
Frequency Modulation

Chapter

6

Trigonometric Functions

GOALS

You will be able to

- Understand radian measure and its relationship to degree measure
- Use radian measure with trigonometric functions
- Make connections between trigonometric ratios and the graphs of the primary and reciprocal trigonometric functions
- Pose, model, and solve problems that involve trigonometric functions
- Solve problems that involve rates of change in trigonometric functions

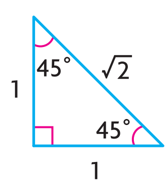
? FM radio stations and many other wireless technologies (such as the sound portion of a television signal, cordless phones, and cell phones) transmit information using sine waves. The equations that are used to model these sine waves, however, do not use angles that are measured in degrees. What is an alternative way to measure angles, and how does this affect the graphs of trigonometric functions?

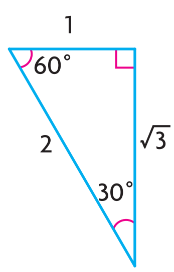
6 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

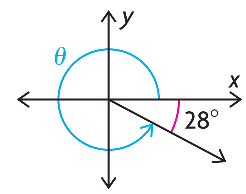
Question	Appendix
1, 2, 3, 4	R-10
5	R-11
6, 7	R-12





SKILLS AND CONCEPTS You Need

1. For angle θ, determine
a) the size of the **related acute angle**
b) the size of the **principal angle**



2. Point $P(3, -4)$ lies on the terminal arm of an angle in standard position.
a) Sketch the angle, and determine the values of the primary and reciprocal ratios.
b) Determine the measure of the principal angle, to the nearest degree.

3. Draw each angle in standard position. Then, using the **special triangles** as required, determine the exact value of the trigonometric ratio.
a) $\sin 60°$ **c)** $\sin 120°$ **e)** $\sec 135°$
b) $\tan 180°$ **d)** $\cos 300°$ **f)** $\csc 270°$

4. Determine the value(s) of θ, if $0° \leq \theta \leq 360°$.
a) $\cos \theta = \frac{1}{2}$ **c)** $\tan \theta = 1$ **e)** $\cot \theta = -1$
b) $\tan \theta = \frac{1}{\sqrt{3}}$ **d)** $\cos \theta = -1$ **f)** $\sin \theta = 1$

5. For each of the following, state the **period**, **amplitude**, **equation of the axis**, and range of the function. Then sketch its graph.
a) $y = \sin \theta$, where $-360° \leq \theta \leq 360°$.
b) $y = \cos \theta$, where $-360° \leq \theta \leq 360°$.

6. State the period, equation of the axis, horizontal shift, and amplitude of each function. Then sketch one cycle.
a) $y = 2 \sin (3(x + 45°))$ **b)** $y = -\sin \left(\frac{1}{2}(x - 60°)\right) - 1$

7. Identify the transformation that is associated with each of the parameters (a, k, d, and c) in the graphs defined by $y = a \sin (k(x - d)) + c$ and $y = a \cos (k(x - d)) + c$. Discuss which graphical feature (period, amplitude, equation of the axis, or horizontal shift) is associated with each parameter.

APPLYING What You Know

Using a Sinusoidal Model

YOU WILL NEED

- graph paper

A Ferris wheel has a diameter of 20 m, and its axle is located 15 m above the ground. Once the riders are loaded, the Ferris wheel accelerates to a steady speed and rotates 10 times in 4 min. The height, h metres, of a rider above the ground during a ride on this Ferris wheel can be modelled by a sinusoidal function of the form $h(t) = a \sin (k(t - d)) + c$, where t is the time in seconds.

The height of a rider begins to be tracked when the rider is level with the axis of the Ferris wheel on the first rotation.

? What does the graph of the rider's height versus time, for three complete revolutions, look like? What equation can be used to describe this graph?

A. Determine the maximum and minimum heights of a rider above the ground during the ride.

B. How many seconds does one complete revolution take? What part of the graph represents this?

C. On graph paper, sketch a graph of the rider's height above the ground versus time for three revolutions of the Ferris wheel.

