{j} - \vec{k}$
b. $5\vec{i} + \vec{j} - \vec{k}$
c. $12\vec{i} + \vec{j} - 2\vec{k}$

Section 7.1, pp. 362–364

1. **a.** 10 N is a melon, 50 N is a chair, 100 N is a computer
b. Answers will vary.

2. **a.** 

b. 180°

3. a line along the same direction

4. For three forces to be in equilibrium, they must form a triangle, which is a planar figure.

5. **a.** The resultant is 13 N at an angle of N 22.6° W. The equilibrant is 13 N at an angle of S 22.6° W.
b. The resultant is 15 N at an angle of S 36.9° W. The equilibrant is 15 N at N 36.9° E.

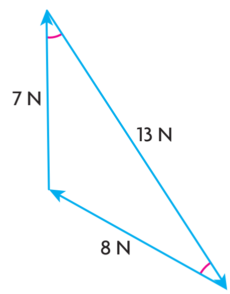
6. **a.** yes **b.** yes **c.** no **d.** yes

7. Arms 90 cm apart will yield a resultant with a smaller magnitude than at 30 cm apart. A resultant with a smaller magnitude means less force to counter your weight, hence a harder chin-up.

8. The resultant would be 12.17 N at 34.7° from the 6 N force toward the 8 N force. The equilibrant would be 12.17 N at 145.3° from the 6 N force away from the 8 N force.

9. 9.66 N 15° from given force, 2.95 N perpendicular to 9.66 N force

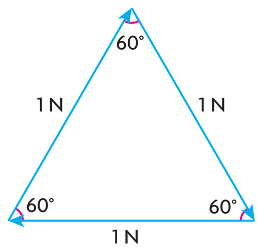
10. 49 N directed up the ramp

11. **a.** 

b. 60°

12. approximately 7.1 N 45° south of east

13. **a.** 7
b. The angle between f_1 and the resultant is 16.3°. The angle between $\vec{f}_1$ and the equilibrant is 163.7°.

14. **a.** 

For these three equal forces to be in equilibrium, they must form an equilateral triangle. Since the resultant will lie along one of these lines, and since all angles of an equilateral triangle are 60°, the resultant will be at a 60° angle with the other two vectors.

b. Since the equilibrant is directed opposite the resultant, the angle between the equilibrant and the other two vectors is $180° - 60° = 120°$.

15. 7.65 N, 67.5° from $\vec{f_2}$ toward $\vec{f_3}$

16. 45° rope: 175.73 N
30° rope: 143.48

17. 24 cm string: approximately 39.2 N,
32 cm string: approximately 29.4 N

18. 8.5° to the starboard side

19. **a.** magnitude for resultant and equilbrant $\doteq 13.75$ N
b. $\theta_{5N} \doteq 111.3°$, $\theta_{8N} \doteq 125.6°$, $\theta_{10N} \doteq 136.7°$

20. We know that the resultant of these two forces is equal in magnitude and angle to the diagonal line of the parallelogram formed with $\vec{f_1}$ and $\vec{f_2}$ as legs and has diagonal length $|\vec{f_1} + \vec{f_2}|$. We also know from the cosine rule that $|\vec{f_1} + \vec{f_2}|^2$
$= |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$,
where ϕ is the supplement to θ in, our parallelogram. Since we know $\phi = 180 - \theta$, then $\cos\phi = \cos(180 - \theta) = -\cos\theta$.
Thus, we have

$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$$
$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$$
$$|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$$

Section 7.2, pp. 369–370

1. **a.** 84 km/h in the direction of the train's movement
b. 76 km/h in the direction of the train's movement

2. **a.** 500 km/h north
b. 700 km/h north

3. 304.14, W 9.5° S

4. 60° upstream

5. **a.** 2 m/s forward
b. 22 m/s in the direction of the car

6. 13 m/s, N 37.6° W

7. **a.** 732.71 km/h, N 5.5° W
b. about 732.71 km

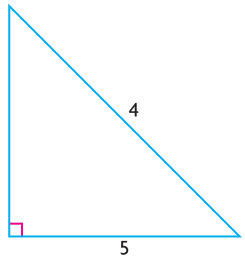
8. **a.** about 1383 km
b. about 12.5° east of north

9. **a.** about 10.4° south of west
b. 2 h, 53.1° downstream to the bank

10. **a.** 5 km/h
b. about 0.67 km
c. 20 min

11. **a.** about 18.4° west of north
b. about 108 km/h

12.



Since her swimming speed is a maximum of 4 km/h, this is her maximum resultant magnitude, which is also the hypotenuse of the triangle formed by her and the river's velocity vector. Since one of these legs is 5 km/h, we have a triangle with a leg larger than its hypotenuse, which is impossible.

13. **a.** about 68 m
b. 100 s

14. **a.** about 58.5°, upstream
b. about 58.6 s

15. 35 h

Section 7.3, pp. 377–378

1. To be guaranteed that the two vectors are perpendicular, the vectors must be nonzero.

2. $\vec{a} \cdot \vec{b}$ is a scalar, and a dot product is only defined for vectors.

