Chapter 5

DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

The world's population experiences exponential growth—the rate of growth becomes more rapid as the size of the population increases. Can this be explained in the language of calculus? Well, the rate of growth of the population is described by an exponential function, and the derivative of the population with respect to time is a constant multiple of the population at any time t. There are also many situations that can be modelled by trigonometric functions, whose derivative also provides a model for instantaneous rate of change at any time t. By combining the techniques in this chapter with the derivative rules seen earlier, we can find the derivative of an exponential or trigonometric function that is combined with other functions. Logarithmic functions and exponential functions are inverses of each other, and, in this chapter, you will also see how their graphs and properties are related to each other.

CHAPTER EXPECTATIONS

In this chapter, you will

- define e and the derivative of $y = e^x$, **Section 5.1**
- determine the derivative of the general exponential function $y = b^x$, **Section 5.2**
- compare the graph of an exponential function with the graph of its derivative, **Sections 5.1, 5.2**
- solve optimization problems using exponential functions, **Section 5.3**
- investigate and determine the derivatives of sinusoidal functions, **Section 5.4**
- determine the derivative of the tangent function, **Section 5.5**
- solve rate of change problems involving exponential and trigonometric function models using their derivatives, **Sections 5.1 to 5.5**

Review of Prerequisite Skills

In Chapter 5, you will be studying the derivatives of two classes of functions that occur frequently in calculus problems: exponential functions and trigonometric functions. To begin, we will review some of the properties of exponential and trigonometric functions.

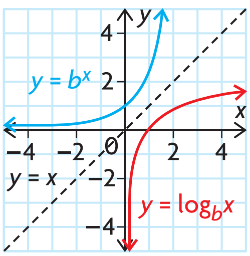
Properties of Exponents

- $b^m b^n = b^{m+n}$
- $\dfrac{b^m}{b^n} = b^{m-n}, b^n \neq 0$
- $(b^m)^n = b^{mn}$
- $b^{\log_b m} = m$
- $\log_b b^m = m$

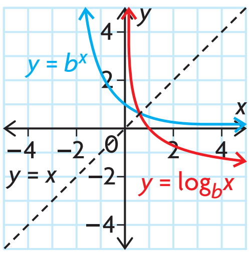
Properties of the Exponential Function, $y = b^x$

- The base b is positive and $b \neq 1$.
- The y-intercept is 1.
- The x-axis is a horizontal asymptote.
- The domain is the set of real numbers, **R**.
- The range is the set of positive real numbers.
- The exponential function is always increasing if $b > 1$.
- The exponential function is always decreasing if $0 < b < 1$.
- The inverse of $y = b^x$ is $x = b^y$.
- The inverse is called the logarithmic function and is written as $\log_b x = y$.

Graphs of $y = \log_b x$ and $y = b^x$



for $b > 1$



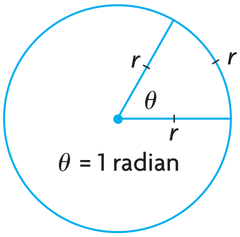
for $0 < b < 1$

- If $b^m = n$ for $b > 0$, then $\log_b n = m$.

Radian Measure

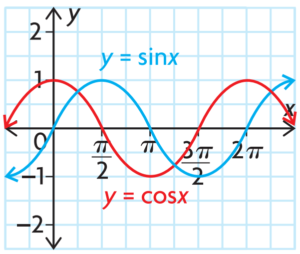
A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

π radians $= 180°$



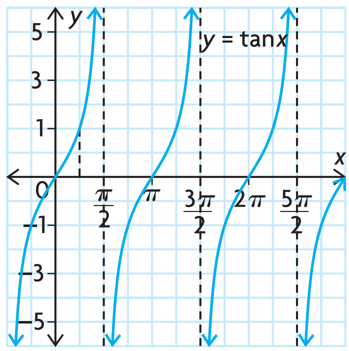
Sine and Cosine Functions
$f(x) = \sin x$ and $f(x) = \cos x$

Domain	$x \in \mathbf{R}$
Range	$-1 \leq \sin x \leq 1$
	$-1 \leq \cos x \leq 1$
Periodicity	$\sin(x + 2\pi) = \sin x$
	$\cos(x + 2\pi) = \cos x$



Tangent Function
$f(x) = \tan x$

Domain	$\left\{x \in \mathbf{R} \mid x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$
Range	$\{y \in \mathbf{R}\}$
Periodicity	$\tan(x + \pi) = \tan x$



Transformations of Sinusoidal Functions

For $y = a \sin k(x - d) + c$ and $y = a \cos k(x - d) + c$,

- the amplitude is $|a|$
- the period is $\frac{2\pi}{|k|}$
- the horizontal shift is d, and
- the vertical translation is c

Trigonometric Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reflection Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Cofunction Identities

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

Exercise

1. Evaluate each of the following:

a. 3^{-2} b. $32^{\frac{2}{5}}$ c. $27^{-\frac{2}{3}}$ d. $\left(\frac{2}{3}\right)^{-2}$

2. Express each of the following in the equivalent logarithmic form:

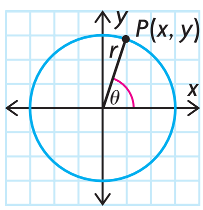
a. $5^4 = 625$ c. $x^3 = 3$ e. $3^8 = z$

b. $4^{-2} = \frac{1}{16}$ d. $10^w = 450$ f. $a^b = T$

3. Sketch the graph of each function, and state its x-intercept.

a. $y = \log_{10}(x + 2)$ b. $y = 5^{x+3}$

4. Refer to the following figure. State the value of each trigonometric ratio below.



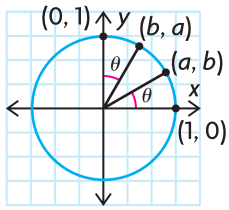
a. $\sin\theta$ b. $\cos\theta$ c. $\tan\theta$

5. Convert the following angles to radian measure:

a. $360°$ c. $-90°$ e. $270°$ g. $225°$

b. $45°$ d. $30°$ f. $-120°$ h. $330°$

6. Refer to the following figure. State the value of each trigonometric ratio below.



a. $\sin \theta$ c. $\cos \theta$ e. $\cos\left(\frac{\pi}{2} - \theta\right)$

b. $\tan \theta$ d. $\sin\left(\frac{\pi}{2} - \theta\right)$ f. $\sin(-\theta)$

7. The value of $\sin \theta$, $\cos \theta$, or $\tan \theta$ is given. Determine the values of the other two functions if θ lies in the given interval.

a. $\sin \theta = \frac{5}{13}, \frac{\pi}{2} \le \theta \le \pi$ c. $\tan \theta = -2, \frac{3\pi}{2} \le \theta \le 2\pi$

b. $\cos \theta = -\frac{2}{3}, \pi \le \theta \le \frac{3\pi}{2}$ d. $\sin \theta = 1, 0 \le \theta \le \pi$

8. State the period and amplitude of each of the following:

a. $y = \cos 2x$ d. $y = \frac{2}{7}\cos(12x)$

b. $y = 2 \sin \frac{x}{2}$ e. $y = 5 \sin\left(\theta - \frac{\pi}{6}\right)$

c. $y = -3 \sin(\pi x) + 1$ f. $y = |3 \sin x|$

9. Sketch the graph of each function over two complete periods.

a. $y = \sin 2x + 1$ b. $y = 3 \cos\left(x + \frac{\pi}{2}\right)$

10. Prove the following identities:

a. $\tan x + \cot x = \sec x \csc x$ b. $\frac{\sin x}{1 - \sin^2 x} = \tan x \sec x$

11. Solve the following equations, where $x \in [0, 2\pi]$.

a. $3 \sin x = \sin x + 1$ b. $\cos x - 1 = -\cos x$

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

While many real-life situations can be modelled fairly well by polynomial functions, there are some situations that are best modelled by other types of functions, including exponential, logarithmic, and trigonometric functions. Because determining the derivative of a polynomial function is simple, finding the rate of change for models described by polynomial functions is also simple. Often the rate of change at various times is more important to the person studying the scenario than the value of the function is. In this chapter, you will learn how to differentiate exponential and trigonometric functions, increasing the number of function types you can use to model real-life situations and, in turn, analyze using rates of change.

Case Study—Microbiologist

Microbiologists contribute their expertise to many fields, including medicine, environmental science, and biotechnology. Enumerating, the process of counting bacteria, allows microbiologists to build mathematical models that predict populations after a given amount of time has elapsed. Once they can predict a population accurately, the model can be used in medicine, for example, to predict the dose of medication required to kill a certain bacterial infection. The data in the table shown was used by a microbiologist to produce a polynomial-based mathematical model to predict population $p(t)$ as a function of time t, in hours, for the growth of a certain strain of bacteria:

$$p(t) = 1000\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right)$$

Time (h)	Population
0	1000
0.5	1649
1.0	2718
1.5	4482
2.0	7389

DISCUSSION QUESTIONS

1. How well does the function fit the data? Use the data, the equation, a graph, and/or a graphing calculator to comment on the "goodness of fit."

2. Use $p(t)$ and $p'(t)$ to determine the following:

a) the population after 0.5 h and the rate at which the population is growing at this time.

b) the population after 1.0 h and the rate at which the population is growing at this time.

3. What pattern did you notice in your calculations? Explain this pattern by examining the terms of the equation to find the reason why.

The polynomial function in this case study is an approximation of a special function in mathematics, natural science, and economics, $f(x) = e^x$, where e has a value of 2.718 28…. At the end of this chapter, you will complete a task on rates of change of exponential growth in a biotechnology case study.

Section 5.1—Derivatives of Exponential Functions, $y = e^x$

Many mathematical relations in the world are nonlinear. We have already studied various polynomial and rational functions and their rates of change. Another type of nonlinear model is the exponential function. Exponential functions are often used to model rapid change. Compound interest, population growth, the intensity of an earthquake, and radioactive decay are just a few examples of exponential change.

In this section, we will study the exponential function $y = e^x$ and its derivative. The number e is a special irrational number, like the number π. It is called the natural number, or Euler's number in honour of the Swiss mathematician Leonhard Euler (pronounced "oiler"), who lived from 1707 to 1783. We use a rational approximation for e of about 2.718. The rules developed thus far have been applied to polynomial functions and rational functions. We are now going to show how the derivative of an exponential function can be found.

INVESTIGATION

Tech Support

To evaluate powers of e, such as e^{-2}, press 2ND LN – 2 ENTER.

In this investigation, you will

- graph the exponential function $f(x) = e^x$ and its derivative
- determine the relationship between the exponential function and its derivative

A. Consider the function $f(x) = e^x$. Create a table similar to the one shown below. Complete the $f(x)$ column by using a graphing calculator to calculate the values of e^x for the values of x provided. Round all values to three decimal places.

x	$f(x)$	$f'(x)$
−2	0.135	
−1		
0		
1		
2		
3		

B. Graph the function $f(x) = e^x$.

C. Use a graphing calculator to calculate the value of the derivative $f'(x)$ at each of the given points.
To calculate $f'(x)$, press MATH and scroll down to **8:nDeriv(** under the MATH menu. Press ENTER and the display on the screen will be **nDeriv(**. To find the derivative, key in the expression e^x, the variable x, and the x-value

at which you want the derivative, for example, to determine $\frac{d}{dx}(e^x)$ at $x = -2$, the display will be **nDeriv(** e^x, X, -2). Press ENTER, and the approximate value of $f'(-2)$ will be returned.

D. What do you notice about the values of $f(x)$ and $f'(x)$?

E. Draw the graph of the derivative function $f'(x)$ on the same set of axes as $f(x)$. How do the two graphs compare?

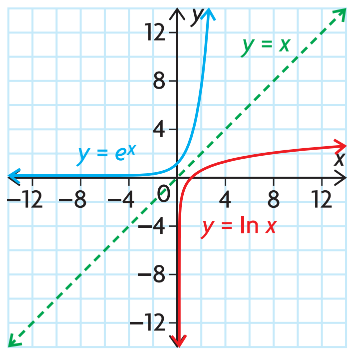
F. Try a few other values of x to see if the pattern continues.

G. What conclusion can you make about the function $f(x) = e^x$ and its derivative?

Properties of $y = e^x$

Since $y = e^x$ is an exponential function, it has the same properties as other exponential functions you have studied.

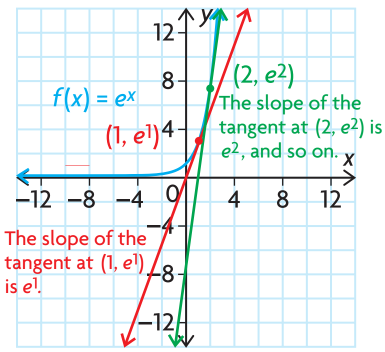
Recall that the logarithm function is the inverse of the exponential function. For example, $y = \log_2 x$ is the inverse of $y = 2^x$. The function $y = e^x$ also has an inverse, $y = \log_e x$. Their graphs are reflections in the line $y = x$. The function $y = \log_e x$ can be written as $y = \ln x$ and is called the **natural logarithm function.**



All the properties of exponential functions and logarithmic functions you are familiar with also apply to $y = e^x$ and $y = \ln x$.

$y = e^x$	$y = \ln x$
• The domain is $\{x \in \mathbf{R}\}$.	• The domain is $\{x \in \mathbf{R} \mid x > 0\}$.
• The range is $\{y \in \mathbf{R} \mid y > 0\}$.	• The range is $\{y \in \mathbf{R}\}$.
• The function passes through (0, 1).	• The function passes through (1, 0).
• $e^{\ln x} = x, x > 0$.	• $\ln e^x = x, x \in \mathbf{R}$.
• The line $y = 0$ is the horizontal asymptote.	• The line $x = 0$ is the vertical asymptote.

From the investigation, you should have noticed that all the values of the derivative $f'(x)$ were exactly the same as those of the original function $f(x) = e^x$. This is a very significant result, since this function is its own derivative—that is, $f(x) = f'(x)$. Since the derivative also represents the slope of the tangent at any given point, the function $f(x) = e^x$ has the special property that the slope of the tangent at a point is the value of the function at this point.



Derivative of $f(x) = e^x$
For the function $f(x) = e^x$, $f'(x) = e^x$.

EXAMPLE 1 Selecting a strategy to differentiate a composite function involving e^x

Determine the derivative of $f(x) = e^{3x}$.

Solution

To find the derivative, use the chain rule.

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d(e^{3x})}{d(3x)}\frac{d(3x)}{dx}\\ &= e^{3x} \times 3\\ &= 3e^{3x}\end{aligned}$$

Derivative of a Composite Function Involving e^x
In general, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)}\, g'(x)$ by the chain rule.

EXAMPLE 2 Derivatives of exponential functions involving e^x

Determine the derivative of each function.

a. $g(x) = e^{x^2 - x}$

b. $f(x) = x^2 e^x$

Solution

a. To find the derivative of $g(x) = e^{x^2 - x}$, we use the chain rule.

$$\begin{aligned}\frac{dg(x)}{dx} &= \frac{d(e^{x^2-x})}{dx}\\ &= \frac{d(e^{x^2-x})}{d(x^2-x)} \times \frac{d(x^2-x)}{dx} && \text{(Chain rule)}\\ &= e^{x^2-x}(2x-1)\end{aligned}$$

b. Using the product rule,

$$\begin{aligned}f'(x) &= \frac{d(x^2)}{dx} \times e^x + x^2 \times \frac{de^x}{dx} && \text{(Product rule)}\\ &= 2xe^x + x^2e^x && \text{(Factor)}\\ &= e^x(2x + x^2)\end{aligned}$$

EXAMPLE 3 Selecting a strategy to determine the value of the derivative

Given $f(x) = 3e^{x^2}$, determine $f'(-1)$.