D. What type of curve does your graph resemble?

E. Is this function a periodic function? Explain.

F. What is the amplitude of this function?

G. What is the period of this function?

H. What is the equation of the axis of this function?

I. Assign appropriate values to each parameter in $h(t)$ for this situation.

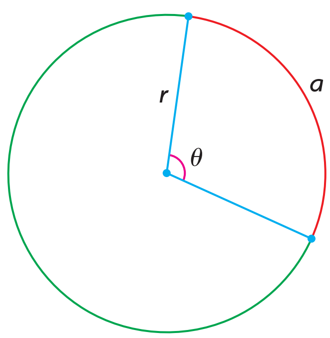
J. Write the equation of a sine function that describes the graph you sketched in part C.

6.1 Radian Measure

GOAL

Use radian measurement to represent the size of an angle.

LEARN ABOUT the Math



Angles are commonly measured in degrees. In mathematics and physics, however, there are many applications in which expressing the size of an angle as a pure number, without units, is more convenient than using degrees. In these applications, the size of an angle is expressed in terms of the length of an arc, a, that subtends the angle, θ, at the centre of a circle with radius r. In this situation, a is proportional to r and also to θ, where $\theta = \frac{a}{r}$. The unit of measure is the radian.

radian

the size of an angle that is subtended at the centre of a circle by an arc with a length equal to the radius of the circle; both the arc length and the radius are measured in units of length (such as centimetres) and, as a result, the angle is a real number without any units

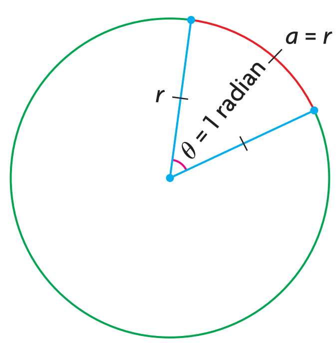
$\theta = \frac{a}{r} = \frac{r}{r} = 1$

? How are radians and degrees related to each other?

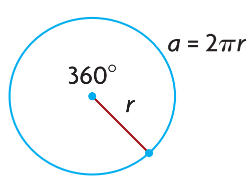
EXAMPLE 1 Connecting radians and degrees

How many degrees is 1 radian?

Solution



1 radian is defined as the angle subtended by an arc length, a, equal to the radius, r. It appears as though 1 radian should be a little less than 60°, since the sector formed resembles an equilateral triangle, with one side that is curved slightly.



Consider the arc length created by an angle of 360°. This arc length is $2\pi r$, the circumference of the circle.

$$\theta = \frac{2\pi \cancel{r}^{1}}{\cancel{r}_{1}} = 2\pi \text{ radians}$$

Using the relationship $\theta = \frac{a}{r}$, the size of the angle can be expressed in radians.

$\theta = 360°$

θ is also 360°, the size of the central angle with an arc length of $2\pi r$.

$2\pi \text{ radians} = 360°$

$\pi \text{ radians} = 180°$

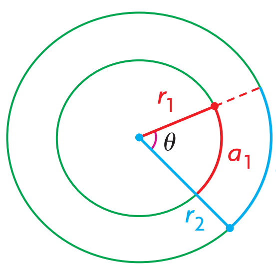
Equating the measures of θ gives an expression relating degrees and radians.

Dividing both sides by 2 gives a simpler relationship between degrees and radians.

$$\frac{\pi \text{ radians}}{\pi} = \frac{180°}{\pi}$$

$$1 \text{ radian} = \frac{180°}{\pi} \doteq 57.3°$$

Dividing both sides by π gives the value of 1 radian in degree measure.



It is important to note that the size of an angle in radians is not affected by the size of the circle. The diagram shows that a_1 and a_2 subtend the same angle θ, so $\theta = \frac{a_1}{r_1} = \frac{a_2}{r_2}$.

The relationship π radians $= 180°$ can be used to convert between degrees and radians.