3. Answers may vary. For example, let $\vec{a} = \hat{i}, \vec{b} = \hat{j}, \vec{c} = -\vec{i}\ \hat{i} \cdot \vec{a} \cdot \vec{b} \cdot \vec{b} = 0$, $\vec{a} \cdot \vec{c} \cdot \vec{c} = 0$, but $\vec{a} = -\vec{c}$.

4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{c}$ because $\vec{c} = \vec{a}$

5. -1

6. **a.** 16
b. -6.93
c. 0
d. -1
e. 0
f. -26.2

7. **a.** 30°
b. 80°
c. 53°
d. 127°
e. 60°
f. 120°

8. 22.5

9. **a.** $2|\vec{a}|^2 - 15|\vec{b}|^2 + 7\vec{a} \cdot \vec{b}$
b. $6|\vec{x}|^2 - 19\vec{x} \cdot \vec{y} + 3|\vec{y}|^2$

10. 0

11. 1

12. **a.** $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$
$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$
$= |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} + |\vec{b}|^2$
$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$
b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$
$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$
$= |\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - |\vec{b}|^2$
$= |\vec{a}|^2 - |\vec{b}|^2$

13. **a.** $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$
$= (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$
$= |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$
b. $\vec{b} \cdot \vec{c} = 0$, vectors are perpendicular. Therefore $|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2$, which is the Pythagorean theory.

14. 14

15. $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2$
$= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$
$+ (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$
$= |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 + |\vec{u}|^2$
$- 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
$= 2|\vec{u}|^2 + 2|\vec{v}|^2$

16. 3

17. -7

18. $\vec{d} = \vec{b} - \vec{c}$
$\vec{b} = \vec{d} + \vec{c}$
$\vec{c} \cdot \vec{a} = ((\vec{b} \cdot \vec{a})\vec{a}) \cdot \vec{a}$
$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})(\vec{a} \cdot \vec{a})$ because
$\vec{b} \cdot \vec{a}$ is a scalar
$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})|\vec{a}|^2$
$\vec{c} \cdot \vec{a} = (\vec{d} + \vec{c}) \cdot \vec{a}$ because $|\vec{a}| = 1$
$\vec{c} \cdot \vec{a} = \vec{d} \cdot \vec{a} + \vec{c} \cdot \vec{a}$
$\vec{d} \cdot \vec{a} = 0$

Section 7.4, pp. 385–387

1. Any vector of the form (c, c) is perpendicular to $\vec{a}$. Therefore, there are infinitely many vectors perpendicular to $\vec{a}$. Answers may vary. For example: $(1, 1), (2, 2), (3, 3)$.

2. **a.** 0; 90°
b. $34 > 0$; acute
c. $-3 < 0$; obtuse

3. Answer may vary. For example:
a. $(0, 0, 1)$ is perpendicular to every vector in the xy-plane.
b. $(0, 1, 0)$ is perpendicular to every vector in the xz-plane.
c. $(1, 0, 0)$ is perpendicular to every vector in the yz-plane.

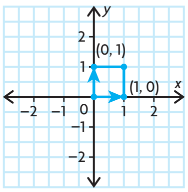
4. **a.** $(1, 2, -1)$ and $(4, 3, 10)$;
$(-4, -5, -6)$ and $\left(5, -3, -\frac{5}{6}\right)$
b. no

5. a. The vectors must be in R^3, which is impossible
b. This is not possible since R^3 does not exist in R^2.

6. a. about 148°
b. about 123°
c. about 64°
d. about 154°

7. a. $k = \frac{2}{3}$
b. $k \geq 0$

8. a.



b. $(1, 1)$ and $(1, -1)$; $(1, 1)$ and $(-1, 1)$
c. $(1, 1) \cdot (1, -1)$
$= 1 - 1$
$= 0$
or
$(1, 1) \cdot (-1, 1)$
$= -1 + 1$
$= 0$

9. a. 90°
b. 30°

10. a. i. $p = \frac{8}{3}$; $q = 3$
ii. Answers may vary. For example, $p = 1, q = -50$.
b. Unique for collinear vectors; not unique for perpendicular vectors

11. $\theta_A = 90°$; $\theta_B \doteq 26.6°$; $\theta_C \doteq 63.4°$

12. a. $O = (0, 0, 0), A = (7, 0, 0),$
$B = (7, 4, 0), C = (0, 4, 0),$
$D = (7, 0, 5), E = (0, 4, 5),$
$F = (0, 0, 5)$
b. 50°

13. a. Answers may vary. For example, $(3, 1, 1)$.
b. Answers may vary. For example, $(1, 1, 1)$.

14. 3 or -1

15. a. $3 + 4p + q = 0$
b. 0

16. Answers may vary. For example, $(1, 0, 1)$ and $(1, 1, 3)$.
$(x, y, z)(1, 2, -1) = 0$
$x + 2y - z = 0$
Let $x = z = 1$.
$(1, 0, 1)$ is perpendicular to $(1, 2, -1)$ and $(-2, -4, 2)$.
Let $x = y = 1$.
$(1, 1, 3)$ is perpendicular to $(1, 2, -1)$ and $(-2, -4, 2)$.