Solution

First, find an expression for the derivative of $f'(x)$.

$$\begin{aligned}f'(x) &= \frac{d(3e^{x^2})}{d(x^2)}\frac{dx^2}{dx} && \text{(Chain rule)}\\ &= 3e^{x^2}(2x)\\ &= 6xe^{x^2}\end{aligned}$$

Then $f'(-1) = -6e$.

Answers are usually left as exact values in this form. If desired, numeric approximations can be obtained from a calculator. Here, using the value of e provided by the calculator, we obtain the answer -16.3097, rounded to four decimal places.

EXAMPLE 4 Connecting the derivative of an exponential function to the slope of a tangent

Determine the equation of the line tangent to $y = \frac{e^x}{x^2}$, where $x = 2$.

Solution

Use the derivative to determine the slope of the required tangent.

$$y = \frac{e^x}{x^2} \qquad \text{(Rewrite as a product)}$$

$$= x^{-2}e^x$$

$$\frac{dy}{dx} = (-2x^{-3})e^x + x^{-2}e^x \qquad \text{(Product rule)}$$

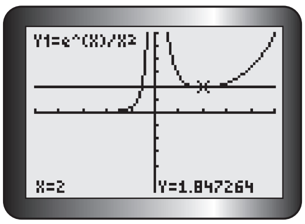
$$= \frac{-2e^x}{x^3} + \frac{e^x}{x^2} \qquad \text{(Determine a common denominator)}$$

$$= \frac{-2e^x}{x^3} + \frac{xe^x}{x^3} \qquad \text{(Simplify)}$$

$$= \frac{-2e^x + xe^x}{x^3} \qquad \text{(Factor)}$$

$$= \frac{(-2 + x)e^x}{x^3}$$

When $x = 2$, $y = \frac{e^2}{4}$. When $x = 2$, $\frac{dy}{dx} = 0$ and the tangent is horizontal. Therefore, the equation of the required tangent is $y = \frac{e^2}{4}$. A calculator yields the following graph for $y = \frac{e^x}{x^2}$, and we see the horizontal tangent at $x = 2$. The number Y $= 1.847264$ in the display is an approximation to the exact number $\frac{e^2}{4}$.



How does the derivative of the general exponential function $g(x) = b^x$ compare with the derivative of $f(x) = e^x$? We will answer this question in Section 5.2.

IN SUMMARY

Key Ideas

- For $f(x) = e^x$, $f'(x) = e^x$.

  In Leibniz notation, $\frac{d}{dx}(e^x) = e^x$.
- For $f(x) = e^{g(x)}$, $f'(x) = e^{g(x)} \times g'(x)$.

  In Leibniz notation, $\frac{d(e^{g(x)})}{dx} = \frac{d(e^{g(x)})}{d(g(x))} \frac{d(g(x))}{dx}$.
- The slope of the tangent at a point on the graph of $y = e^x$ equals the value of the function at this point.

Need to Know

- The rules for differentiating functions, such as the product, quotient, and chain rules, also apply to combinations involving exponential functions of the form $f(x) = e^{g(x)}$.
- e is called Euler's number or the natural number, where $e \doteq 2.718$.

Exercise 5.1

PART A

1. Why can you not use the power rule for derivatives to differentiate $y = e^x$?

2. Differentiate each of the following:

   a. $y = e^{3x}$

   b. $s = e^{3t-5}$

   c. $y = 2e^{10t}$

   d. $y = e^{-3x}$

   e. $y = e^{5-6x+x^2}$

   f. $y = e^{\sqrt{x}}$

K 3. Determine the derivative of each of the following:

   a. $y = 2e^{x^3}$

   b. $y = xe^{3x}$

   c. $f(x) = \frac{e^{-x^3}}{x}$

   d. $f(x) = \sqrt{x}e^x$

   e. $h(t) = et^2 + 3e^{-t}$

   f. $g(t) = \frac{e^{2t}}{1 + e^{2t}}$

4. a. If $f(x) = \frac{1}{3}(e^{3x} + e^{-3x})$, calculate $f'(1)$.

   b. If $f(x) = e^{-\left(\frac{1}{x+1}\right)}$, calculate $f'(0)$.

   c. If $h(z) = z^2(1 + e^{-z})$, calculate $h'(-1)$.

5. a. Determine the equation of the tangent to the curve defined by $y = \frac{2e^x}{1 + e^x}$ at the point $(0, 1)$.

   b. Use graphing technology to graph the function in part a., and draw the tangent at $(0, 1)$.

   c. Compare the equation in part a. with the equation generated by graphing technology. Do they agree?

PART B

6. Determine the equation of the tangent to the curve $y = e^{-x}$ at the point where $x = -1$. Graph the original curve and the tangent.

7. Determine the equation of the tangent to the curve defined by $y = xe^{-x}$ at the point $A(1, e^{-1})$.

8. Determine the coordinates of all points at which the tangent to the curve defined by $y = x^2e^{-x}$ is horizontal.

9. If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, prove that $y'' = \frac{y}{25}$.

10. a. For the function $y = e^{-3x}$, determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

    b. From the pattern in part a., state the value of $\frac{d^ny}{dx^n}$.

11. Determine the first and second derivatives of each function.

    a. $y = -3e^x$ b. $y = xe^{2x}$ c. $y = e^x(4 - x)$

A 12. The number, N, of bacteria in a culture at time t, in hours, is $N(t) = 1000\left[30 + e^{-\frac{t}{30}}\right]$

    a. What is the initial number of bacteria in the culture?

    b. Determine the rate of change in the number of bacteria at time t.

    c. How fast is the number of bacteria changing when $t = 20$?

    d. Determine the largest number of bacteria in the culture during the interval $0 \leq t \leq 50$.

    e. What is happening to the number of bacteria in the culture as time passes?

13. The distance s, in metres, fallen by a skydiver t seconds after jumping (and before the parachute opens) is $s = 160\left(\frac{1}{4}t - 1 + e^{-\frac{t}{4}}\right)$.

    a. Determine the velocity, v, at time t.

    b. Show that acceleration is given by $a = 10 - \frac{1}{4}v$.

    c. Determine $v_T = \lim\limits_{t\to\infty} v$. This is the "terminal" velocity, the constant velocity attained when the air resistance balances the force of gravity.

    d. At what time is the velocity 95% of the terminal velocity? How far has the skydiver fallen at that time?

C 14. a. Use a table of values and successive approximation to evaluate each of the following:

    i. $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)$ ii. $\lim\limits_{x\to 0}(1 + x)^{\frac{1}{x}}$

    b. Discuss your results.

PART C

T 15. Use the definition of the derivative to evaluate each limit.

a. $\lim_{h \to 0} \frac{e^h - 1}{h}$

b. $\lim_{h \to 0} \frac{e^{2+h} - e^2}{h}$

16. For what values of m does the function $y = Ae^{mt}$ satisfy the following equation?

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

17. The hyperbolic functions are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

a. Prove $\frac{d(\sinh x)}{dx} = \cosh x$.

b. Prove $\frac{d(\cosh x)}{dx} = \sinh x$.

c. Prove $\frac{d(\tanh x)}{dx} = \frac{1}{(\cosh x)^2}$ if $\tanh x = \frac{\sinh x}{\cosh x}$.

Extension: Graphing the Hyperbolic Function

1. Use graphing technology to graph $y = \cosh x$ by using the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

CATALOG

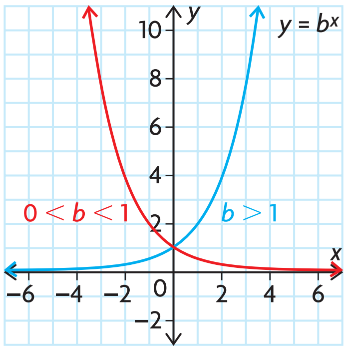
2. Press 2ND 0 for the list of CATALOG items, and select **cosh(** to investigate if cosh is a built-in function.
3. In the same window as problem 1, graph $y = 1.25x^2 + 1$ and $y = 1.05x^2 + 1$. Investigate changes in the coefficient a in the equation $y = ax^2 + 1$ to see if you can create a parabola that will approximate the hyperbolic cosine function.

18. a. Another expression for e is $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. Evaluate this expression using four, five, six, and seven consecutive terms of this expression. (*Note:* 2! is read "two factorial"; $2! = 2 \times 1$ and $5! = 5 \times 4 \times 3 \times 2 \times 1$.)

b. Explain why the expression for e in part a. is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$. What is the value of x?

Section 5.2—The Derivative of the General Exponential Function, $y = b^x$

In the previous section, we investigated the exponential function $y = e^x$ and its derivative. The exponential function has a special property—it is its own derivative. The graph of the derivative function is the same as the graph of $y = e^x$. In this section, we will look at the general exponential function $y = b^x$ and its derivative.



INVESTIGATION

In this investigation, you will

- graph and compare the general exponential function and its derivative using the slopes of the tangents at various points and with different bases
- determine the relationship between the general exponential function and its derivative by means of a special ratio

A. Consider the function $f(x) = 2^x$. Create a table with the headings shown below. Use the equation of the function to complete the $f(x)$ column.

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$
−2			
−1			
0			
1			
2			
3			

B. Graph the function $f(x) = 2^x$.

C. Calculate the value of the derivative $f'(x)$ at each of the given points to three decimal places. To calculate $f'(x)$, use the **nDeriv(** function. (See the investigation in Section 5.1 for detailed instructions.)

D. Draw the graph of the derivative function on the same set of axes as $f(x)$ using the given x values and the corresponding values of $f'(x)$.

E. Compare the graph of the derivative with the graph of $f(x)$.

F. i. Calculate the ratio $\frac{f'(x)}{f(x)}$, and record these values in the last column of your table.

ii. What do you notice about this ratio for the different values of x?

iii. Is the ratio greater or less than 1?

G. Repeat parts A to F for the function $f(x) = 3^x$.

H. Compare the ratio $\frac{f'(x)}{f(x)}$ for the functions $f(x) = 2^x$ and $f(x) = 3^x$.

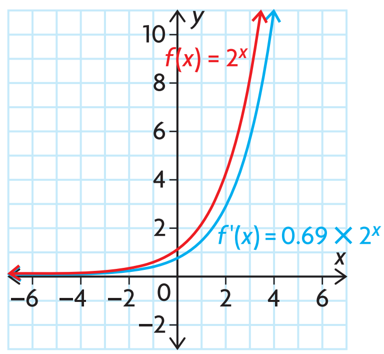
I. Repeat parts A to F for the function $f(x) = b^x$ using different values of b. Does the pattern you found for $f(x) = 2^x$ and $f(x) = 3^x$ continue?

J. What conclusions can you make about the general exponential function and its derivative?

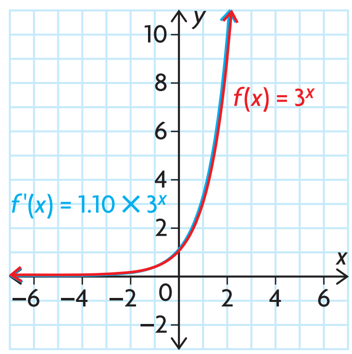
Properties of $y = b^x$

In this investigation, you worked with the functions $f(x) = 2^x$ and $f(x) = 3^x$, and their derivatives. You should have made the following observations:

- For the function $f(x) = 2^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 0.69.
- The derivative of $f(x) = 2^x$ is approximately equal to 0.69×2^x.
- For the function $f(x) = 3^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 1.10.
- The derivative of $f(x) = 3^x$ is approximately equal to 1.10×3^x.



The derivative of $f(x) = 2^x$ is an exponential function.
The graph of $f'(x)$ is a vertical compression of the graph of $f(x)$.



The derivative of $f(x) = 3^x$ is an exponential function.
The graph of $f'(x)$ is a vertical stretch of the graph of $f(x)$.

In general, for the exponential function $f(x) = b^x$, we can conclude that

- $f(x)$ and $f'(x)$ are both exponential functions
- the slope of the tangent at a point on the curve is proportional to the value of the function at this point
- $f'(x)$ is a vertical stretch or compression of $f(x)$, dependent on the value of b
- the ratio $\frac{f'(x)}{f(x)}$ is a constant and is equivalent to the stretch/compression factor

We can use the definition of the derivative to determine the derivative of the exponential function $f(x) = b^x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0}\frac{b^{x+h} - b^x}{h} && \text{(Substitution)} \\
&= \lim_{h\to 0}\frac{b^x \times b^h - b^x}{h} && \text{(Properties of the exponential function)} \\
&= \lim_{h\to 0}\frac{b^x(b^h - 1)}{h} && \text{(Common factor)}
\end{aligned}$$

The factor b^x is constant as $h \to 0$ and does not depend on h. Therefore, $f'(x) = b^x \lim_{h\to 0}\frac{b^h - 1}{h}$.

Consider the functions from our investigation:

- For $f(x) = 2^x$, we determined that $f'(x) \doteq 0.69 \times 2^x$ and so $\lim_{h\to 0}\frac{2^h - 1}{h} \doteq 0.69$.
- For $f(x) = 3^x$, we determined that $f'(x) \doteq 1.10 \times 3^x$ and so $\lim_{h\to 0}\frac{3^h - 1}{h} \doteq 1.10$.

In the previous section, for $f(x) = e^x$, we determined that $f'(x) = e^x$ and so $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$.

Can we find a way to determine this constant of proportionality without using a table of values?

The derivative of $f(x) = e^x$ might give us a hint at the answer to this question. From the previous section, we know that $f'(x) = 1 \times e^x$.

We also know that $\log_e e = 1$, or $\ln e = 1$.
Now consider ln 2 and ln 3.

$\ln 2 \doteq 0.693\ 147$ and $\ln 3 \doteq 1.098\ 612$

These match the constants $\frac{f'(x)}{f(x)}$ that we determined in our investigation.

This leads to the following conclusion:

Derivative of $f(x) = b^x$

$\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b$ and if $f(x) = b^x$, then $f'(x) = (\ln b) \times b^x$

EXAMPLE 1 Selecting a strategy to determine derivatives involving b^x

Determine the derivative of

a. $f(x) = 5^x$

b. $f(x) = 5^{3x-2}$

Solution

a. $f(x) = 5^x$

Use the derivative of $f(x) = b^x$.

$f'(x) = (\ln 5) \times 5^x$

b. To differentiate $f(x) = 5^{3x-2}$, use the chain rule and the derivative of $f(x) = b^x$.

$f(x) = 5^{3x-2}$

We have $f(x) = 5^{g(x)}$ with $g(x) = 3x - 2$.

Then $g'(x) = 3$

Now, $f'(x) = 5^{3x-2} \times (\ln 5) \times 3$

$= 3(5^{3x-2})\ln 5$

Derivative of $f(x) = b^{g(x)}$

For $f(x) = b^{g(x)}$, $f'(x) = b^{g(x)}(\ln b)(g'(x))$

EXAMPLE 2 Solving a problem involving an exponential model

On January 1, 1850, the population of Goldrushtown was 50 000. The size of the population since then can be modelled by the function $P(t) = 50\ 000(0.98)^t$, where t is the number of years since January 1, 1850.

a. What was the population of Goldrushtown on January 1, 1900?

b. At what rate was the population of Goldrushtown changing on January 1, 1900? Was it increasing or decreasing at that time?

Solution

a. January 1, 1900, is exactly 50 years after January 1, 1850, so we let $t = 50$.