EXAMPLE 2 Reasoning how to convert degrees to radians

Convert each of the following angles to radians.

a) 20° **b)** 225°

Solution

a) $\pi \text{ radians} = 180°$

Divide both sides by 180° to get an equivalent expression that is equal to 1.

$$\frac{\pi}{180°} = 1$$

Multiplying by 1 creates an equivalent expression, so multiply by $\frac{\pi}{180°}$ to convert degrees to radians.

$$20° = (\overset{1}{\cancel{20°}})\left(\frac{\pi}{\underset{9}{\cancel{180°}}}\right)$$

Simplify by dividing by the common factor of 20. Notice that the units cancel out.

$$= \frac{\pi}{9}$$

$$\doteq 0.35$$

Express the answer as an exact value in terms of π or as an approximate decimal value, as required.

Communication *Tip*

Whenever an angle is expressed without a unit (that is, as a real number), it is understood to be in radians. We often write "radians" after the number, as a reminder that we are discussing an angle.

b) $225° = (225°)\left(\frac{\pi}{180°}\right)$ ← Multiply by $\frac{\pi}{180°}$ to convert degrees to radians.

$= \frac{225°\pi}{180°} \div \frac{45}{45}$ ← Simplify by dividing by the common factor of 45. (Note that the degree symbols cancel.)

$= \frac{5\pi}{4}$

EXAMPLE 3 Reasoning how to convert radians to degrees

Convert each radian measure to degrees.

a) $\frac{5\pi}{6}$ **b)** 1.75 radians

Solution

a) π radians $= 180°$

$\frac{5\pi}{6} = \frac{5(180°)}{6}$ ← Substitute 180° for π.

$= 5(30°)$ ← Evaluate.

$= 150°$

b) π radians $= 180°$

$1 = \frac{180°}{\pi \text{ radians}}$ ← Divide both sides by π radians to get an equivalent expression that is equal to 1.

$1.75 \text{ radians} = 1.75\ \cancel{\text{radians}}^{1} \times \frac{180°}{\pi\ \cancel{\text{radians}}_{1}}$ ← Multiplying by 1 creates an equivalent expression, so multiply by $\frac{180°}{\pi \text{ radians}}$ to convert radians to degrees.

$\doteq 100.3°$

Reflecting

A. Consider the formula $\theta = \frac{a}{r}$. Explain why angles can be described as having no unit when they are measured in radians.

B. Explain how to convert any angle measure that is given in degrees to radians.

C. Explain how to convert any angle measure that is given in radians to degrees.

APPLY the Math

EXAMPLE 4 Solving a problem that involves radians

The London Eye Ferris wheel has a diameter of 135 m and completes one revolution in 30 min.

a) Determine the angular velocity, ω, in radians per second.
b) How far has a rider travelled at 10 min into the ride?

Solution

a) $30 \text{ min} = 30 \cancel{\text{min}} \times \dfrac{60 \text{ s}}{1 \cancel{\text{min}}}$

$= 1800 \text{ s}$

Since the question asks for angular velocity in radians per second, convert the time to seconds.

Angular velocity, $\omega = \dfrac{2\pi}{1800}$ radians/s

$= \dfrac{\pi}{900}$ radians/s

$\doteq 0.003\ 49$ radians/s

Each revolution of the Ferris wheel represents an angular motion through an angle of 2π radians. Therefore, the Ferris wheel moves through 2π radians every 30 min.

b) Radius, $r = \dfrac{135}{2}$ m

$= 67.5$ m

The rider moves in a circular motion on the edge of a circle that has a radius of 67.5 m.

Number of revolutions, $n = \dfrac{10 \cancel{\text{min}}}{30 \cancel{\text{min}}}$

$= \dfrac{1}{3}$ revolution

The wheel turns through one revolution every 30 min, so the rider has gone through $\frac{1}{3}$ of a revolution at 10 min.