17. 4 or $-\frac{44}{65}$

18. a. $\vec{a} \cdot \vec{b} = 0$
Therefore, since the two diagonals are perpendicular, all the sides must be the same length.
b. $\overrightarrow{AB} = (1, 2, -1),$
$\overrightarrow{BC} = (2, 1, 1),$
$|\overrightarrow{AB}| = |\overrightarrow{BC}| = \sqrt{6}$
c. $\theta_1 = 60°$; $\theta_2 = 120°$

19. a. $(6, 18, -4)$
b. 87.4°

20. $\alpha \doteq 109.5°$ or $\theta \doteq 70.5°$

Mid-Chapter Review, pp. 388–389

1. a. 3
b. 81

2. 15 cm cord: 117.60 N;
20 cm cord: 88.20 N

3. 0

4. a. about 575.1 km/h at S 7.06° E
b. about 1.74 h

5. a. about 112.61 N
b. about 94.49 N

6. 4.5

7. a. 34
b. $\frac{34}{63}$

8. a. 0
b. 5
c. $5\vec{i} - 4\vec{j} + 3\vec{k}$
d. 0
e. 34
f. 9

9. a. $x = -3$ or $x = -\frac{1}{3}$
b. no value

10. a. $\vec{i} - 4\vec{j} - \vec{k}$
b. 24
c. $\sqrt{2}$ or 1.41
d. -4
e. -12

11. about 126.9°

12. $\vec{F} \doteq 6.08$ N, 25.3° from the 4 N force towards the 3 N force. $\vec{E} \doteq 6.08$ N, $180° - 25.3° = 154.7°$ from the 4 N force away from the 3 N force.

13. a. about 109.1°
b. about 87.9°

14. a. about N 2.6° E
b. about 2.17 h

15. $\vec{x} = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ or $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

16. a. about 6.12 m
b. about 84.9 s

17. a. when $\vec{x}$ and $\vec{y}$ have the same length
b. Vectors $\vec{a}$ and $\vec{b}$ determine a parallelogram. Their sum $\vec{a} + \vec{b}$ is one diagonal of the parallelogram formed, with its tail in the same location as the tails of $\vec{a}$ and $\vec{b}$. Their difference $\vec{a} - \vec{b}$ is the other diagonal of the parallelogram.

18. about 268.12 N

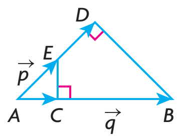
Section 7.5, pp. 398–400

1. a. scalar projection $= 2$,
vector projection $= 2\vec{i}$
b. scalar projection $= 3$,
vector projection $= 3\vec{j}$

2. Using the formula would cause a division by 0. Generally the $\vec{0}$ has any direction and 0 magnitude. You cannot project onto nothing.

3. You are projecting $\vec{a}$ onto the tail of $\vec{b}$, which is a point with magnitude 0. Therefore, it is $\vec{0}$; the projections of $\vec{b}$ onto the tail of $\vec{a}$ are also 0 and $\vec{0}$.

4. Answers may vary. For example,
$\vec{p} = \overrightarrow{AE}, \vec{q} = \overrightarrow{AB}$



scalar projection $\vec{p}$ on $\vec{q} = |\overrightarrow{AC}|$,
vector projection $\vec{p}$ on $\vec{q} = \overrightarrow{AC}$,
scalar projection $\vec{q}$ on $\vec{p} = |\overrightarrow{AD}|$,
vector projection $\vec{q}$ on $\vec{p} = \overrightarrow{AD}$

5. scalar projection of $\vec{a}$ on $\vec{i} = -1$,
vector projection of $\vec{a}$ on $\vec{i} = -\vec{i}$,
scalar projection of $\vec{a}$ on $\vec{j} = 2$,
vector projection of $\vec{a}$ on $\vec{j} = 2\vec{j}$,
scalar projection of $\vec{a}$ on $\vec{k} = -5$,
vector projection of $\vec{a}$ on $\vec{k} = -5\vec{k}$;
Without having to use formulae, a projection of $(-1, 2, 5)$ on $\vec{i}, \vec{j}$, or $\vec{k}$ is the same as a projection of $(-1, 0, 0)$ on $\vec{i}$, $(0, 2, 0)$ on $\vec{j}$, and $(0, 0, 5)$ on $\vec{k}$, which intuitively yields the same result.