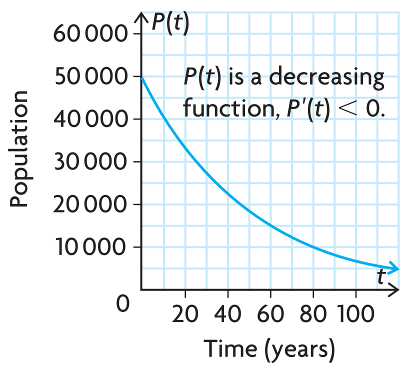
$$P(50) = 50\,000(0.98)^{50}$$
$$= 18\,208.484$$

The population on January 1, 1900, was approximately 18 208.

b. To determine the rate of change in the population, we require the derivative of P.

$$P'(t) = 50\,000(0.98)^t \ln(0.98)$$
$$P'(50) = 50\,000(0.98)^{50} \ln(0.98)$$
$$\doteq -367.861$$

Hence, after 50 years, the population was decreasing at a rate of approximately 368 people per year. (We expected the rate of change to be negative, because the original population function was a decaying exponential function since the base was less than 1.)



IN SUMMARY

Key Ideas

- If $f(x) = b^x$, then $f'(x) = b^x \times \ln b$.

  In Leibniz notation, $\frac{d}{dx}(b^x) = b^x \times \ln b$.
- If $f(x) = b^{g(x)}$, then $f'(x) = b^{g(x)} \times \ln b \times g'(x)$.

  In Leibniz notation, $\frac{d}{dx}(b^{g(x)}) = \frac{d(b^{g(x)})}{d(g(x))} \frac{d(g(x))}{dx}$.

Need to Know

- $\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b$
- When you are differentiating a function that involves an exponential function, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

Exercise 5.2

PART A

K

1. Differentiate each of the following functions:

   a. $y = 2^{3x}$

   b. $y = 3.1^x + x^3$

   c. $s = 10^{3t-5}$

   d. $w = 10^{(5-6n+n^2)}$

   e. $y = 3^{x^2+2}$

   f. $y = 400(2)^{x+3}$

2. Determine the derivative of each function.

   a. $y = x^5 \times (5)^x$

   b. $y = x(3)^{x^2}$

   c. $v = \dfrac{2^t}{t}$

   d. $f(x) = \dfrac{\sqrt{3^x}}{x^2}$

3. If $f(t) = 10^{3t-5} \times e^{2t^2}$, determine the values of t so that $f'(t) = 0$.

PART B

4. Determine the equation of the tangent to $y = 3(2^x)$ at $x = 3$.

5. Determine the equation of the tangent to $y = 10^x$ at $(1, 10)$.

A

6. A certain radioactive material decays exponentially. The percent, P, of the material left after t years is given by $P(t) = 100(1.2)^{-t}$.

   a. Determine the half-life of the substance.

   b. How fast is the substance decaying at the point where the half-life is reached?

T

7. Historical data show that the amount of money sent out of Canada for interest and dividend payments during the period from 1967 to 1979 can be approximated by the model $P = (5 \times 10^8)e^{0.200\,15t}$, where t is measured in years $(t = 0$ in $1967)$ and P is the total payment in Canadian dollars.

   a. Determine and compare the rates of increase for the years 1968 and 1978.

   b. Assuming this trend continues, compare the rate of increase for 1988 with the rate of increase for 1998.

   c. Check the Statistics Canada website to see if the rates of increase predicted by this model were accurate for 1988 and 1998.

8. Determine the equation of the tangent to the curve $y = 2^{-x^2}$ at the point on the curve where $x = 0$. Graph the curve and the tangent at this point.

PART C

C

9. The velocity of a car is given by $v(t) = 120(1 - 0.85^t)$. Graph the function. Describe the acceleration of the car.

Section 5.3—Optimization Problems Involving Exponential Functions

In earlier chapters, you considered numerous situations in which you were asked to optimize a given situation. As you learned, to optimize means to determine values of variables so that a function representing quantities such as cost, area, number of objects, or distance can be minimized or maximized.

Here we will consider further optimization problems, using exponential function models.

EXAMPLE 1 **Solving an optimization problem involving an exponential model**

The effectiveness of studying for an exam depends on how many hours a student studies. Some experiments show that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$, where t is the number of hours spent studying for an examination. If a student has up to 30 h for studying, how many hours are needed for maximum effectiveness?

Solution

We wish to find the maximum value of the function $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$, on the interval $0 \leq t \leq 30$.

First find critical numbers by determining $E'(t)$.

$$\begin{aligned} E'(t) &= 0.5\left(e^{-\frac{t}{20}} + t\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)\right) \\ &= 0.5e^{-\frac{t}{20}}\left(1 - \frac{t}{20}\right) \end{aligned}$$

(Product and chain rules)

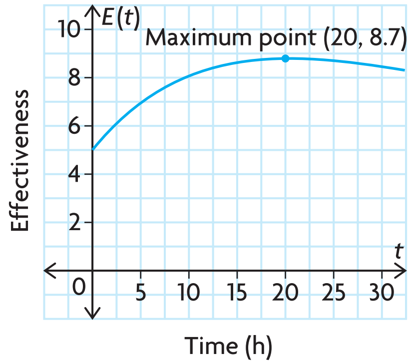
E' is defined for $t \in \mathbf{R}$, and $e^{-\frac{t}{20}} > 0$ for all values of t. So, $E'(t) = 0$ when $1 - \frac{t}{20} = 0$.
Therefore, $t = 20$ is the only critical number.

To determine the maximum effectiveness, we use the algorithm for finding extreme values.

$$\begin{aligned} E(0) &= 0.5(10 + 0e^{0}) = 5 \\ E(20) &= 0.5(10 + 20e^{-1}) \doteq 8.7 \\ E(30) &= 0.5(10 + 30e^{-1.5}) \doteq 8.3 \end{aligned}$$

Therefore, the maximum effectiveness measure of 8.7 is achieved when a student studies 20 h for the exam.

Examining the graph of the function $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$ confirms our result.



EXAMPLE 2 Using calculus techniques to analyze an exponential business model

A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for t days is given by $f(t) = 0.7(1 - e^{-0.2t})$. The area covered by the advertisement contains 10 million potential customers, and each response to the advertisement results in revenue to the company of \$0.70 (on average), excluding the cost of advertising. The advertising costs \$30 000 to produce and a further \$5000 per day to run.

a. Determine $\lim_{t \to \infty} f(t)$, and interpret the result.

b. What percent of potential customers have responded after seven days of advertising?

c. Write the function $P(t)$ that represents the average profit after t days of advertising. What is the average profit after seven days?

d. For how many full days should the advertising campaign be run in order to maximize the average profit? Assume an advertising budget of \$200 000.

Solution

a. As $t \to \infty$, $e^{-0.2t} \to 0$, so $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} 0.7(1 - e^{-0.2t}) = 0.7$. This result means that if the advertising is left in place indefinitely (forever), 70% of the population will respond.

b. $f(7) = 0.7(1 - e^{-0.2(7)}) \doteq 0.53$

After seven days of advertising, about 53% of the population has responded.

c. The average profit is the difference between the average revenue received from all customers responding to the ad and the advertising costs. Since the area covered by the ad contains 10 million potential customers, the number of customers responding to the ad after t days is $10^7[0.7(1 - e^{-0.2t})] = 7 \times 10^6(1 - e^{-0.2t})$.

The average revenue to the company from these respondents is $R(t) = 0.7[7 \times 10^6(1 - e^{-0.2t})] = 4.9 \times 10^6(1 - e^{-0.2t})$.
The advertising costs for t days are $C(t) = 30\,000 + 5000t$.
Therefore, the average profit earned after t days of advertising is given by

$$\begin{aligned} P(t) &= R(t) - C(t) \\ &= 4.9 \times 10^6(1 - e^{-0.2t}) - 30\,000 - 5000t \end{aligned}$$

After seven days of advertising, the average profit is

$$\begin{aligned} P(7) &= 4.9 \times 10^6(1 - e^{-0.2(7)}) - 30\,000 - 5000(7) \\ &\doteq 3\,627\,000 \end{aligned}$$

d. If the total advertising budget is \$200 000, then we require that

$$\begin{aligned} 30\,000 + 5000t &\le 200\,000 \\ 5000t &\le 170\,000 \\ t &\le 34 \end{aligned}$$

We wish to maximize the average profit function $P(t)$ on the interval $0 \le t \le 34$.
For critical numbers, determine $P'(t)$.

$$\begin{aligned} P'(t) &= 4.9 \times 10^6(0.2e^{-0.2t}) - 5000 \\ &= 9.8 \times 10^5 e^{-0.2t} - 5000 \end{aligned}$$

$P'(t)$ is defined for $t \in \mathbf{R}$. Let $P'(t) = 0$.

$$9.8 \times 10^5 e^{-0.2t} - 5000 = 0$$

$$e^{-0.2t} = \frac{5000}{9.8 \times 10^5} \quad \text{(Isolate } e^{-0.2t}\text{)}$$

$$e^{-0.2t} \doteq 0.005\,102\,04 \quad \text{(Take the ln of both sides)}$$

$$-0.2t = \ln(0.005\,102\,04) \quad \text{(Solve)}$$

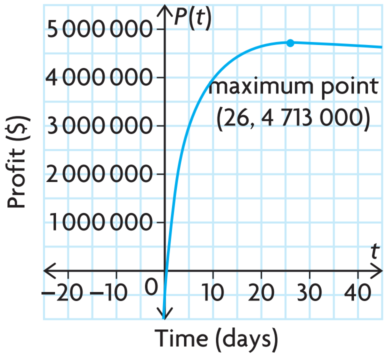
$$t \doteq 26$$

To determine the maximum average profit, we evaluate.

$$\begin{aligned} P(26) &= 4.9 \times 10^6(1 - e^{-0.2(26)}) - 30\,000 - 5000(26) \\ &\doteq 4\,713\,000 \\ P(0) &= 4.9 \times 10^6(1 - e^0) - 30\,000 - 0 \\ &= -30\,000 \text{ (They're losing money!)} \\ P(34) &= 4.9 \times 10^6(1 - e^{-0.2(34)}) - 30\,000 - 5000(34) \\ &\doteq 4\,695\,000 \end{aligned}$$

The maximum average profit of \$4 713 000 occurs when the ad campaign runs for 26 days.

Examining the graph of the function $P(t)$ confirms our result.



IN SUMMARY

Key Ideas

- Optimizing means determining the values of the independent variable so that the values of a function that models a situation can be minimized or maximized.
- The techniques used to optimize an exponential function model are the same as those used to optimize polynomial and rational functions.

Need to Know

- Apply the algorithm introduced in Chapter 3 to solve an optimization problem:

1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
2. Determine the domain of the function to be optimized, using the information given in the problem.
3. Use the algorithm for finding extreme values (from Chapter 3) to find the absolute maximum or minimum value of the function on the domain.
4. Use your result from step 3 to answer the original problem.
5. Graph the original function using technology to confirm your results.

Exercise 5.3

PART A

1. Use graphing technology to graph each of the following functions. From the graph, find the absolute maximum and absolute minimum values of the given functions on the indicated intervals.

   a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq x \leq 10$

   b. $m(x) = (x + 2)e^{-2x}$ on $x \in [-4, 4]$

2. a. Use the algorithm for finding extreme values to determine the absolute maximum and minimum values of the functions in question 1.

   b. Explain which approach is easier to use for the functions in question 1.

3. The squirrel population in a small self-contained forest was studied by a biologist. The biologist found that the squirrel population, P, measured in hundreds, is a function of time, t, where t is measured in weeks. The function is $P(t) = \frac{20}{1 + 3e^{-0.02t}}$.

   a. Determine the population at the start of the study, when $t = 0$.

   b. The largest population the forest can sustain is represented mathematically by the limit as $t \to \infty$. Determine this limit.

   c. Determine the point of inflection.

   d. Graph the function.

   e. Explain the meaning of the point of inflection in terms of squirrel population growth.

PART B

4. The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, where x is the number of items sold.

   a. Determine the number of items that yield the maximum profit. At full capacity, the factory can produce 2000 items per month.

   b. Repeat part a., assuming that, at most, 500 items can be produced per month.

K 5. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

6. A rumour spreads through a population in such a way that t hours after the rumour starts, the percent of people involved in passing it on is given by $P(t) = 100\left(e^{-t} - e^{-4t}\right)$. What is the highest percent of people involved in spreading the rumour within the first 3 h? When does this occur?

7. Small countries trying to develop an industrial economy rapidly often try to achieve their objectives by importing foreign capital and technology. Statistics Canada data show that when Canada attempted this strategy from 1867 to 1967, the amount of U.S. investment in Canada increased from about $\$15 \times 10^6$ to $\$280\,305 \times 10^6$. This increase in foreign investment can be represented by the simple mathematical model $C(t) = 0.015 \times 10^9 e^{0.075\,33t}$, where t represents the number of years (starting with 1867 as zero) and C represents the total capital investment from U.S. sources in dollars.
   a. Graph the curve for the 100-year period.
   b. Compare the growth rate of U.S. investment in 1947 with the rate in 1967.
   c. Determine the growth rate of investment in 1967 as a percent of the amount invested.
   d. If this model is used up to 1977, calculate the total U.S. investment and the growth rate in this year.
   e. Use the Internet to determine the actual total U.S. investment in 1977, and calculate the error in the model.
   f. If the model is used up to 2007, calculate the expected U.S. investment and the expected growth rate.

A 8. A colony of bacteria in a culture grows at a rate given by $N(t) = 2^{\frac{t}{5}}$, where N is the number of bacteria t minutes from the beginning. The colony is allowed to grow for 60 min, at which time a drug is introduced to kill the bacteria. The number of bacteria killed is given by $K(t) = e^{\frac{t}{3}}$, where K bacteria are killed at time t minutes.
   a. Determine the maximum number of bacteria present and the time at which this occurs.
   b. Determine the time at which the bacteria colony is obliterated.

9. Lorianne is studying for two different exams. Because of the nature of the courses, the measure of study effectiveness on a scale from 0 to 10 for the first course is $E_1 = 0.6(9 + te^{-\frac{t}{20}})$, while the measure for the second course is $E_2 = 0.5(10 + te^{-\frac{t}{10}})$. Lorianne is prepared to spend up to 30 h, in total, studying for the exams. The total effectiveness is given by $f(t) = E_1 + E_2$. How should this time be allocated to maximize total effectiveness?

C 10. Explain the steps you would use to determine the absolute extrema of $f(x) = x - e^{2x}$ on the interval $x \in [-2, 2]$.

T 11. a. For $f(x) = x^2e^x$, determine the intervals of increase and decrease.
   b. Determine the absolute minimum value of $f(x)$.

12. Find the maximum and minimum values of each function. Graph each function.

a. $y = e^x + 2$

b. $y = xe^x + 3$

c. $y = 2xe^{2x}$

d. $y = 3xe^{-x} + x$

13. The profit function of a commodity is $P(x) = xe^{-0.5x^2}$, where $x > 0$. Find the maximum value of the function if x is measured in hundreds of units and P is measured in thousands of dollars.

14. You have just walked out the front door of your home. You notice that it closes quickly at first and then closes more slowly. In fact, a model of the movement of the door is given by $d(t) = 200\,t(2)^{-t}$, where d is the number of degrees between the door frame and the door at t seconds.

a. Graph this relation.

b. Determine when the speed of the moving door is increasing and decreasing.

c. Determine the maximum speed of the moving door.

d. At what point would you consider the door closed?

PART C

15. Suppose that, in question 9, Lorianne has only 25 h to study for the two exams. Is it possible to determine the time to be allocated to each exam? If so, how?

16. Although it is true that many animal populations grow exponentially for a period of time, it must be remembered that the food available to sustain the population is limited and the population will level off because of this. Over a period of time, the population will level out to the maximum attainable value, L. One mathematical model to describe a population that grows exponentially at the beginning and then levels off to a limiting value, L, is the **logistic model**. The equation for this model is $P = \frac{aL}{a + (L - a)e^{-kLt}}$, where the independent variable t represents the time and P represents the size of the population. The constant a is the size of the population at $t = 0$, L is the limiting value of the population, and k is a mathematical constant.

a. Suppose that a biologist starts a cell colony with 100 cells and finds that the limiting size of the colony is 10 000 cells. If the constant $k = 0.0001$, draw a graph to illustrate this population, where t is in days.

b. At what point in time does the cell colony stop growing exponentially? How large is the colony at this point?

c. Compare the growth rate of the colony at the end of day 3 with the growth rate at the end of day 8. Explain what is happening.