Distance travelled, $d = \dfrac{1}{3}(2\pi \times 67.5 \text{ m})$

$= 45\pi$ m

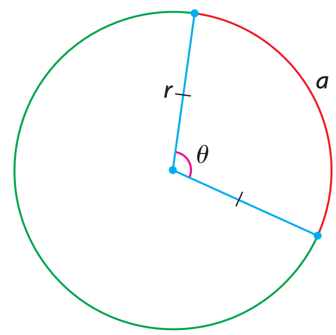
$\doteq 141.4$ m

The rider travels $\frac{1}{3}$ of the circumference in 10 min.

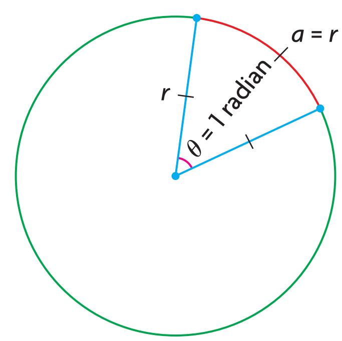
In Summary

Key Ideas

- The radian is an alternative way to represent the size of an angle. The arc length, a, of a circle is proportional to its radius, r, and the central angle that it subtends, θ, by the formula $\theta = \frac{a}{r}$.



- One radian is defined as the angle subtended by an arc that is the same length as the radius. $\theta = \frac{a}{r} = \frac{r}{r} = 1$. 1 radian is about 57.3°.



Need to Know

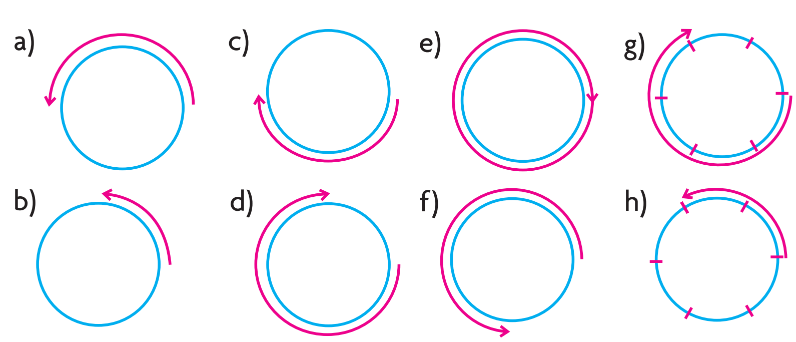
- Using radians enables you to express the size of an angle as a real number without any units, often in terms of π. It is related to degree measure by the following conversion factor: π radians $= 180°$.
- To convert from degree measure to radians, multiply by $\frac{\pi}{180°}$.
- To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

CHECK Your Understanding

Communication *Tip*

Recall that counterclockwise rotation is represented using positive angles, while clockwise rotation is represented using **negative angles**.

1. A point is rotated about a circle of radius 1. Its start and finish are shown. State the rotation in radian measure and in degree measure.



2. Sketch each rotation about a circle of radius 1.

a) π c) $\frac{2\pi}{3}$ e) $\frac{5\pi}{3}$ g) $-\frac{\pi}{2}$

b) $\frac{\pi}{3}$ d) $\frac{4\pi}{3}$ f) $-\pi$ h) $-\frac{\pi}{4}$

3. Convert each angle from degrees to radians, in exact form.

a) $75°$ b) $200°$ c) $400°$ d) $320°$

4. Convert each angle from radians to degrees. Express the measure correct to two decimal places, if necessary.

a) $\frac{5\pi}{3}$ b) 0.3π c) 3 d) $\frac{11\pi}{4}$

PRACTISING

5. a) Determine the measure of the central angle that is formed by an arc length of 5 cm in a circle with a radius of 2.5 cm. Express the measure in both radians and degrees, correct to one decimal place.
   b) Determine the arc length of the circle in part a) if the central angle is 200°.