6. a. scalar projection: $\frac{\vec{p} \cdot \vec{q}}{|\vec{q}|} = \frac{458}{21}$,
vector projection: $\frac{458}{441}(-4, 5, -20)$
b. about 82.5°, about 74.9°, about 163.0°

7. a. scalar projection: 0, vector projection: $\vec{0}$

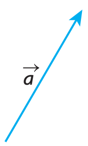
b. scalar projection: 2, vector projection: $2\vec{i}$

c. scalar projection: $\frac{50}{13}$, vector projection: $\frac{50}{169}(-5, 12)$

8. a. The scalar projection of $\vec{a}$ on the x-axis $(X, 0, 0)$ is -1; The vector projection of $\vec{a}$ on the x-axis is $-\vec{i}$; The scalar projection of $\vec{a}$ on the y-axis $(0, Y, 0)$ is 2; The vector projection of $\vec{a}$ on the y-axis is $2\vec{j}$; The scalar projection of $\vec{a}$ on the z-axis $(0, 0, Z)$ is 4; The vector projection of $\vec{a}$ on the z-axis is $4\vec{k}$.

b. The scalar projection of $m\,\vec{a}$ on the x-axis $(X, 0, 0)$ is $-m$; The vector projection of $m\,\vec{a}$ on the x-axis is $-m\vec{i}$; The scalar projection of $m\,\vec{a}$ on the y-axis $(0, Y, 0)$ is $2m$; The vector projection $m\,\vec{a}$ on the y-axis $(0, Y, 0)$ is $2m\vec{j}$; The scalar projection of $m\,\vec{a}$ on the z-axis $(0, 0, Z)$ is $4m$; The vector projection of $m\,\vec{a}$ on the z-axis is $4m\vec{k}$.

9. a.



vector projection: $\vec{a}$
scalar projection: $|\vec{a}|$

b. $|\vec{a}|\cos\theta = |\vec{a}|\cos 0$
$= |\vec{a}|(1)$
$= |\vec{a}|$.

The vector projection is the scalar projection multiplied by $\frac{\vec{a}}{|\vec{a}|}$,

$|\vec{a}| \times \frac{\vec{a}}{|\vec{a}|} = \vec{a}$.

10. a.

b. $\frac{(-\vec{a})\cdot\vec{a}}{|\vec{a}|} = \frac{-|\vec{a}|^2}{|\vec{a}|}$
$= -|\vec{a}|$

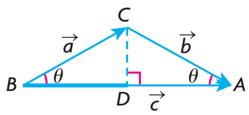
So, the vector projection is

$-|\vec{a}|\left(\frac{\vec{a}}{|\vec{a}|}\right) = -\vec{a}$.

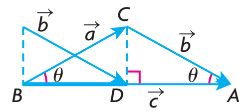
11. a. scalar projection of $\overrightarrow{AB}$ on the x-axis is -2; vector projection of $\overrightarrow{AB}$ on the x-axis is $-2\vec{i}$; scalar projection of $\overrightarrow{AB}$ on the y-axis is 1; vector projection of $\overrightarrow{AB}$ on the y-axis is $\vec{j}$; scalar projection of $\overrightarrow{AB}$ on the z-axis is 2; vector projection of $\overrightarrow{AB}$ on the z-axis is $2\vec{k}$.

b. 70.5°

12. a. $|\overrightarrow{BD}|$



b. $|\overrightarrow{BD}|$

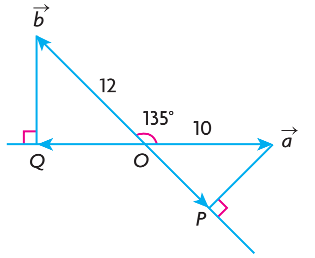


c. In an isosceles triangle, CD is a median and a right bisector of BA.

d. Yes

13. a. -7.07, -8.49

b.



$\overrightarrow{OQ}$ is the vector projection of $\vec{b}$ on $\vec{a}$
$\overrightarrow{OP}$ is the vector projection of $\vec{a}$ on $\vec{b}$

14. a. $-\frac{1}{3}$

b. The scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ is $\frac{19}{3}\cdot -\frac{1}{3} + \frac{19}{13} = 6$

The scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$ is 6.

c. Same lengths and both are in the direction of $\overrightarrow{OD}$. Add to get one vector.

15. a.

$$\begin{aligned}1 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma\\ &= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)^2\\ &\quad + \left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)^2\\ &\quad + \left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)^2\\ &= \frac{a^2}{a^2+b^2+c^2}\\ &\quad + \frac{b^2}{a^2+b^2+c^2}\\ &\quad + \frac{c^2}{a^2+b^2+c^2}\\ &= \frac{a^2+b^2+c^2}{a^2+b^2+c^2}\\ &= 1\end{aligned}$$

b. Answers may vary. For example: $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $(0, \sqrt{3}, 1)$

c. If two angles add to 90°, then all three will add to 180°.

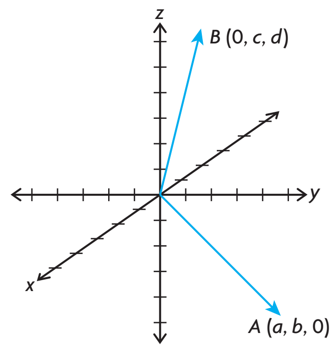
16. a. about 54.7°

b. about 125.3°

17.