Mid-Chapter Review

1. Determine the derivative of each function.
   a. $y = 5e^{-3x}$
   b. $y = 7e^{\frac{1}{7}x}$
   c. $y = x^3e^{-2x}$
   d. $y = (x - 1)^2e^x$
   e. $y = (x - e^{-x})^2$
   f. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

2. A certain radioactive substance decays exponentially over time. The amount of a sample of the substance that remains, P, after t years is given by $P(t) = 100e^{-5t}$, where P is expressed as a percent.
   a. Determine the rate of change of the function, $\frac{dP}{dt}$.
   b. What is the rate of decay when 50% of the original sample has decayed?

3. Determine the equation of the tangent to the curve $y = 2 - xe^x$ at the point where $x = 0$.

4. Determine the first and second derivatives of each function.
   a. $y = -3e^x$
   b. $y = xe^{2x}$
   c. $y = e^x(4 - x)$

5. Determine the derivative of each function.
   a. $y = 8^{2x+5}$
   b. $y = 3.2(10)^{0.2x}$
   c. $f(x) = x^22^x$
   d. $H(x) = 300(5)^{3x-1}$
   e. $q(x) = 1.9^x + x^{1.9}$
   f. $f(x) = (x - 2)^2 \times 4^x$

6. The number of rabbits in a forest at time t, in months, is $R(t) = 500[10 + e^{-\frac{t}{10}}]$.
   a. What is the initial number of rabbits in the forest?
   b. Determine the rate of change of the number of rabbits at time t.
   c. How fast is the number of rabbits changing after one year?
   d. Determine the largest number of rabbits in the forest during the first three years.
   e. Use graphing technology to graph R versus t. Give physical reasons why the population of rabbits might behave this way.

7. A drug is injected into the body in such a way that the concentration, C, in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

8. Given $y = c(e^{kx})$, for what values of k does the function represent growth? For what values of k does the function represent decay?

9. The rapid growth in the number of a species of insect is given by $P(t) = 5000e^{0.02t}$, where t is the number of days.
   a. What is the initial population $(t = 0)$?
   b. How many insects will there be after a week?
   c. How many insects will there be after a month (30 days)?
10. If you have ever travelled in an airplane, you probably noticed that the air pressure in the airplane varied. The atmospheric pressure, y, varies with the altitude, x kilometres, above Earth. For altitudes up to 10 km, the pressure in millimetres of mercury (mm Hg) is given by $y = 760e^{-0.125x}$. What is the atmospheric pressure at each distance above Earth?
    a. 5 km b. 7 km c. 9 km
11. A radioactive substance decays in such a way that the amount left after t years is given by $A = 100e^{-0.3t}$. The amount, A, is expressed as a percent. Find the function, A', that describes the rate of decay. What is the rate of decay when 50% of the substance is gone?
12. Given $f(x) = xe^x$, find all the x values for which $f'(x) > 0$. What is the significance of this?
13. Find the equation of the tangent to the curve $y = 5^{-x^2}$ at the point on the curve where $x = 1$. Graph the curve and the tangent at this point.
14. a. Determine an equation for $A(t)$, the amount of money in the account at any time t.
    b. Find the derivative $A'(t)$ of the function.
    c. At what rate is the amount growing at the end of two years? At what rate is it growing at the end of five years and at the end of 10 years?
    d. Is the rate constant?
    e. Determine the ratio of $\frac{A'(t)}{A(t)}$ for each value that you determined for $A'(t)$.
    f. What do you notice?
15. The function $y = e^x$ is its own derivative. It is not the only function, however, that has this property. Show that for every value of c, $y = c\,(e^x)$ has the same property.

Section 5.4—The Derivatives of $y = \sin x$ and $y = \cos x$

In this section, we will investigate to determine the derivatives of $y = \sin x$ and $y = \cos x$.

INVESTIGATION 1

A. Using a graphing calculator, graph $y = \sin x$, where x is measured in radians. Use the following WINDOW settings:

- Xmin = 0, Xmax = 9.4, Xscl = $\pi \div 2$
- Ymin = −3.1, Ymax = 3.1, Yscl = 1

Enter $y = \sin x$ into Y1, and graph the function.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute y and $\frac{dy}{dx}$, respectively, for $y = \sin x$. Record these values in a table like the following (correct to four decimal places):

x	$\sin x$	$\frac{d}{dx}(\sin x)$
0		
0.5		
1.0		
:		
:		
:		
6.5		

Tech Support

To calculate $\frac{dy}{dx}$ at a point, press 2ND TRACE 6 and enter the desired x-coordinate of your point. Then press ENTER.

C. Create another column, to the right of the $\frac{d}{dx}(\sin x)$ column, with $\cos x$ as the heading. Using your graphing calculator, graph $y = \cos x$ with the same window settings as above.

D. Compute the values of $\cos x$ for $x = 0, 0.5, 1.0, \ldots, 6.5$, correct to four decimal places. Record the values in the $\cos x$ column.

E. Compare the values in the $\frac{d}{dx}(\sin x)$ column with those in the $\cos x$ column, and write a concluding equation.

Tech Support

For help calculating a value of a function using a graphing calculator, see Technical Appendix p. 598.

INVESTIGATION 2

A. Using your graphing calculator, graph $y = \cos x$, where x is measured in radians. Use the following WINDOW settings:

- Xmin = 0, Xmax = 9.4, Xscl = $\pi \div 2$
- Ymin = -3.1, Ymax = 3.1, Yscl = 1

Enter $y = \cos x$ into Y1, and graph the function.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute y and $\frac{dy}{dx}$, respectively, for $y = \cos x$. Record these values, correct to four decimal places, in a table like the following:

x	$\cos x$	$\frac{d}{dx}(\cos x)$
0		
0.5		
1.0		
:		
:		
:		
6.5		

C. Create another column to the right of the $\frac{d}{dx}(\cos x)$ column with $-\sin x$ as the heading. Using your graphing calculator, graph $y = -\sin x$ with the same window settings as above.

D. Compute the values of $-\sin x$ for $x = 0, 0.5, 1.0, \dots, 6.5$, correct to four decimal places. Record the values in the $-\sin x$ column.

E. Compare the values in the $\frac{d}{dx}(\cos x)$ column with those in the $-\sin x$ column, and write a concluding equation.

Investigations 1 and 2 lead to the following conclusions:

Derivatives of Sinusoidal Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 1 Selecting a strategy to determine the derivative of a sinusoidal function

Determine $\frac{dy}{dx}$ for each function.

a. $y = \cos 3x$

b. $y = x \sin x$

Solution

a. To differentiate this function, use the chain rule.

$$y = \cos 3x$$

$$\frac{dy}{dx} = \frac{d(\cos 3x)}{d(3x)} \times \frac{d(3x)}{dx} \quad \text{(Chain rule)}$$

$$= -\sin 3x \times (3)$$

$$= -3 \sin 3x$$

b. To find the derivative, use the product rule.

$$y = x \sin x$$

$$\frac{dy}{dx} = \frac{dx}{dx} \times \sin x + x\frac{d(\sin x)}{dx} \quad \text{(Product rule)}$$

$$= (1) \times \sin x + x \cos x$$

$$= \sin x + x \cos x$$

EXAMPLE 2 Reasoning about the derivatives of sinusoidal functions

Determine $\frac{dy}{dx}$ for each function.

a. $y = \sin x^2$

b. $y = \sin^2 x$

Solution

a. To differentiate this composite function, use the chain rule and change of variable.

Here, the inner function is $u = x^2$, and the outer function is $y = \sin u$.

$$\text{Then, } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

$$= (\cos u)(2x) \quad \text{(Substitute)}$$

$$= 2x \cos x^2$$

b. Since $y = \sin^2 x = (\sin x)^2$, we use the chain rule with $y = u^2$, where $u = \sin x$.

$$\text{Then, } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

$$= (2u)(\cos x) \quad \text{(Substitute)}$$

$$= 2 \sin x \cos x$$

With practice, you will learn how to apply the chain rule without the intermediate step of introducing the variable u. For $y = \sin x^2$, for example, you can skip this step and immediately write $\frac{dy}{dx} = (\cos x^2)(2x)$.

Derivatives of Composite Sinusoidal Functions

If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\sin f(x)) = \frac{d(\sin f(x))}{d(f(x))} \times \frac{d(f(x))}{dx} = \cos f(x) \times \frac{d(f(x))}{dx}$.

If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\cos f(x)) = \frac{d(\cos f(x))}{d(f(x))} \times \frac{d(f(x))}{dx} = -\sin f(x) \times \frac{d(f(x))}{dx}$.

EXAMPLE 3 Differentiating a composite cosine function

Determine $\frac{dy}{dx}$ for $y = \cos(1 + x^3)$.

Solution

$$y = \cos(1 + x^3)$$

$$\frac{dy}{dx} = \frac{d[\cos(1 + x^3)]}{d(1 + x^3)} \times \frac{d(1 + x^3)}{dx} \qquad \text{(Chain rule)}$$

$$= -\sin(1 + x^3)(3x^2)$$

$$= -3x^2 \sin(1 + x^3)$$

EXAMPLE 4 Differentiating a combination of functions

Determine y' for $y = e^{\sin x + \cos x}$.

Solution

$$y = e^{\sin x + \cos x}$$

$$y' = \frac{d(e^{\sin x + \cos x})}{d(\sin x + \cos x)} \times \frac{d(\sin x + \cos x)}{dx} \qquad \text{(Chain rule)}$$

$$= e^{\sin x + \cos x}(\cos x - \sin x)$$

EXAMPLE 5 Connecting the derivative of a sinusoidal function to the slope of a tangent

Determine the equation of the tangent to the graph of $y = x\cos 2x$ at $x = \frac{\pi}{2}$.

Solution

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}\cos \pi = -\frac{\pi}{2}$.

The point of tangency is $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

The slope of the tangent at any point on the graph is given by

$$\frac{dy}{dx} = \frac{dx}{dx} \times \cos 2x + x \times \frac{d(\cos 2x)}{dx}$$ (Product and chain rules)

$$= (1)(\cos 2x) + x(-\sin 2x)(2)$$ (Simplify)

$$= \cos 2x - 2x\sin 2x$$

At $x = \frac{\pi}{2}$, $\frac{dy}{dx} = \cos \pi - \pi(\sin \pi)$ (Evaluate)

$$= -1$$

The equation of the tangent is

$$y + \frac{\pi}{2} = -\left(x - \frac{\pi}{2}\right) \text{ or } y = -x.$$

EXAMPLE 6 Connecting the derivative of a sinusoidal function to its extreme values

Determine the maximum and minimum values of the function $f(x) = \cos^2 x$ on the interval $x \in [0, 2\pi]$.

Solution

By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$ or at endpoints of the interval. The derivative of $f(x)$ is

$$f'(x) = 2(\cos x)(-\sin x)$$ (Chain rule)

$$= -2\sin x\cos x$$

$$= -\sin 2x$$ (Using the double angle identity)

Solving $f'(x) = 0$,

$$-\sin 2x = 0$$

$$\sin 2x = 0$$

$$2x = 0, \pi, 2\pi, 3\pi, \text{ or } 4\pi$$

$$\text{so } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi$$

We evaluate $f(x)$ at the critical numbers. (In this example, the endpoints of the interval are included.)

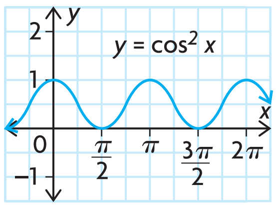
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) = \cos^2 x$	1	0	1	0	1

The maximum value is 1 when $x = 0$, π, or 2π. The minimum value is 0 when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The above solution is verified by our knowledge of the cosine function. For the function $y = \cos x$,
- the domain is $x \in \mathbf{R}$
- the range is $-1 \leq \cos x \leq 1$

For the given function $y = \cos^2 x$,
- the domain is $x \in \mathbf{R}$
- the range is $0 \leq \cos^2 x \leq 1$

Therefore, the maximum value is 1 and the minimum value is 0.



IN SUMMARY

Key Idea

- The derivatives of sinusoidal functions are found as follows:
  - $\frac{d(\sin x)}{dx} = \cos x$ and $\frac{d(\cos x)}{dx} = -\sin x$
  - If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.
  - If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

Need to Know

- When you are differentiating a function that involves sinusoidal functions, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

Exercise 5.4

PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \sin 2x$
b. $y = 2 \cos 3x$
c. $y = \sin (x^3 - 2x + 4)$
d. $y = 2 \cos (-4x)$
e. $y = \sin 3x - \cos 4x$
f. $y = 2^x + 2 \sin x - 2 \cos x$
g. $y = \sin (e^x)$
h. $y = 3 \sin (3x + 2\pi)$
i. $y = x^2 + \cos x + \sin \frac{\pi}{4}$
j. $y = \sin \frac{1}{x}$

2. Differentiate the following functions:

a. $y = 2 \sin x \cos x$
b. $y = \frac{\cos 2x}{x}$
c. $y = \cos (\sin 2x)$
d. $y = \frac{\sin x}{1 + \cos x}$
e. $y = e^x(\cos x + \sin x)$
f. $y = 2x^3 \sin x - 3x \cos x$

PART B

3. Determine an equation for the tangent at the point with the given x-coordinate for each of the following functions:

a. $f(x) = \sin x, x = \frac{\pi}{3}$
b. $f(x) = x + \sin x, x = 0$
c. $f(x) = \cos (4x), x = \frac{\pi}{4}$
d. $f(x) = \sin 2x + \cos x, x = \frac{\pi}{2}$
e. $f(x) = \cos \left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$
f. $f(x) = 2 \sin x \cos x, x = \frac{\pi}{2}$

C 4. a. If $f(x) = \sin^2 x$ and $g(x) = 1 - \cos^2 x$, explain why $f'(x) = g'(x)$.
b. If $f(x) = \sin^2 x$ and $g(x) = 1 + \cos^2 x$, how are $f'(x)$ and $g'(x)$ related?

5. Differentiate each function.

a. $v(t) = \sin^2 (\sqrt{t})$
b. $v(t) = \sqrt{1 + \cos t + \sin^2 t}$
c. $h(x) = \sin x \sin 2x \sin 3x$
d. $m(x) = (x^2 + \cos^2 x)^3$

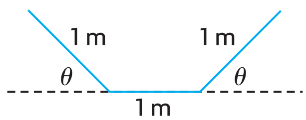
6. Determine the absolute extreme values of each function on the given interval. (Verify your results with graphing technology.)
   a. $y = \cos x + \sin x, 0 \leq x \leq 2\pi$
   b. $y = x + 2\cos x, -\pi \leq x \leq \pi$
   c. $y = \sin x - \cos x, x \in [0, 2\pi]$
   d. $y = 3\sin x + 4\cos x, x \in [0, 2\pi]$

A

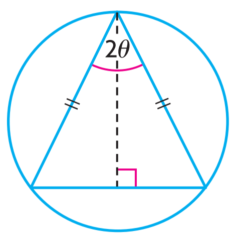
7. A particle moves along a line so that, at time t, its position is $s(t) = 8\sin 2t$.
   a. For what values of t does the particle change direction?
   b. What is the particle's maximum velocity?
8. a. Graph the function $f(x) = \cos x + \sin x$.
   b. Determine the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal, on the interval $0 \leq x \leq \pi$.
9. Determine expressions for the derivatives of csc x and sec x.
10. Determine the slope of the tangent to the curve $y = \cos 2x$ at point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.
11. A particle moves along a line so that at time t, its position is $s = 4\sin 4t$.
    a. When does the particle change direction?
    b. What is the particle's maximum velocity?
    c. What is the particle's minimum distance from the origin? What is its maximum distance from the origin?