6. Determine the arc length of a circle with a radius of 8 cm if
   a) the central angle is 3.5
   b) the central angle is 300°

7. Convert to radian measure.

K a) $90°$ c) $-180°$ e) $-135°$ g) $240°$

b) $270°$ d) $45°$ f) $60°$ h) $-120°$

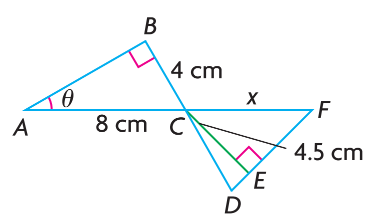
8. Convert to degree measure.

a) $\frac{2\pi}{3}$ c) $\frac{\pi}{4}$ e) $\frac{7\pi}{6}$ g) $\frac{11\pi}{6}$

b) $-\frac{5\pi}{3}$ d) $-\frac{3\pi}{4}$ f) $-\frac{3\pi}{2}$ h) $-\frac{9\pi}{2}$

9. If a circle has a radius of 65 m, determine the arc length for each of the following central angles.

a) $\frac{19\pi}{20}$ b) 1.25 c) $150°$



10. Given $\angle DCE = \frac{\pi}{12}$ radians and $CE = 4.5$ cm, determine the size of
A θ and x .

11. A wind turbine has three blades, each measuring 3 m from centre to tip. At a particular time, the turbine is rotating four times a minute.

a) Determine the angular velocity of the turbine in radians/second.

b) How far has the tip of a blade travelled after 5 min?

12. A wheel is rotating at an angular velocity of 1.2π radians/s, while a point on the circumference of the wheel travels 9.6π m in 10 s.

a) How many revolutions does the wheel make in 1 min?

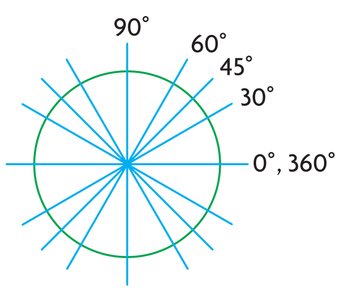
b) What is the radius of the wheel?

13. **T** Two pieces of mud are stuck to the spoke of a bicycle wheel. Piece A is closer to the circumference of the tire, while piece B is closer to the centre of the wheel.

a) Is the angular velocity at which piece A is travelling greater than, less than, or equal to the angular velocity at which piece B is travelling?

b) Is the velocity at which piece A is travelling greater than, less than, or equal to the velocity at which piece B is travelling?

c) If the angular velocity of the bicycle wheel increased, would the velocity at which piece A is travelling as a percent of the velocity at which piece B is travelling increase, decrease, or stay the same?



14. **C** In your notebook, sketch the diagram shown and label each angle, in degrees, for one revolution. Then express each of these angles in exact radian measure.

Extending

15. Circle A has a radius of 15 cm and a central angle of $\frac{\pi}{6}$ radians, circle B has a radius of 17 cm and a central angle of $\frac{\pi}{7}$ radians, and circle C has a radius of 14 cm and a central angle of $\frac{\pi}{5}$ radians. Put the circles in order, from smallest to largest, based on the lengths of the arcs subtending the central angles.

16. The members of a high-school basketball team are driving from Calgary to Vancouver, which is a distance of 675 km. Each tire on their van has a radius of 32 cm. If the team members drive at a constant speed and cover the distance from Calgary to Vancouver in 6 h 45 min, what is the angular velocity, in radians/second, of each tire during the drive?

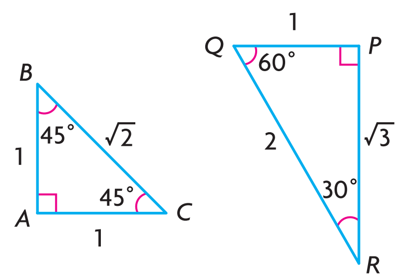
6.2 Radian Measure and Angles on the Cartesian Plane

GOAL

Use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

LEARN ABOUT the Math

Recall that the special triangles shown can be used to determine the exact values of the primary and reciprocal trigonometric ratios for some angles measured in degrees.



? How can these special triangles be used to determine the exact values of the trigonometric ratios for angles expressed in radians?