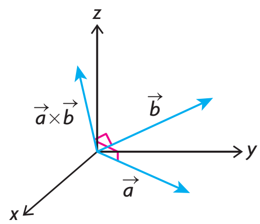
$$\begin{aligned}&\cos^2 x + \sin^2 x = 1\\ &\cos^2 x = 1 - \sin^2 x\\ &1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma\\ &1 = (1 - \sin^2\alpha) + (1 - \sin^2\beta)\\ &\qquad + (1 - \sin^2\gamma)\\ &1 = 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma)\\ &\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2\end{aligned}$$

18. Answers may vary. For example:

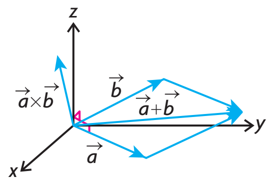


Section 7.6, pp. 407–408

1. a.



$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Thus, their dot product must equal 0. The same applies to the second case.



b. $\vec{a} + \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

c. Once again, $\vec{a} - \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} - \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

2. $\vec{a} \times \vec{b}$ produces a vector, not a scalar. Thus, the equality is meaningless.

3. **a.** It's possible because there is a vector crossed with a vector, then dotted with another vector, producing a scalar.
b. This is meaningless because $\vec{a} \cdot \vec{b}$ produces a scalar. This results in a scalar crossed with a vector, which is meaningless.
c. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} + \vec{d}$ also produces a vector. The result is a vector dotted with a vector producing a scalar.
d. This is possible. $\vec{a} \times \vec{b}$ produces a scalar, and $\vec{c} \times \vec{d}$ produces a vector. The product of a scalar and vector produces a vector.
e. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} \times \vec{d}$ produces a vector. The cross product of a vector and vector produces a vector.
f. This is possible. $\vec{a} \times \vec{b}$ produces a vector. When added to another vector, it produces another vector.

4. **a.** $(-7, -8, -2)$
b. $(1, 5, 1)$
c. $(-11, -33, 22)$
d. $(-19, -22, 7)$
e. $(3, 3, -1)$
f. $(-8, -26, 11)$

5. 1

6. **a.** $(-4, 0, 0)$
b. Vectors of the form $(0, b, c)$ are in the yz-plane. Thus, the only vectors perpendicular to the yz-plane are those of the form $(a, 0, 0)$ because they are parallel to the x-axis.

7. **a.** $(1, 2, 1) \times (2, 4, 2)$
$= (2(2) - 1(4), 1(2) - 1(2), 1(4) - 2(2))$
$= (0, 0, 0)$
b. $(a, b, c) \times (ka, kb, kc)$
$= (b(kc) - c(kb), c(ka) - a(kc), a(kb) - b(ka))$
Using the associative law of multiplication, we can rearrange this:
$= (bck - bck, ack - ack, abk - abk)$
$= (0, 0, 0)$

8. **a.** $\vec{p} \times (\vec{q} + \vec{r}) = (-26, -7, 3)$
$\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-26, -7, 3)$
b. $\vec{p} \times (\vec{q} + \vec{r}) = (-3, 2, 5)$
$\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-3, 2, 5)$

9. **a.** $\vec{i} \times \vec{j} = (1, 0, 0) \times (0, 1, 0) = \vec{k}$
$-\vec{j} \times \vec{i} = (0, -1, 0) \times (1, 0, 0) = \vec{k}$
b. $\vec{j} \times \vec{k} = (0, 1, 0) \times (0, 0, 1) = \vec{i}$
$-\vec{k} \times \vec{j} = (0, 0, -1) \times (0, 1, 0) = \vec{i}$
c. $\vec{k} \times \vec{i} = (0, 0, 1) \times (1, 0, 0) = \vec{j}$
$-\vec{i} \times \vec{k} = (-1, 0, 0) \times (0, 0, 1) = \vec{j}$

10. $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)(a_1, a_2, a_3)$
$= k(a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1)$
$= k(0)$
$= 0$
$\vec{a}$ is perpendicular to $k(\vec{a} \times \vec{b})$.

11. **a.** $(0, 0, 6), (0, 0, -6)$
b. $(0, 0, 0)$
c. All the vectors are in the xy-plane. Thus, their cross product in part b. is between vectors parallel to the z-axis and so parallel to each other. The cross product of parallel vectors is $\vec{0}$.

12. Let $\vec{x} = (1, 0, 1)$, $\vec{y} = (1, 1, 1)$, and $\vec{z} = (1, 2, 3)$
Then, $\vec{x} \times \vec{y} = (0 - 1, 1 - 1, 1 - 0)$
$= (-1, 0, 1)$
$(\vec{x} \times \vec{y}) \times \vec{z}$
$= (0 - 2, 1 - (-3), -3 - 0)$
$= (-2, 4, -3)$
$\vec{y} \times \vec{z} = (3 - 2, 1 - 3, 2 - 1)$
$= (1, -2, 1)$
$\vec{x} \times (\vec{y} \times \vec{z}) = (0 + 2, 1 - 1, -2 - 0)$
$= (2, 0, -2)$
Thus, $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

13. By the distributive property of cross product:
$= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$
By the distributive property again:
$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$
A vector crossed with itself equals $\to 0$, thus:
$= -\vec{b} \times \vec{a} + \vec{a} \times \vec{b}$
$= \vec{a} \times \vec{b} - \vec{b} \times \vec{a}$
$= \vec{a} \times \vec{b} - (-\vec{a} \times \vec{b})$
$= 2\vec{a} \times \vec{b}$

Section 7.7, pp. 414–415

1. By pushing as far away from the hinge as possible, $|\vec{r}|$ is increased, making the cross product bigger. By pushing at right angles, sine is its largest value, 1, making the cross product larger.