T

12. An irrigation channel is constructed by bending a sheet of metal that is 3 m wide, as shown in the diagram. What angle θ will maximize the cross-sectional area (and thus the capacity) of the channel?



13. An isosceles triangle is inscribed in a circle of radius **R**. Find the value of θ that maximizes the area of the triangle.



PART C

14. If $y = A\cos kt + B\sin kt$, where A, B, and k are constants, show that $y'' + k^2y = 0$.

Section 5.5—The Derivative of $y = \tan x$

In this section, we will study the derivative of the remaining primary trigonometric function—tangent.

Since this function can be expressed in terms of sine and cosine, we can find its derivative using the product rule.

EXAMPLE 1 **Reasoning about the derivative of the tangent function**

Determine $\frac{dy}{dx}$ for $y = \tan x$.

Solution

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \\ &= (\sin x)(\cos x)^{-1} \\ \frac{dy}{dx} &= \frac{d(\sin x)}{dx} \times (\cos x)^{-1} + \sin x \times \frac{d(\cos x)^{-1}}{dx} && \text{(Product rule)} \\ &= (\cos x)(\cos x)^{-1} + \sin x\,(-1)(\cos x)^{-2}(-\sin x) && \text{(Chain rule)} \\ &= 1 + \frac{\sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x && \text{(Using the Pythagorean identity)} \\ &= \sec^2 x \end{aligned}$$

Therefore, $\frac{d(\tan x)}{dx} = \sec^2 x$

EXAMPLE 2 **Selecting a strategy to determine the derivative of a composite tangent function**

Determine $\frac{dy}{dx}$ for $y = \tan(x^2 + 3x)$.

Solution

$$\begin{aligned} y &= \tan(x^2 + 3x) \\ \frac{dy}{dx} &= \frac{d\tan(x^2 + 3x)}{d(x^2 + 3x)} \times \frac{d(x^2 + 3x)}{dx} && \text{(Chain rule)} \\ &= \sec^2(x^2 + 3x) \times (2x + 3) \\ &= (2x + 3)\sec^2(x^2 + 3x) \end{aligned}$$

Derivatives of Composite Functions Involving $y = \tan x$

If $y = \tan f(x)$, then $\frac{dy}{dx} = \sec^2 f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\tan f(x)) = \frac{d(\tan f(x))}{d(f(x))} \times \frac{df(x))}{dx} = \sec^2(f(x)) \times \frac{d(f(x))}{dx}$.

EXAMPLE 3

Determining the derivative of a combination of functions

Determine $\frac{dy}{dx}$ for $y = (\sin x + \tan x)^4$.

Solution

$$y = (\sin x + \tan x)^4$$

$$\frac{dy}{dx} = 4(\sin x + \tan x)^3(\cos x + \sec^2 x) \qquad \text{(Chain rule)}$$

EXAMPLE 4

Determining the derivative of a product involving the tangent function

Determine $\frac{dy}{dx}$ for $y = x\tan(2x - 1)$.

Solution

$$y = x\tan(2x - 1)$$

$$\frac{dy}{dx} = (1)\tan(2x - 1) + (x)\sec^2(2x - 1)\frac{d(2x - 1)}{dx} \qquad \text{(Product and chain rules)}$$

$$= \tan(2x - 1) + 2x\sec^2(2x - 1)$$

IN SUMMARY

Key Idea

- The derivatives of functions involving the tangent function are found as follows:
  - $\frac{d(\tan x)}{dx} = \sec^2 x$
  - $\frac{d}{dx}(\tan f(x)) = \sec^2 f(x) \times f'(x)$

Need to Know

- Trigonometric identities can be used to write one expression as an equivalent expression and then differentiate. In some cases, the new function will be easier to work with.

Exercise 5.5

PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \tan 3x$

b. $y = 2\tan x - \tan 2x$

c. $y = \tan^2(x^3)$

d. $y = \frac{x^2}{\tan \pi x}$

e. $y = \tan(x^2) - \tan^2 x$

f. $y = 3\sin 5x \tan 5x$

A 2. Determine an equation for the tangent to each function at the point with the given x-coordinate.

a. $f(x) = \tan x,\ x = \frac{\pi}{4}$

b. $f(x) = 6\tan x - \tan 2x,\ x = 0$

PART B

3. Determine y' for each of the following:

a. $y = \tan(\sin x)$

b. $y = [\tan(x^2 - 1)]^{-2}$

c. $y = \tan^2(\cos x)$

d. $y = (\tan x + \cos x)^2$

e. $y = \sin^3 x \tan x$

f. $y = e^{\tan\sqrt{x}}$

4. Determine $\frac{d^2y}{dx^2}$ for each of the following:

a. $y = \sin x \tan x$

b. $y = \tan^2 x$

5. Determine all the values of x, $0 \le x \le 2\pi$, for which the slope of the tangent to $f(x) = \sin x \tan x$ is zero.

6. Determine the local maximum point on the curve $y = 2x - \tan x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

T 7. Prove that $y = \sec x + \tan x$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

8. Determine the equation of the line that is tangent to $y = 2\tan x$, where $x = \frac{\pi}{4}$.

C 9. If you forget the expression that results when differentiating the tangent function, explain how you can derive this derivative using an identity.

PART C

10. Determine the derivative of $\cot x$.

11. Determine $f''(x)$, where $f(x) = \cot 4x$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

A simplified model for bacterial growth is $P(t) = P_0e^{rt}$, where $P(t)$ is the population of the bacteria colony after t hours, P_0 is the initial population of the colony (the population at $t = 0$), and r determines the growth rate of the colony. The model is simple because it does not account for limited resources, such as space and nutrients. As time increases, so does the population, but there is no bound on the population. While a model like this can describe a population for a short period of time or can be made to describe a population for a longer period of time by adjusting conditions in a laboratory experiment, in general, populations are better described by more complex models.

To determine how the population of a particular type of bacteria will grow over time under controlled conditions, a microbiologist observes the initial population and the population every half hour for 8 h. (The microbiologist also controls the environment in which the colony is growing to make sure that temperature and light conditions remain constant and ensures that the amount of nutrients available to the colony as it grows is sufficient for the increasing population.)

After analyzing the population data, the microbiologist determines that the population of the bacteria colony can be modelled by the equation $P(t) = 500\,e^{0.1t}$.

a. What is the initial population of the bacteria colony?

b. What function describes the instantaneous rate of change in the bacteria population after t hours?

c. What is the instantaneous rate of change in the population after 1 h? What is the instantaneous rate of change after 8 h?

d. How do your answers for part c. help you make a prediction about how long the bacteria colony will take to double in size? Make a prediction for the number of hours the population will take to double, using your answers for part c. and/or other information.

e. Determine the actual doubling time—the time that the colony takes to grow to twice its initial population. (***Hint:*** Solve for t when $P(t) = 1000$.)

f. Compare your prediction for the doubling time with the calculated value. If your prediction was not close to the actual value, what factors do you think might account for the difference?

g. When is the instantaneous rate of change equal to 500 bacteria per hour?

Key Concepts Review

In this chapter, we introduced a new base for exponential functions, namely the number e, where $e \doteq 2.718\ 281$. We examined the derivatives of the exponential functions along with the primary trigonometric functions. You should now be able to apply all the rules of differentiation that you learned in Chapter 2 to expressions that involve the exponential, sine, cosine, and tangent functions combined with polynomial and rational functions.

We also examined some applications of exponential and trigonometric functions. The calculus techniques that are used to determine instantaneous rates of change, equations of tangent lines, and absolute extrema for polynomial and rational functions, can also be used for exponential and trigonometric functions.

Derivative Rules for Exponential Functions

- $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \times g'(x)$
- $\frac{d}{dx}(b^x) = b^x \ln b$ and $\frac{d}{dx}(b^{g(x)}) = b^{g(x)}(\ln b)g'(x)$

Derivative Rules for Primary Trigonometric Functions

- $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\sin f(x)) = \cos f(x) \times f'(x)$
- $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\cos f(x)) = -\sin f(x) \times f'(x)$
- $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(\tan f(x)) = \sec^2 f(x) \times f'(x)$

Review Exercise

1. Differentiate each of the following:
   a. $y = 6 - e^x$
   b. $y = 2x + 3e^x$
   c. $y = e^{2x+3}$
   d. $y = e^{-3x^2+5x}$
   e. $y = xe^x$
   f. $s = \dfrac{e^t - 1}{e^t + 1}$
2. Determine $\frac{dy}{dx}$ for each of the following:
   a. $y = 10^x$
   b. $y = 4^{3x^2}$
   c. $y = (5x)(5^x)$
   d. $y = (x^4)\,2^x$
   e. $y = \dfrac{4x}{4^x}$
   f. $y = \dfrac{5^{\sqrt{x}}}{x}$
3. Differentiate each of the following:
   a. $y = 3\sin 2x - 4\cos 2x$
   b. $y = \tan 3x$
   c. $y = \dfrac{1}{2 - \cos x}$
   d. $y = x\tan 2x$
   e. $y = (\sin 2x)e^{3x}$
   f. $y = \cos^2 2x$
4. a. Given the function $f(x) = \frac{e^x}{x}$, solve the equation $f'(x) = 0$.
   b. Discuss the significance of the solution you found in part a.
5. a. If $f(x) = xe^{-2x}$, find $f'\left(\frac{1}{2}\right)$.
   b. Explain what this number represents.
6. Determine the second derivative of each of the following:
   a. $y = xe^x - e^x$
   b. $y = xe^{10x}$
7. If $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, prove that $\frac{dy}{dx} = 1 - y^2$.
8. Determine the equation of the tangent to the curve defined by $y = x - e^{-x}$ that is parallel to the line represented by $3x - y - 9 = 0$.
9. Determine the equation of the tangent to the curve $y = x\sin x$ at the point where $x = \frac{\pi}{2}$.
10. An object moves along a line so that, at time t, its position is $s = \frac{\sin t}{3 + \cos 2t}$, where s is the displacement in metres. Calculate the object's velocity at $t = \frac{\pi}{4}$.

11. The number of bacteria in a culture, N, at time t is given by $N(t) = 2000\left[30 + te^{-\frac{t}{20}}\right]$.

    a. When is the rate of change of the number of bacteria equal to zero?

    b. If the bacterial culture is placed in a colony of mice, the number of mice that become infected, M, is related to the number of bacteria present by the equation $M(t) = \sqrt[3]{N + 1000}$. After 10 days, how many mice are infected per day?

12. The concentrations of two medicines in the bloodstream t hours after injection are $c_1(t) = te^{-t}$ and $c_2(t) = t^2e^{-t}$.

    a. Which medicine has the larger maximum concentration?

    b. Within the first half hour, which medicine has the larger maximum concentration?

13. Differentiate.

    a. $y = (2 + 3e^{-x})^3$

    b. $y = x^e$

    c. $y = e^{e^x}$

    d. $y = (1 - e^{5x})^5$

14. Differentiate.

    a. $y = 5^x$

    b. $y = (0.47)^x$

    c. $y = (52)^{2x}$

    d. $y = 5(2)^x$

    e. $y = 4(e)^x$

    f. $y = -2(10)^{3x}$

15. Determine y'.

    a. $y = \sin 2^x$

    b. $y = x^2 \sin x$

    c. $y = \sin\left(\frac{\pi}{2} - x\right)$

    d. $y = \cos x \sin x$

    e. $y = \cos^2 x$

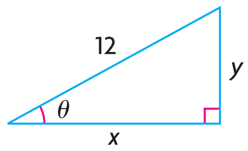
    f. $y = \cos x \sin^2 x$

16. Determine the equation of the tangent to the curve $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$.

17. An object is suspended from the end of a spring. Its displacement from the equilibrium position is $s = 8 \sin (10\pi t)$ at time t. Calculate the velocity and acceleration of the object at any time t, and show that $\frac{d^2s}{dt^2} + 100\pi^2 s = 0$.

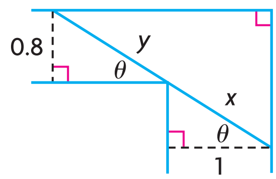
18. The position of a particle is given by $s = 5\cos\left(2t + \frac{\pi}{4}\right)$ at time t. What are the maximum values of the displacement, the velocity, and the acceleration?

19. The hypotenuse of a right triangle is 12 cm in length. Calculate the measures of the unknown angles in the triangle that will maximize its perimeter.

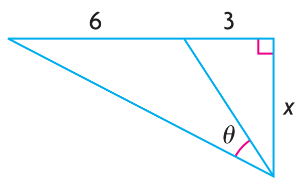


20. A fence is 1.5 m high and is 1 m from a wall. A ladder must start from the ground, touch the top of the fence, and rest somewhere on the wall. Calculate the minimum length of the ladder.

21. A thin rigid pole needs to be carried horizontally around a corner joining two corridors, which are 1 m and 0.8 m wide. Calculate the length of the longest pole that can be carried around this corner.



22. When the rules of hockey were developed, Canada did not use the metric system. Thus, the distance between the goal posts was designated to be six feet (slightly less than 2 m). If Sidney Crosby is on the goal line, three feet outside one of the goal posts, how far should he go out (perpendicular to the goal line) to maximize the angle in which he can shoot at the goal?
Hint: Determine the values of x that maximize θ in the following diagram.



23. Determine $f''(x)$
 a. $f(x) = 4\sin^2(x - 2)$
 b. $f(x) = 2(\cos x)(\sec^2 x)$

Chapter 5 Test

1. Determine the derivative $\frac{dy}{dx}$ for each of the following:

   a. $y = e^{-2x^2}$
   b. $y = 3^{x^2+3x}$
   c. $y = \dfrac{e^{3x} + e^{-3x}}{2}$
   d. $y = 2 \sin x - 3 \cos 5x$
   e. $y = \sin^3 (x^2)$
   f. $y = \tan \sqrt{1 - x}$

2. Determine the equation of the tangent to the curve defined by $y = 2e^{3x}$ that is parallel to the line defined by $-6x + y = 2$.

3. Determine the equation of the tangent to $y = e^x + \sin x$ at $(0, 1)$.

4. The velocity of a certain particle that moves in a straight line under the influence of forces is given by $v(t) = 10e^{-kt}$, where k is a positive constant and $v(t)$ is in centimetres per second.

   a. Show that the acceleration of the particle is proportional to a constant multiple of its velocity. Explain what is happening to the particle.
   b. What is the initial velocity of the particle?
   c. At what time is the velocity equal to half the initial velocity? What is the acceleration at this time?

5. Determine $f''(x)$.

   a. $f(x) = \cos^2 x$
   b. $f(x) = \cos x \cot x$

6. Determine the absolute extreme values of $f(x) = \sin^2 x$, where $x \in [0, \pi]$.

7. Calculate the slope of the tangent line that passes through $y = 5^x$, where $x = 2$. Express your answer to two decimal places.

8. Determine all the maximum and minimum values of $y = xe^x + 3e^x$.

9. $f(x) = 2 \cos x - \sin 2x$ where $x \in [-\pi, \pi]$

   a. Determine all critical number for $f(x)$ on the given interval.
   b. Determine the intervals where $f(x)$ is increasing and where it is decreasing.
   c. Determine all local maximum and minimum values of $f(x)$ on the given interval.
   d. Use the information you found above to sketch the curve.