2. **a.** 0
b. This makes sense because the vectors lie on the same line. Thus, the parallelogram would just be a line making its area 0.

3. **a.** 450 J
b. about 10 078.91 J
c. about 32 889.24 J
d. 35 355.34 J

4. **a.** $\vec{k}$
b. $-\vec{k}$
c. $-\vec{j}$
d. $\vec{j}$

5. **a.** $\sqrt{3}$ square units
b. $\sqrt{213}$ square units

6. $2, \frac{-12}{5}$

7. **a.** $\frac{5\sqrt{6}}{2}$ square units
b. $\frac{5\sqrt{6}}{2}$ square units.
c. Any two sides of a triangle can be used to calculate its area.

8. about 0.99 J

9. $\frac{6}{\sqrt{26}}$ or about 1.18

10. **a.** $\vec{p} \times \vec{q} = (-6 - 3, 6 - 3, 1 + 4)$
$= (-9, 3, 5)$
$(\vec{p} \times \vec{q}) \times \vec{r}$
$= (0 - 5, 5 + 0, -9 - 3)$
$= (-5, 5, -12)$
$a(1, -2, 3) + b(2, 1, 3)$
$= (-5, 5, -12)$
Looking at x-components:
$a + 2b = -5$
$a = -5 - 2b$
y-components:
$-2a + b = 5$
Substitute a:
$10 + 4b + b = 5$
$5b = -5$
$b = -1$
Substitute b back into the x-components:
$a = -5 + 2$
$a = -3$
Check in z-components:
$3a + 3b = -12$
$-9 - 3 = -12$
b. $\vec{p} \cdot \vec{r} = 1 - 2 + 0 = -1$
$\vec{q} \cdot \vec{r} = 2 + 1 + 0 = 3$
$(\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p}$
$= -1(2, 1, 3) - 3(1, -2, 3)$
$= (-2, -1, -3) - (3, -6, 9)$
$= (-2 - 3, -1 + 6, -3 - 9)$
$= (-5, 5, -12)$

Review Exercise, pp. 418–421

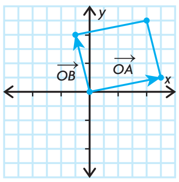
1. **a.** $(2, 0, 2)$
b. $(-4, 0, -4)$
c. 16
d. The cross products are parallel, so the original vectors are in the same plane.

2. **a.** 3
b. 7
c. $4\sqrt{3}$
d. $2\sqrt{17}$
e. 5
f. -1

3. **a.** 6
b. $-\frac{54}{5}$

4. about 18.52°

5. **a.**

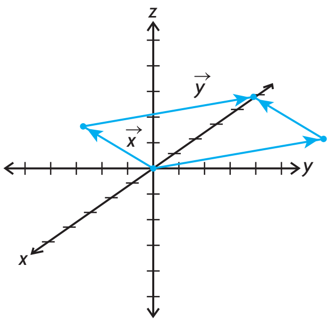


b. about 77.9°

6. rope at 45°: about 87.86 N,
rope at 30°: about 71.74 N

7. 304.14 km/h, W 9.46° N

8. **a.**



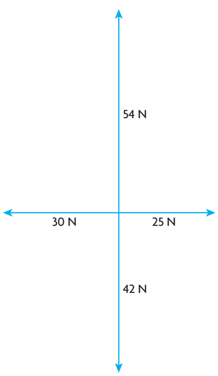
b. approximately 56.78

9. $\left(\frac{9}{\sqrt{115}}, -\frac{5}{\sqrt{115}}, -\frac{3}{\sqrt{115}}\right)$

10. **a.** about 77.64° is the largest angle
b. 36.50

11. 30 cm string: 78.4 N;
40 cm string: 58.8 N

12. **a.**



b. The resultant is 13 N in a direction N22.6°W. The equilibrant is 13 N in a direction S22.6°E.

13. **a.** Let D be the origin, then:
$A = (2, 0, 0)$, $B = (2, 4, 0)$,
$C = (0, 4, 0)$, $D = (0, 0, 0)$,
$E = (2, 0, 3)$, $F = (2, 4, 3)$,
$G = (0, 4, 3)$, $H = (0, 0, 3)$
b. about 44.31°
c. about 3.58

14. 7.5

15. **a.** about 48.2°
b. about 8 min 3 s
c. Such a situation would have resulted in a right triangle where one of the legs is longer than the hypotenuse, which is impossible.