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Cumulative Review of Calculus

1. Using the limit definition of the slope of a tangent, determine the slope of the tangent to each curve at the given point.

   a. $f(x) = 3x^2 + 4x - 5, (2, 15)$

   b. $f(x) = \dfrac{2}{x - 1}, (2, 2)$

   c. $f(x) = \sqrt{x + 3}, (6, 3)$

   d. $f(x) = 2^{5x}, (1, 32)$

2. The position, in metres, of an object is given by $s(t) = 2t^2 + 3t + 1$, where t is the time in seconds.

   a. Determine the average velocity from $t = 1$ to $t = 4$.

   b. Determine the instantaneous velocity at $t = 3$.

3. If $\lim\limits_{h \to 0} \dfrac{(4 + h)^3 - 64}{h}$ represents the slope of the tangent to $y = f(x)$ at $x = 4$, what is the equation of $f(x)$?

4. An object is dropped from the observation deck of the Skylon Tower in Niagara Falls, Ontario. The distance, in metres, from the deck at t seconds is given by $d(t) = 4.9t^2$.

   a. Determine the average rate of change in distance with respect to time from $t = 1$ to $t = 3$.

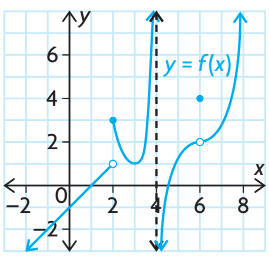
   b. Determine the instantaneous rate of change in distance with respect to time at 2 s.

   c. The height of the observation deck is 146.9 m. How fast is the object moving when it hits the ground?

5. The model $P(t) = 2t^2 + 3t + 1$ estimates the population of fish in a reservoir, where P represents the population, in thousands, and t is the number of years since 2000.

   a. Determine the average rate of population change between 2000 and 2008.

   b. Estimate the rate at which the population was changing at the start of 2005.



6. a. Given the graph of $f(x)$ at the left, determine the following:

   i. $f(2)$

   ii. $\lim\limits_{x \to 2^-} f(x)$

   iii. $\lim\limits_{x \to 2^+} f(x)$

   iv. $\lim\limits_{x \to 6} f(x)$

   b. Does $\lim\limits_{x \to 4} f(x)$ exist? Justify your answer.

7. Consider the following function:

$$f(x) = \begin{cases} x^2 + 1, \text{ if } x < 2 \\ 2x + 1, \text{ if } x = 2 \\ -x + 5, \text{ if } x > 2 \end{cases}$$

   Determine where $f(x)$ is discontinuous, and justify your answer.

8. Use algebraic methods to evaluate each limit (if it exists).

a. $\lim_{x \to 0} \frac{2x^2 + 1}{x - 5}$

b. $\lim_{x \to 3} \frac{x - 3}{\sqrt{x + 6} - 3}$

c. $\lim_{x \to -3} \frac{\frac{1}{x} + \frac{1}{3}}{x + 3}$

d. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2}$

e. $\lim_{x \to 2} \frac{x - 2}{x^3 - 8}$

f. $\lim_{x \to 0} \frac{\sqrt{x + 4} - \sqrt{4 - x}}{x}$

9. Determine the derivative of each function from first principles.

a. $f(x) = 3x^2 + x + 1$

b. $f(x) = \frac{1}{x}$

10. Determine the derivative of each function.

a. $y = x^3 - 4x^2 + 5x + 2$

b. $y = \sqrt{2x^3 + 1}$

c. $y = \frac{2x}{x + 3}$

d. $y = (x^2 + 3)^2(4x^5 + 5x + 1)$

e. $y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$

f. $y = [x^2 + (2x + 1)^3]^5$

11. Determine the equation of the tangent to $y = \frac{18}{(x + 2)^2}$ at the point $(1, 2)$.

12. Determine the slope of the tangent to $y = x^2 + 9x + 9$ at the point where the curve intersects the line $y = 3x$.

13. In 1980, the population of Littletown, Ontario, was 1100. After a time t, in years, the population was given by $p(t) = 2t^2 + 6t + 1100$.

a. Determine $p'(t)$, the function that describes the rate of change of the population at time t.

b. Determine the rate of change of the population at the start of 1990.

c. At the beginning of what year was the rate of change of the population 110 people per year?

14. Determine f' and f'' for each function.

a. $f(x) = x^5 - 5x^3 + x + 12$

b. $f(x) = \frac{-2}{x^2}$

c. $f(x) = \frac{4}{\sqrt{x}}$

d. $f(x) = x^4 - \frac{1}{x^4}$

15. Determine the extreme values of each function on the given interval.

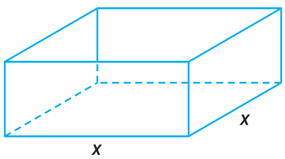
a. $f(x) = 1 + (x + 3)^2, -2 \le x \le 6$

b. $f(x) = x + \frac{1}{\sqrt{x}}, 1 \le x \le 9$

c. $f(x) = \frac{e^x}{1 + e^x}, x \in [0, 4]$

d. $f(x) = 2 \sin 4x + 3, x \in [0, \pi]$

16. The position, at time t, in seconds, of an object moving along a line is given by $s(t) = 3t^3 - 40.5t^2 + 162t$ for $0 \leq t \leq 8$.
    a. Determine the velocity and the acceleration at any time t.
    b. When is the object stationary? When is it advancing? When is it retreating?
    c. At what time, t, is the velocity not changing?
    d. At what time, t, is the velocity decreasing?
    e. At what time, t, is the velocity increasing?
17. A farmer has 750 m of fencing. The farmer wants to enclose a rectangular area on all four sides, and then divide it into four pens of equal size with the fencing parallel to one side of the rectangle. What is the largest possible area of each of the four pens?
18. A cylindrical metal can is made to hold 500 mL of soup. Determine the dimensions of the can that will minimize the amount of metal required. (Assume that the top and sides of the can are made from metal of the same thickness.)
19. A cylindrical container, with a volume of 4000 cm^3, is being constructed to hold candies. The cost of the base and lid is $0.005/$cm^2$, and the cost of the side walls is $0.0025/$cm^2$. Determine the dimensions of the cheapest possible container.
20. An open rectangular box has a square base, with each side measuring x centimetres.



    a. If the length, width, and depth have a sum of 140 cm, find the depth in terms of x.
    b. Determine the maximum possible volume you could have when constructing a box with these specifications. Then determine the dimensions that produce this maximum volume.
21. The price of x MP3 players is $p(x) = 50 - x^2$, where $x \in \mathbf{N}$. If the total revenue, $R(x)$, is given by $R(x) = xp(x)$, determine the value of x that corresponds to the maximum possible total revenue.
22. An express railroad train between two cities carries 10 000 passengers per year for a one-way fare of $50. If the fare goes up, ridership will decrease because more people will drive. It is estimated that each $10 increase in the fare will result in 1000 fewer passengers per year. What fare will maximize revenue?
23. A travel agent currently has 80 people signed up for a tour. The price of a ticket is $5000 per person. The agency has chartered a plane seating 150 people at a cost of $250 000. Additional costs to the agency are incidental fees of $300 per person. For each $30 that the price is lowered, one new person will sign up. How much should the price per person be lowered to maximize the profit for the agency?

24. For each function, determine the derivative, all the critical numbers, and the intervals of increase and decrease.

a. $y = -5x^2 + 20x + 2$

b. $y = 6x^2 + 16x - 40$

c. $y = 2x^3 - 24x$

d. $y = \dfrac{x}{x - 2}$

25. For each of the following, determine the equations of any horizontal, vertical, or oblique asymptotes and all local extrema:

a. $y = \dfrac{8}{x^2 - 9}$

b. $y = \dfrac{4x^3}{x^2 - 1}$

26. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = 4x^3 + 6x^2 - 24x - 2$

b. $y = \dfrac{3x}{x^2 - 4}$

27. Determine the derivative of each function.

a. $f(x) = (-4)e^{5x+1}$

b. $f(x) = xe^{3x}$

c. $y = 6^{3x-8}$

d. $y = e^{\sin x}$

28. Determine the equation of the tangent to the curve $y = e^{2x-1}$ at $x = 1$.

29. In a research laboratory, a dish of bacteria is infected with a particular disease. The equation $N(d) = (15d)e^{-\frac{d}{5}}$ models the number of bacteria, N, that will be infected after d days.

a. How many days will pass before the maximum number of bacteria will be infected?

b. Determine the maximum number of bacteria that will be infected.

30. Determine the derivative of each function.

a. $y = 2\sin x - 3\cos 5x$

b. $y = (\sin 2x + 1)^4$

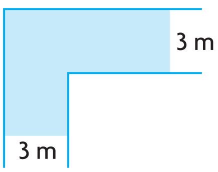
c. $y = \sqrt{x^2 + \sin 3x}$

d. $y = \dfrac{\sin x}{\cos x + 2}$

e. $y = \tan x^2 - \tan^2 x$

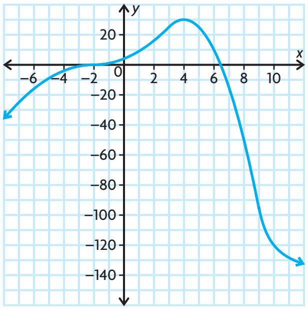
f. $y = \sin(\cos x^2)$

31. A tool shed, 250 cm high and 100 cm deep, is built against a wall. Calculate the shortest ladder that can reach from the ground, over the shed, to the wall behind.



32. A corridor that is 3 m wide makes a right-angle turn, as shown on the left. Find the longest rod that can be carried horizontally around this corner. Round your answer to the nearest tenth of a metre.

4. hole at $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote;
$y = 1$;
Domain: $\{x \in \mathbf{R} \mid x \neq -2, x \neq 3\}$

5. 

6. There are discontinuities at $x = -3$ and $x = 3$.

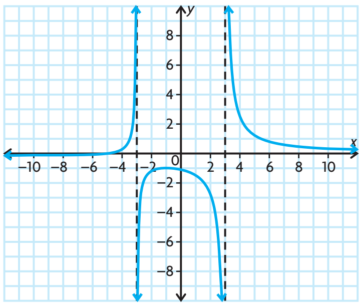
$$\left.\begin{aligned}\lim_{x\to 3^-} f(x) &= \infty\\ \lim_{x\to 3^+} f(x) &= -\infty\end{aligned}\right\} x = -3 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x\to 3^-} f(x) &= -\infty\\ \lim_{x\to 3^+} f(x) &= \infty\end{aligned}\right\} x = 3 \text{ is a vertical asymptote.}$$

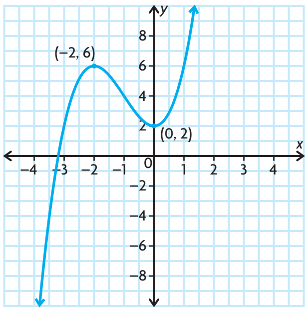
The y-intercept is $-\frac{10}{9}$ and x-intercept is -5.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$y = 0$ is a horizontal asymptote.



7. $b = 3, c = 2$

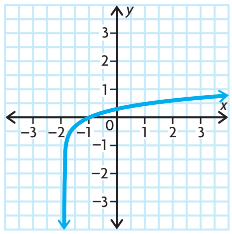


Chapter 5

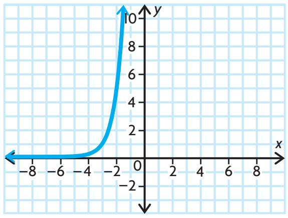
Review of Prerequisite Skills, pp. 224–225

1. a. $\frac{1}{9}$ c. $\frac{1}{9}$
b. 4 d. $\frac{9}{4}$

2. a. $\log_5 625 = 4$
b. $\log_4 \frac{1}{16} = -2$
c. $\log_x 3 = 3$
d. $\log_{10} 450 = w$
e. $\log_3 z = 8$
f. $\log_a T = b$

3. a. 

x-intercept: $(-1, 0)$

b. 

no x-intercept

4. a. $\frac{y}{r}$ b. $\frac{x}{r}$ c. $\frac{y}{x}$

5. a. 2π e. $\frac{3\pi}{2}$
b. $\frac{\pi}{4}$ f. $-\frac{2\pi}{3}$
c. $-\frac{\pi}{2}$ g. $\frac{5\pi}{4}$
d. $\frac{\pi}{6}$ h. $\frac{11\pi}{6}$

6. a. b c. a e. b
b. $\frac{b}{a}$ d. a f. $-b$

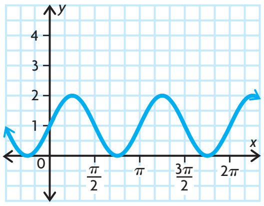
7. a. $\cos\theta = -\frac{12}{13}$,
$\tan\theta = -\frac{5}{12}$

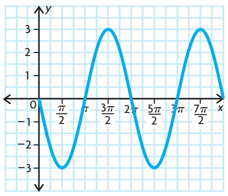
b. $\sin\theta = -\frac{\sqrt{5}}{3}$,
$\tan\theta = \frac{\sqrt{5}}{2}$

c. $\sin\theta = -\frac{2}{\sqrt{5}}$,
$\cos\theta = \frac{1}{\sqrt{5}}$

d. $\cos\frac{\pi}{2} = 0$,
$\tan\frac{\pi}{2}$ is undefined

8. a. period: π, amplitude: 1
b. period: 4π, amplitude: 2
c. period: 2, amplitude: 3
d. period: $\frac{\pi}{6}$, amplitude: $\frac{2}{7}$
e. period: 2π, amplitude: 5
f. period: 2π, amplitude: $\frac{3}{2}$

9. a. 

b. 

10. a. $\tan x + \cot x = \sec x \csc x$

$$\begin{aligned}
\text{LS} &= \tan x + \cot x\\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\
&= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}\\
&= \frac{1}{\cos x + \sin x}\\
\text{RS} &= \sec x \csc x\\
&= \frac{1}{\cos x} \times \frac{1}{\sin x}\\
&= \frac{1}{\cos x \sin x}
\end{aligned}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

b. $\dfrac{\sin x}{1-\sin^2 x} = \tan x + \sec x$

$$\begin{aligned} \text{LS} &= \frac{\sin x}{1-\sin^2 x} \\ &= \frac{\sin x}{\cos^2 x} \\ \text{RS} &= \tan x \sec x \\ &= \frac{\sin x}{\cos x} \times \frac{1}{\cos x} \\ &= \frac{\sin x}{\cos^2 x} \end{aligned}$$

Therefore, $\dfrac{\sin x}{1-\sin^2 x} = \tan x \sec x$.

11. **a.** $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b. $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Section 5.1, pp. 232–234

1. You can only use the power rule when the term containing variables is in the base of the exponential expression. In the case of $y = e^x$, the exponent contains a variable.

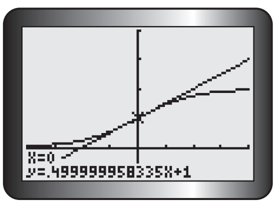
2. **a.** $3e^{3x}$
b. $3e^{3t-5}$
c. $20e^{10t}$
d. $-3e^{-3x}$
e. $(-6+2x)e^{5-6x+x^2}$
f. $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$

3. **a.** $6x^2e^{x^3}$
b. $e^{3x}(3x+1)$
c. $\dfrac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$
d. $\sqrt{x}e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right)$
e. $2te^{t^2} - 3e^{-t}$
f. $\dfrac{2e^{2t}}{(1+e^{2t})^2}$

4. **a.** $e^3 - e^{-3}$
b. $\dfrac{1}{e}$
c. $-2-3e$

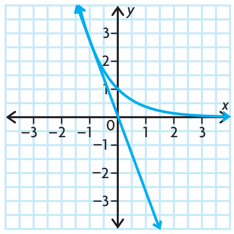
5. **a.** $y = \dfrac{1}{2}x + 1$

b.



c. The answers agree very well; the calculator does not show a slope of exactly 0.5, due to internal rounding.

6. $ex + y = 0$



7. $y = \dfrac{1}{e}$

8. $(0, 0)$ and $\left(2, \dfrac{4}{e^2}\right)$

9. If $y = \dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, then

$$y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right), \text{ and}$$

$$\begin{aligned} y'' &= \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right) \\ &= \frac{1}{25}\left[\frac{5}{2}\left(e^{\frac{x}{5}} + e^{-\frac{x}{5}}\right)\right] \\ &= \frac{1}{25}y \end{aligned}$$

10. **a.** $\dfrac{dy}{dx} = -3e^{-3x}$,

$\dfrac{d^2y}{dx^2} = 9e^{-3x}$,

$\dfrac{d^3y}{dx^3} = -27e^{-3x}$

b. $\dfrac{d^ny}{dx^n} = (-1)^n(3^n)\,e^{-3x}$

11. **a.** $\dfrac{dy}{dx} = -3e^{3x}$,

$\dfrac{d^2y}{dx^2} = -3e^x$

b. $\dfrac{dy}{dx} = e^{2x}(2x+1)$,

$\dfrac{d^2y}{dx^2} = 4xe^{2x} + 4e^{2x}$

c. $\dfrac{dy}{dx} = e^x(3-x)$,

$\dfrac{d^2y}{dx^2} = e^x(2-x)$

12. **a.** 31 000
b. $-\dfrac{100}{3}e^{-\frac{t}{30}}$
c. -17 bacteria/h
d. 31 000 at time $t = 0$
e. The number of bacteria is constantly decreasing as time passes.

13. **a.** $40\left(1 - e^{-\frac{t}{4}}\right)$

b. $a = \dfrac{dv}{dt} = 40\left(\dfrac{1}{4} - e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a,** $v = 40\left(1 - e^{-\frac{t}{4}}\right)$, which gives $e^{\frac{t}{4}} = 1 - \dfrac{v}{40}$. Thus,

$a = 10\left(1 - \dfrac{v}{40}\right) = 10 - \dfrac{1}{4}v.$

c. 40 m/s
d. about 12 s, about 327.3 m

14. **a.** **i.** e
ii. e
b. The limits have the same value because as $x \to \infty, \frac{1}{x} \to 0$.

15. **a.** 1
b. e^2

16. $m = -3$ or $m = 2$

17. **a.** $$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right] \\ &= \frac{1}{2}(e^x + e^{-x}) \\ &= \cosh x \end{aligned}$$

b. $$\begin{aligned} \frac{d}{dx}(\cosh x) &= \frac{1}{2}(e^t - e^{-t}) \\ &= \sinh x \end{aligned}$$

c. Since $\tanh x = \dfrac{\sinh x}{\cosh x}$,

$$\begin{aligned} \frac{d}{dx}(\tanh x) &= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x)}{(\cosh x)^2} - \frac{(\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2} \\ &= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(e^x + e^{-x})}{(\cosh x)^2} - \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2} \\ &= \frac{\frac{1}{4}[(e^{2x} + 2 + e^{-2x})}{(\cosh x)^2} - \frac{(e^{2t} - 2 + e^{-2t})]}{(\cosh x)^2} \\ &= \frac{\frac{1}{4}(4)}{(\cosh x)^2} \\ &= \frac{1}{(\cosh x)^2} \end{aligned}$$

18. **a.** Four terms: 2.666 66$\overline{6}$
Five terms: 2.708 33$\overline{3}$
Six terms: 2.716 66$\overline{6}$
Seven terms: 2.718 05$\overline{5}$

b. The expression for e in part a is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. The value of x is 1.

Section 5.2, p. 240

1. **a.** $3(2^{3x})\ln 2$
b. $\ln 3.1(3.1)^x + 3x^2$
c. $3(10^{3t-5})\ln 10$
d. $(-6 + 2n)(10^{5-6n+n^2})\ln 10$
e. $2x(3^{x^2+2})\ln 3$
f. $400(2)^{x+3}\ln 2$

2. **a.** $5^x[(x^5 \times \ln 5) + 5x^4]$
b. $(3)^{x^2}[(2x^2 \ln 3) + 1]$
c. $-\frac{2^t}{t^2} + \frac{2^t \ln 2}{t}$
d. $\frac{3^{\frac{x}{2}}[x \ln 3 - 4]}{x^3}$

3. $-\frac{3 \ln 10}{4}$

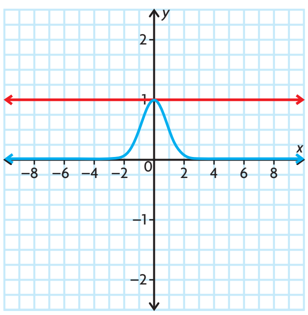
4. $-16.64x + y + 25.92 = 0$

5. $-23.03x + y + 13.03 = 0$

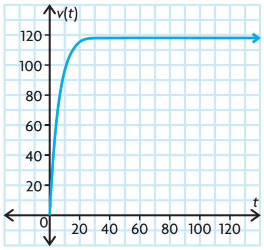
6. **a.** about 3.80 years
b. about -9.12%/year

7. **a.** In 1978, the rate of increase of debt payments was \$904 670 000/annum compared to \$122 250 000/annum in 1968. The rate of increase for 1978 is 7.4 times larger than that for 1968.
b. The rate of increase for 1998 is 7.4 times larger than that for 1988.
c. Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

8. $y = 1$



9.



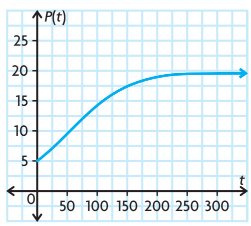
From the graph, the values of $v(t)$ quickly rise in the range of about $0 \le t \le 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$, the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

Section 5.3, pp. 245–247

1. **a.** absolute max: about 0.3849, absolute min: 0
b. absolute max: about 10.043, absolute min: about -5961.916

2. **a.** $f(x)$: max: 0.3849, min: 0; $m(x)$: max: about 10, min: about -5961
b. The graphing approach seems to be easier to use for the functions. It is quicker and it gives the graphs of the functions in a good viewing rectangle. The only problem may come in the second function, $m(x)$, because for $x < 1.5$, the function quickly approaches values in the negative thousands.

3. **a.** 500 squirrels
b. 2000 squirrels
c. (54.9, 10)
d.



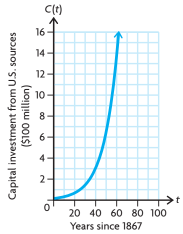
e. P grows exponentially until the point of inflection, then the growth rate decreases and the curve becomes concave down.

4. **a.** 1001 items
b. 500 items

5. 500 units

6. 47.2%; 0.462 h

7. **a.**



b. The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.
c. 7.5%
d. $C = 59.537 \times 10^9$ dollars, $\frac{dC}{dt} = 4.4849 \times 10^9$ dollars/year
e. Statistics Canada data shows the actual amount of U.S. investment in 1977 was 62.5×10^9 dollars. The error in the model is 3.5%.
f. $C = 570.490 \times 10^9$ dollars, $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year

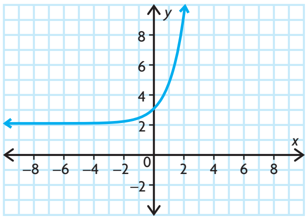
8. **a.** 478 158; 38.2 min after the drug was introduced
b. 42.72 min after the drug was introduced

9. 10 h of study should be assigned to the first exam and 20 h of study for the second exam.

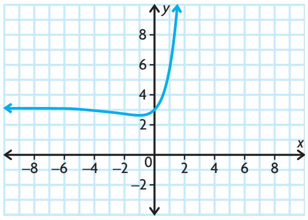
10. Use the algorithm for finding extreme values. First, find the derivate $f'(x)$. Then find any critical points by setting $f'(x) = 0$ and solving for x. Also find the values of x for which $f'(x)$ is undefined. Together these are the critical values. Now evaluate $f(x)$ for the critical values and the endpoints 2 and -2. The highest value will be the absolute maximum on the interval, and the lowest value will be the absolute minimum on the interval.

11. **a.** $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$. Also, $f(x)$ is decreasing on the interval $(-2, 0)$.
b. 0

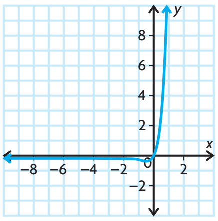
12. **a.** no maximum or minimum value



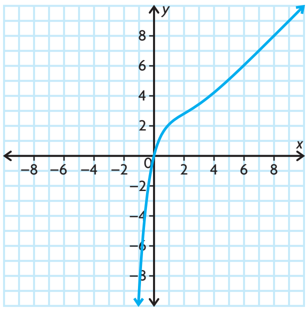
b. min: $-e^{-1} + 3 \doteq 2.63$, no max



c. min: $-e^{-1} \doteq -0.37$, no max

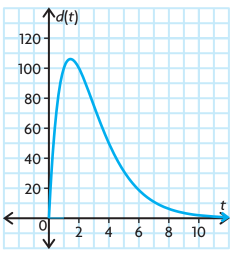


d. no max or min



13. about 0.61

14. **a.**



b. increasing when $0 < t < \frac{1}{\ln 2}$ and decreasing when $t > \frac{1}{\ln 2}$

c. about 106.15°/s

d. $t > 10$ s

15. The solution starts in a similar way to that of question 9. The effectiveness function is

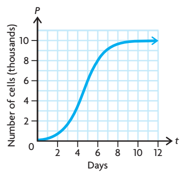
$$E(t) = 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right)$$

The derivative simplifies to

$$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{10}}(5 - t)$$

This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine that the maximum effectiveness occurs when $t = 8.16$ h.

16. **a.**



b. after 4.6 days, 5012

c. The rate of growth is slowing down as the colony is getting closer to its limiting value.

Mid-Chapter Review, pp. 248–249

1. **a.** $-15e^{-3x}$

b. $e^{\frac{1}{7}x}$

c. $e^{-2x}(-2x^3 + 3x^2)$

d. $(e^x)(x^2 - 1)$

e. $2(x + xe^{-x} - e^{-x} - e^{-2x})$

f. $\dfrac{4}{(e^x + e^{-x})^2}$

2. **a.** $-500e^{-5t}$

b. -2.5

3. $x + y - 2 = 0$

4. **a.** $y' = -3e^x$, $y'' = -3e^x$

b. $y' = 2xe^{2x} + e^{2x}$, $y'' = 4xe^{2x} + 4e^{2x}$

c. $y' = 3e^x - xe^x$, $y'' = 2e^x - xe^x$

5. **a.** $2(\ln 8)(8^{2x+5})$

b. $0.64(\ln 10)((10)^{2x})$

c. $2^x((\ln 2)(x^2) + 2x)$

d. $900(\ln 5)(5)^{3x-1}$

e. $(\ln 1.9)(1.9)^x + 1.9x^{0.9}$

f. $4^x((\ln 4)(x - 2)^2 + 2x - 4)$

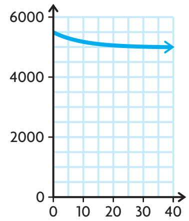
6. **a.** 5500

b. $-50\left(e^{-\frac{t}{10}}\right)$

c. decreasing by about 15 rabbits/month

d. 5500

e.



The graph is constantly decreasing. The y-intercept is (0, 5500). Rabbit populations normally grow exponentially, but this population is shrinking exponentially. Perhaps a large number of rabbit predators, such as snakes, recently began to appear in the forest. A large number of predators would quickly shrink the rabbit population.

7. at about 0.41 h

8. The original function represents growth when $ck > 0$, meaning that c and k must have the same sign. The original function represents decay when c and k have opposite signs.

9. **a.** 5000

b. 5751

c. 9111

10. **a.** 406.80 mm Hg

b. 316.82 mm Hg

c. 246.74 mm Hg

11. 15% per year

12. $f(x) = xe^x$

$$f'(x) = xe^x + (1)e^x = e^x(x + 1)$$

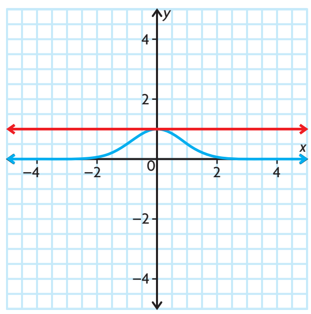
So, $e^x > 0$

$x + 1 > 0$

$x > -1$

This means that the function is increasing when $x > -1$.

13.



14. **a.** $A(t) = 1000(1.06)^t$

b. $A'(t) = 1000(1.06)^t \ln 1.06$

c. $A'(2) = \$65.47$, $A'(5) = \$77.98$, $A'(10) = \$104.35$

d. No

e. $\frac{A'(2)}{A(2)} = \ln 1.06$,
$\frac{A'(5)}{A(5)} = \ln 1.06$,
$\frac{A'(10)}{A(10)} = \ln 1.06$

f. All the ratios are equivalent (they equal ln 1.06, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

15. $y = ce^x$
$y' = c(e^x) + (0)(e^x)$
$= ce^x$
$y = y' = ce^x$

Section 5.4, pp. 256–257

1. **a.** $2\cos 2x$
b. $-6\sin 3x$
c. $(3x^2 - 2)(\cos(x^3 - 2x + 4))$
d. $8\sin(-4x)$
e. $3\cos(3x) + 4\sin(4x)$
f. $2^x(\ln 2) + 2\cos x + 2\sin x$
g. $e^x\cos(e^x)$
h. $9\cos(3x + 2\pi)$
i. $2x - \sin x$
j. $-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

2. **a.** $2\cos(2x)$
b. $-\frac{2\sin 2x}{x} - \frac{\cos 2x}{x^2}$
c. $-\sin(\sin 2x) \times 2\cos 2x$
d. $\frac{1}{1 + \cos x}$
e. $e^x(2\cos x)$
f. $2x^3\cos x + 6x^2\sin x + 3x\sin x - 3\cos x$

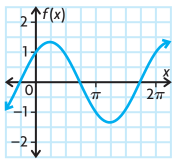
3. **a.** $-x + 2y + \left(\frac{\pi}{3} - \sqrt{3}\right) = 0$
b. $-2x + y = 0$
c. $y = -1$
d. $y = -3\left(x - \frac{\pi}{2}\right)$
e. $y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right)$
f. $2x + y - \pi = 0$

4. **a.** One could easily find $f'(x)$ and $g'(x)$ to see that they both equal $2(\sin x)(\cos x)$. However, it is easier to notice a fundamental trigonometric identity. It is known that $\sin^2 x + \cos^2 x = 1$. So, $\sin^2 x = 1 - \cos^2 x$. Therefore, $f(x)$ is in fact equal to $g(x)$. So, because $f(x) = g(x)$, $f'(x) = g'(x)$.
b. $f'(x)$ and $g'(x)$ are each others' negative. That is,
$f'(x) = (\sin x)(\cos x)$, while
$g'(x) = -2(\sin x)(\cos x)$.

5. **a.** $v'(t) = \frac{\sin(\sqrt{t})\cos(\sqrt{t})}{\sqrt{t}}$
b. $v'(t) = \frac{-\sin t + 2(\sin t)(\cos t)}{2\sqrt{1 + \cos t + \sin^2 t}}$
c. $h'(x) = 3\sin x\sin 2x\cos 3x + 2\sin x\sin 3x\cos 2x + \sin 2x\sin 3x\cos x$
d. $m'(x) = 3(x^2 + \cos^2 x)^2 \times (2x - 2\sin x\cos x)$

6. **a.** absolute max: $\sqrt{2}$, absolute min: $-\sqrt{2}$
b. absolute max: 2.26, absolute min: -5.14
c. absolute max: $\sqrt{2}$, absolute min: $-\sqrt{2}$
d. absolute max: 5, absolute min: -5

7. **a.** $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k
b. 8

8. **a.**


b. $\left(\frac{\pi}{4}, \sqrt{2}\right)$

9. $(\csc x)' = -\csc x\cot x$,
$(\sec x)' = \sec x\tan x$

10. $-\sqrt{3}$

11. **a.** $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k
b. 4
c. minimum: 0 maximum: 4

12. $\theta = \frac{\pi}{3}$

13. $\theta = \frac{\pi}{6}$

14. First find y''.
$y = A\cos kt + B\sin kt$
$y' = -kA\sin kt + kB\cos kt$
$y'' = -k^2A\cos kt - k^2B\sin kt$
So, $y'' + k^2y$
$= -k^2A\cos kt - k^2B\sin kt + k^2(A\cos kt + B\sin kt)$
$= -k^2A\cos kt - k^2B\sin kt + k^2A\cos kt + k^2B\sin kt$
$= 0$
Therefore, $y'' + k^2y = 0$.