16. **a.** $\overrightarrow{OA} + \overrightarrow{OB} = (-3, 8, -8)$,
$\overrightarrow{OA} - \overrightarrow{OB} = (9, -4, -4)$
b. about 84.36°

17. **a.** $a = 4$ and $b = -4$
b. $\vec{p} \cdot \vec{q} = 2a - 2b - 18 = 0$
c. $\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$

18. **a.** about 74.62°
b. about 0.75
c. $(0.1875)(\sqrt{3}, -2, -3)$
d. about 138.59°

19. **a.** special
b. not special

20. **a.** $(-1, 1, 3)$
b. $(-2, 2, 6)$
c. 0
d. $(-1, 1, 3)$

21. about 11.55 N

22. $(2, -8, -10)$

23. -141

24. 5 or -7

25. about 103.34°

26. **a.** $C = (3, 0, 5)$, $F = (0, 4, 0)$
b. $(-3, 4, -5)$
c. about 111.1°

27. **a.** about 7.30
b. about 3.84
c. about 3.84

28. **a.** scalar: 1,
vector: $\vec{i}$
b. scalar: 1,
vector: $\vec{j}$
c. scalar: $\frac{1}{\sqrt{2}}$,
vector: $\frac{1}{2}(\vec{k} + \vec{j})$

29. **a.** $|\vec{b}|, |\vec{c}|$
b. $\vec{a}$; When dotted with $\vec{d}$, it equals 0.

30. 7.50 J

31. **a.** $\vec{a} \cdot \vec{b} = 6 - 5 - 1 = 0$
b. $\vec{a}$ with the x-axis:
$|\vec{a}| = \sqrt{4 + 25 + 1} = \sqrt{30}$
$\cos(\alpha) = \frac{2}{\sqrt{30}}$
$\vec{a}$ with the y-axis:
$\cos(\beta) = \frac{5}{\sqrt{30}}$
$\vec{a}$ with the z-axis:
$\cos(\gamma) = \frac{-1}{\sqrt{30}}$
$|\vec{b}| = \sqrt{9 + 1 + 1} = \sqrt{11}$
$\vec{b}$ with the x-axis:
$\cos(\alpha) = \frac{3}{\sqrt{11}}$
$\vec{b}$ with the y-axis:
$\cos(\beta) = \frac{-1}{\sqrt{11}}$
$\vec{b}$ with the z-axis:
$\cos(\gamma) = \frac{1}{\sqrt{11}}$
c. $\vec{m}_1 \times \vec{m}_2 = \frac{6}{\sqrt{330}} - \frac{5}{\sqrt{330}}$
$- \frac{1}{\sqrt{330}} = 0$

32. $|3\vec{i} + 3\vec{j} + 10\vec{k}| = \sqrt{118}$
$|-\vec{i} + 9\vec{j} - 6\vec{k}| = \sqrt{118}$

33. **a.** $\cos\alpha = \frac{\sqrt{3}}{2}$,
$\cos\beta = \cos\gamma = \pm\frac{1}{2\sqrt{2}}$
b. acute case: 69.3°,
obtuse case: 110.7°

34. -5

35. $|\vec{a} + \vec{b}| = \sqrt{1 + 1 + 64} = \sqrt{66}$
$|\vec{a} - \vec{b}| = \sqrt{1 + 81 + 16} = \sqrt{98}$
$\frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$
$= \frac{66}{4} - \frac{98}{4} = -8$

36. $\vec{c} = \vec{b} - \vec{a}$
$|\vec{c}|^2 = |\vec{b} - \vec{a}|^2$
$= (\vec{b} - \vec{a})(\vec{b} - \vec{a})$
$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$
$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

37. $\overrightarrow{AB} = (2, 0, 4)$
$|\overrightarrow{AB}| = 2\sqrt{5}$
$\overrightarrow{AC} = (1, 0, 2)$
$|\overrightarrow{AC}| = \sqrt{5}$
$\overrightarrow{BC} = (-1, 0, -2)$
$|\overrightarrow{BC}| = \sqrt{5}$

$\cos A = 1$
$\cos B = 1$
$\cos C = -1$
area of triangle $ABC = 0$

Chapter 7 Test, p. 422

1. **a.** $(-4, -1, -3)$
b. $(-4, -1, -3)$
c. 0
d. 0

2. **a.** scalar projection: $\frac{1}{3}$,
vector projection: $\frac{1}{9}(2, -1, -2)$.
b. x-axis: $48.2°$; y-axis: $109.5°$; z-axis: $131.8°$
c. $\sqrt{26}$ or 5.10

3. Both forces have a magnitude of 78.10 N. The resultant makes an angle 33.7° to the 40 N force and 26.3° to the 50 N force. The equilibrant makes an angle 146.3° to the 40 N force and 153.7° to the 50 N force.

4. 1004.99 km/h, N 5.7° W

5. **a.** 96 m downstream
b. 28.7° upstream

6. 3.50 square units.