Section 5.5, p. 260

1. **a.** $3\sec^2 3x$
b. $2\sec^2 x - 2\sec 2x$
c. $6x^2\tan(x^2)\sec^2(x^3)$
d. $\frac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$
e. $2x\sec^2(x^2) - 2\tan x\sec^2 x$
f. $15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$

2. **a.** $y = 2\left(x - \frac{\pi}{4}\right)$
b. $y = -2x$

3. **a.** $\cos x\sec^2(\sin x)$
b. $-4x[\tan(x^2 - 1)]^{-3}\sec^2(x^2 - 1)$
c. $-2\tan(\cos x)\sec^2(\cos x)\sin x$
d. $2(\tan x + \cos x)(\sec^2 x - \sin x)$
e. $\sin^2 x(3\tan x\cos x + \sin x\sec^2 x)$
f. $\frac{1}{2\sqrt{x}}e^{\tan\sqrt{x}}\sec^2 2\sqrt{x}$

4. **a.** $\cos x + \sec x + \frac{2\sin^2 x}{\cos^3 x}$
b. $2\sec^2 x(1 + 3\tan^2 x)$

5. $x = 0, \pi$, and 2π

6. $\left(\frac{\pi}{4}, 0.57\right)$

7. $y = \sec x + \tan x$
$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$
$= \frac{1 + \sin x}{\cos x}$
$\frac{dy}{dx} = \frac{\cos^2 x - (1 + \sin x)(-\sin x)}{\cos^2 x}$
$= \frac{\cos^2 x - (-\sin x - \sin^2 x)}{\cos^2 x}$
$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$
$= \frac{1 + \sin x}{\cos^2 x}$
The denominator is never negative. $1 + \sin x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since $\sin x$ reaches its minimum of -1 at $x = \frac{\pi}{2}$. Since the derivative of the original function is always positive in the specified interval, the function is always increasing in that interval.

8. $-4x + y - (2 - \pi) = 0$

9. Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to derive the derivative of the tangent function.

10. $-\csc^2 x$

11. $f''(x) = 8\csc^2 x\cot x$

Review Exercise, pp. 263–265

1. a. $-e^x$
   b. $2 + 3e^x$
   c. $2e^{2x+3}$
   d. $(-6x + 5)e^{-3x^2+5x}$
   e. $e^x(x + 1)$
   f. $\frac{2e^t}{(e^t + 1)^2}$
2. a. $10^x \ln 10$
   b. $6x(4^{3x^2})\ln 4$
   c. $5 \times 5^x(x \ln 5 + 1)$
   d. $x^3 \times 2^x(x \ln 2 + 4)$
   e. $\frac{4 - 4x \ln 4}{4^x}$
   f. $5^{\sqrt{x}}\left(-\frac{1}{x^2} + \frac{\ln 5}{2x\sqrt{x}}\right)$
3. a. $6 \cos(2x) + 8 \sin(2x)$
   b. $3 \sec^2(3x)$
   c. $-\frac{\sin x}{(2 - \cos x)^2}$
   d. $2x \sec^2(2x) + \tan 2x$
   e. $e^{3x}(3 \sin 2x + 2 \cos 2x)$
   f. $-4 \cos(2x)\sin(2x)$
4. a. $x = 1$
   b. The function has a horizontal tangent at $(1, e)$. So this point could be possible local max or min.
5. a. 0
   b. The slope of the tangent to $f(x)$ at the point with x-coordinate $\frac{1}{2}$ is 0.
6. a. $e^x(x + 1)$
   b. $20e^{10x}(5x + 1)$
7. $y = \frac{e^{2x} - 1}{e^{2x} + 1}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$$
$$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(e^{2x} + 1)^2}$$

Now, $1 - y^2 = 1 - \frac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$

$$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(3^{2x} + 1)^2} = \frac{dy}{dx}$$

8. $3x - y + 2\ln 2 - 2 = 0$
9. $-x + y = 0$
10. about 0.3928 m per unit of time
11. a. $t = 20$
    b. After 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.
12. a. c_2
    b. c_1
13. a. $-9e^{-x}(2 + 3e^{-x})^2$
    b. ex^{e-1}
    c. e^{x+e^x}
    d. $-25e^{5x}(1 - e^{5x})^4$
14. a. $5^x \ln 5$
    b. $(0.47)^x \ln(0.47)$
    c. $2(52)^{2x} \ln 52$
    d. $5(2)^x \ln 2$
    e. $4e^x$
    f. $-6(10)^{3x} \ln 10$
15. a. $2^x \ln 2 \cos 2^x$
    b. $x^2 \cos x + 2x \sin x$
    c. $-\cos\left(\frac{\pi}{2} - x\right)$
    d. $\cos^2 x - \sin^2 x$
    e. $-2 \cos x \sin x$
    f. $2 \sin x \cos^2 x - \sin^3 x$
16. $x + y - \frac{\pi}{2} = 0$
17. $v = \frac{ds}{dt}$;

Thus, $v = 8(\cos(10\pi t))(10\pi)$
$= 80\pi \cos(10\pi t)$

The acceleration at any time t is

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

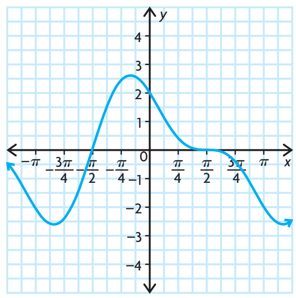
Hence, $a = 80\pi(-\sin(10\pi t))(10\pi)$
$= -800\pi^2 \sin(10\pi t)$. Now,

$\frac{d^2s}{dt^2} + 100\pi^2 s = -800\pi^2 \sin(10\pi t)$
$+ 100\pi^2(8 \sin(10\pi t)) = 0$.

18. displacement: 5,
    velocity: 10,
    acceleration: 20
19. each angle $\frac{\pi}{4}$ rad, or 45°
20. 4.5 m
21. 2.5 m
22. 5.19 ft
23. a. $f''(x) = -8 \sin^2(x - 2) + 8 \cos^2(x - 2)$
    b. $f''(x) = (4 \cos x)(\sec^2 x \tan x) - 2 \sin x(\sec x)^2$

Chapter 5 Test, p. 266

1. a. $-4xe^{-2x^2}$
   b. $3e^{x^2+3x} \cdot \ln 3 \cdot (2x + 3)$
   c. $\frac{3}{2}[e^{3x} - e^{-3x}]$
   d. $2 \cos x + 15 \sin 5x$
   e. $6x \sin^2(x^2)\cos(x^2)$
   f. $-\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$
2. $-6x + y = 2$,
   The tangent line is the given line.
3. $-2x + y = 1$
4. a. $a(t) = v'(t) = -10ke^{-kt}$
   $= -k(10e^{-kt})$
   $= -kv(t)$
   Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.
   b. 10 cm/s
   c. $\frac{\ln 2}{k}$; $-5k$
5. a. $f''(x) = 2(\sin^2 x - \cos^2 x)$
   b. $f''(x) = \csc x \cot^2 x + \csc^3 x + \sin x$
6. absolute max: 1,
   absolute min: 0
7. 40.24
8. minimum: $\left(-4, -\frac{1}{e^4}\right)$, no maximum
9. a. $x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}$
   b. increasing: $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$;
   decreasing: $-\pi \le x < -\frac{5\pi}{6}$ and $-\frac{\pi}{6} < x < \pi$
   c. local maximum at $x = -\frac{\pi}{6}$; local minimum at $x = -\frac{5\pi}{6}$
   d.



Cumulative Review of Calculus, pp. 267–270

1. a. 16
   b. −2
   c. $\frac{1}{6}$
   d. $160 \ln 2$
2. a. 13 m/s
   b. 15 m/s
3. $f(x) = x^3$
4. a. 19.6 m/s
   b. 19.6 m/s
   c. 53.655 m/s

5. **a.** 19 000 fish/year
   b. 23 000 fish/year

6. **a.** **i.** 3
   ii. 1
   iii. 3
   iv. 2
   b. No, $\lim_{x\to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$ must exist and they must be the same. In this case, $\lim_{x\to 4^-} f(x) = \infty$, but $\lim_{x\to 4^+} f(x) = -\infty$, so $\lim_{x\to 4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x\to 2^-} f(x) = 5$, but $\lim_{x\to 2^+} f(x) = 3$.

8. **a.** $-\frac{1}{5}$ **d.** $\frac{4}{3}$
   b. 6 **e.** $\frac{1}{12}$
   c. $-\frac{1}{9}$ **f.** $\frac{1}{2}$

9. **a.** $6x + 1$
   b. $-\frac{1}{x^2}$

10. **a.** $3x^2 - 8x + 5$
    b. $\frac{3x^2}{\sqrt{2x^3 + 1}}$
    c. $\frac{6}{(x + 3)^2}$
    d. $4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$
    e. $\frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$
    f. $5[x^2 + (2x + 1)^3]^4 \times [2x + 6(2x + 1)^2]$

11. $4x + 3y - 10 = 0$

12. 3

13. **a.** $p'(t) = 4t + 6$
    b. 46 people per year
    c. 2006

14. **a.** $f'(x) = 5x^4 - 15x^2 + 1$; $f''(x) = 20x^3 - 30x$
    b. $f'(x) = \frac{4}{x^3}$; $f''(x) = -\frac{12}{x^4}$
    c. $f'(x) = -\frac{2}{\sqrt{x^3}}$; $f''(x) = \frac{3}{\sqrt{x^5}}$
    d. $f'(x) = 4x^3 + \frac{4}{x^5}$; $f''(x) = 12x^2 - \frac{20}{x^6}$

15. **a.** maximum: 82, minimum: 6
    b. maximum: $9\frac{1}{3}$, minimum: 2
    c. maximum: $\frac{e^4}{1 + e^4}$, minimum: $\frac{1}{2}$
    d. maximum: 5, minimum: 1

16. **a.** $v(t) = 9t^2 - 81t + 162$, $a(t) = 18t - 81$
    b. stationary when $t = 6$ or $t = 3$, advancing when $v(t) > 0$, and retreating when $v(t) < 0$
    c. $t = 4.5$
    d. $0 \le t < 4.5$
    e. $4.5 < t \le 8$

17. 14 062.5 m^2

18. $r \doteq 4.3$ cm, $h \doteq 8.6$ cm

19. $r = 6.8$ cm, $h = 27.5$ cm

20. **a.** $140 - 2x$
    b. 101 629.5 cm^3; 46.7 cm by 46.7 cm by 46.6 cm

21. $x = 4$

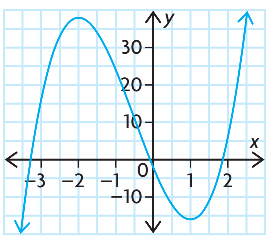
22. \$70 or \$80

23. \$1140

24. **a.** $\frac{dy}{dx} = -10x + 20$, $x = 2$ is critical number, Increase: $x < 2$, Decrease: $x > 2$
    b. $\frac{dy}{dx} = 12x + 16$, $x = -\frac{4}{3}$ is critical number, Increase: $x > -\frac{4}{3}$, Decrease: $x < -\frac{4}{3}$
    c. $\frac{dy}{dx} = 6x^2 - 24$, $x = \pm 2$ are critical numbers, Increase: $x < -2$, $x > 2$, Decrease: $-2 < x < 2$
    d. $\frac{dy}{dx} = -\frac{2}{(x - 2)^2}$. The function has no critical numbers. The function is decreasing everywhere it is defined, that is, $x \ne 2$.

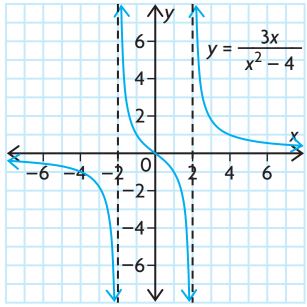
25. **a.** $y = 0$ is a horizontal asymptote. $x = \pm 3$ are the vertical asymptotes. There is no oblique asymptote. $\left(0, -\frac{8}{9}\right)$ is a local maximum.
    b. There are no horizontal asymptotes. $x = \pm 1$ are the vertical asymptotes. $y = 4x$ is an oblique asymptote. $(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum.

26. **a.**



$y = 4x^3 + 6x^2 - 24x - 2$

b.



27. **a.** $(-20)e^{5x+1}$
    b. $e^{3x}(3x + 1)$
    c. $(3\ln 6)6^{3x-8}$
    d. $(\cos x)e^{\sin x}$

28. $y = 2e(x - 1) + e$

29. **a.** 5 days
    b. 27

30. **a.** $2\cos x + 15\sin 5x$
    b. $8\cos 2x(\sin 2x + 1)^3$
    c. $\frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$
    d. $\frac{1 + 2\cos x}{(\cos x + 2)^2}$
    e. $2x\sec^2 x^2 - 2\tan x \sec^2 x$
    f. $-2x\sin x^2 \cos(\cos x^2)$

31. about 4.8 m

32. about 8.5 m

Chapter 6

Review of Prerequisite Skills, p. 273

1. **a.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{\sqrt{3}}{2}$
   b. $-\sqrt{3}$ **e.** $\frac{\sqrt{2}}{2}$
   c. $\frac{1}{2}$ **f.** 1

2. $\frac{4}{3}$

3. **a.** $AB \doteq 29.7$, $\angle B \doteq 36.5°$, $\angle C \doteq 53.5°$
   b. $\angle A \doteq 97.9°$, $\angle B \doteq 29.7°$, $\angle C \doteq 52.4°$

4. $\angle Z \doteq 50°$, $XZ \doteq 7.36$, $YZ \doteq 6.78$

5. $\angle R \doteq 44°$, $\angle S \doteq 102°$, $\angle T \doteq 34°$

6. 5.82 km

7. 8.66 km

8. 21.1 km

9. 59.4 cm^2

5. a. 19 000 fish/year
b. 23 000 fish/year

6. a. i. 3
ii. 1
iii. 3
iv. 2

b. No, $\lim_{x \to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ must exist and they must be the same. In this case, $\lim_{x \to 4^-} f(x) = \infty$, but $\lim_{x \to 4^+} f(x) = -\infty$, so $\lim_{x \to 4} f(x)$ does not exist.

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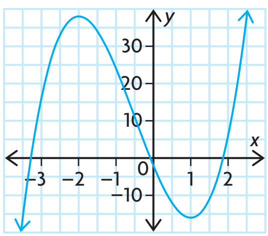

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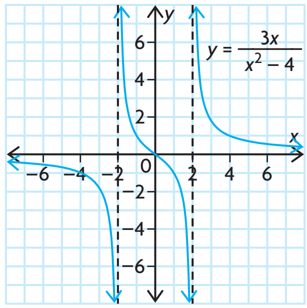
b.



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Chapter 6

Review of Prerequisite Skills, p. 273

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d. $\frac{\sqrt{3}}{2}$
b. $-\sqrt{3}$
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c. $\frac{1}{2}$
f. 1

2. $\frac{4}{3}$

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