7. cord at 45°: about 254.0 N;
cord at 70°: about 191.1 N

8. **a.** 0
33
$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$
$= \frac{1}{4}(33) - \frac{1}{4}(33) = 0$
So, the equation holds for these vectors.

b.
$$\begin{aligned}|\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{x}) + (\vec{y}\cdot\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$
$$\begin{aligned}|\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot -\vec{y}) + (-\vec{y}\cdot\vec{x}) + (-\vec{y}\cdot -\vec{y})\\ &= (\vec{x}\cdot\vec{x}) - 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$
So, the right side of the equation is
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2 = \frac{1}{4}(4(\vec{x}\cdot\vec{y})) = \vec{x}\cdot\vec{y}$$

Chapter 8

Review of Prerequisite Skills, pp. 424–425

1. **a.** $(2, -9, 6)$
b. $(13, -12, -41)$

2. **a.** yes **c.** yes
b. yes **d.** no

3. yes

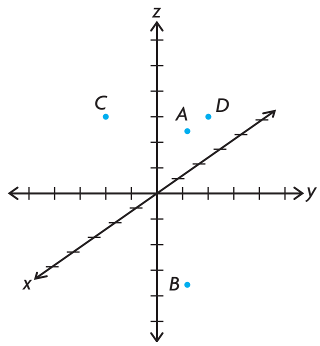
4. $t = 18$

5. **a.** $(3, 1)$
b. $(5, 6)$
c. $(-4, 7, 0)$

6. $\sqrt{2802}$

7. **a.** $(-22, -8, -13)$
b. $(0, 0, -3)$

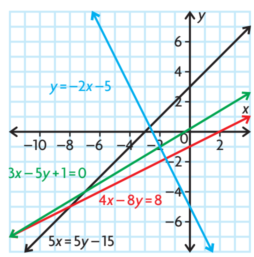
8.



9. **a.** $(-7, -3)$
b. $(10, 14)$
c. $(2, -8, 5)$
d. $(-4, 5, 4)$

10. **a.** $(7, 3)$
b. $(-10, -14)$
c. $(-2, 8, -5)$
d. $(4, -5, -4)$

11. **a.** slope: -2; y-intercept: -5
b. slope: $\frac{1}{2}$; y-intercept: -1
c. slope: $\frac{3}{5}$; y-intercept: $\frac{1}{5}$
d. slope: 1; y-intercept: 3



12. Answers may vary. For example:
a. $(8, 14)$
b. $(-15, 12, 9)$
c. $\vec{i} + 3\vec{j} - 2\vec{k}$
d. $-20\vec{i} + 32\vec{j} + 8\vec{k}$

13. **a.** 33
b. -33
c. 77
d. $(-11, -8, 28)$
e. $(11, 8, -28)$
f. $(55, 40, -140)$

14. The dot product of two vectors yields a real number, while the cross product of two vectors gives another vector.

Section 8.1, pp. 433–434

1. Direction vectors for a line are unique only up to scalar multiplication. So, since each of the given vectors is just a scalar multiple of $\left(\frac{1}{3}, \frac{1}{6}\right)$, each is an acceptable direction vector for the line.

2. **a.** Answers may vary. For example, $(-2, 7)$, $(1, 5)$, and $(4, 3)$.
b. $t = -5$
If $t = -5$, then $x = -14$ and $y = 15$. So $P(-14, 15)$ is a point on the line.

3. Answers may vary. For example:
a. direction vector: $(2, 1)$; point: $(3, 4)$
b. direction vector: $(2, -7)$; point: $(1, 3)$
c. direction vector: $(0, 2)$; point: $(4, 1)$
d. direction vector: $(-5, 0)$; point: $(0, 6)$

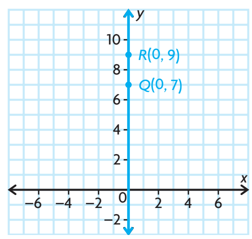
4. Answers may vary. For example:
$\vec{r} = (2, 1) + t(-5, 4), t \in \mathbf{R}$
$\vec{q} = (-3, 5) + s(5, -4), s \in \mathbf{R}$

5. **a.** $R(-9, 18)$ is a point on the line. When $t = 7$, $x = -9$ and $y = 18$.
b. Answers may vary. For example:
$\vec{r} = (-9, 18) + t(-1, 2), t \in \mathbf{R}$
c. Answers may vary. For example:
$\vec{r} = (-2, 4) + t(-1, 2), t \in \mathbf{R}$

6. Answers may vary. For example:
a. $(-3, -4)$, $(0, 0)$, and $(3, 4)$
b. $\vec{r} = t(1, 1), t \in \mathbf{R}$
c. This describes the same line as part a.

7. One can multiply a direction vector by a constant to keep the same line, but multiplying the point yields a different line.

8. **a.**



b. $\vec{r} = (0, 7) + t(0, 2), t \in \mathbf{R}$; $x = 0$, $y = 7 + 2t, t \in \mathbf{R}$