EXAMPLE 1 Connecting radians and the special triangles

Determine the radian measures of the angles in the special triangles, and calculate their primary trigonometric ratios.

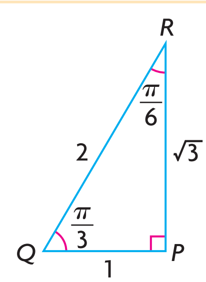
Solution

$\angle Q = 60°$

$$60° = \overset{1}{\cancel{60°}}\left(\frac{\pi}{\underset{3}{\cancel{180°}}}\right) = \frac{\pi}{3}$$

$\angle R = 30°$

$$30° = \overset{1}{\cancel{30°}}\left(\frac{\pi}{\underset{6}{\cancel{180°}}}\right) = \frac{\pi}{6}$$



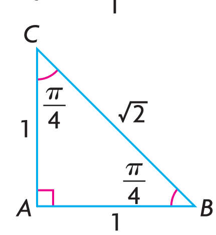
$\triangle PQR$ is the 30°, 60°, 90° special triangle. Multiply each angle by $\frac{\pi}{180°}$ to convert from degrees to radians.

$\angle B = \angle C = 45°$

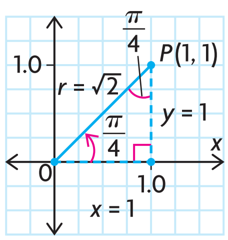
$$45° = \overset{1}{\cancel{45°}}\left(\frac{\pi}{\underset{4}{\cancel{180°}}}\right) = \frac{\pi}{4}$$

$\angle P = \angle A = 90°$

$$90° = \overset{1}{\cancel{90°}}\left(\frac{\pi}{\underset{2}{\cancel{180°}}}\right) = \frac{\pi}{2}$$



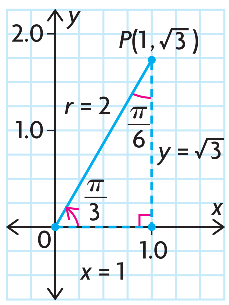
$\triangle ABC$ is the 45°, 45°, 90° special triangle. Multiply each angle by $\frac{\pi}{180°}$ to convert from degrees to radians.



$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \csc\frac{\pi}{4} = \sqrt{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \cot\frac{\pi}{4} = 1$$



$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sec\frac{\pi}{3} = 2$$

$$\tan\frac{\pi}{3} = \sqrt{3} \qquad \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

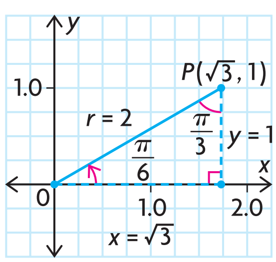
Draw each special angle on the Cartesian plane in **standard position**. Use the trigonometric definitions of angles on the Cartesian plane to determine the exact value of each angle. Recall that

$\sin\theta = \frac{y}{r}$ $\quad\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\quad\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\quad\cot\theta = \frac{x}{y}$

where $x^2 + y^2 = r^2$ and $r > 0$.



$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \csc\frac{\pi}{6} = 2$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \cot\frac{\pi}{6} = \sqrt{3}$$

Reflecting

A. Compare the exact values of the trigonometric ratios in each special triangle when the angles are given in radians and when the angles are given in degrees.

B. Explain why the strategy that is used to determine the value of a trigonometric ratio for a given angle on the Cartesian plane is the same when the angle is expressed in radians and when the angle is expressed in degrees.

APPLY the Math

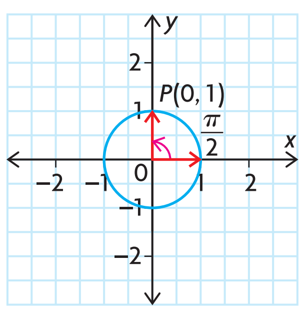
EXAMPLE 2 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of each trigonometric ratio.

a) $\sin\left(\frac{\pi}{2}\right)$ **b)** $\cot\left(\frac{3\pi}{2}\right)$

Solution

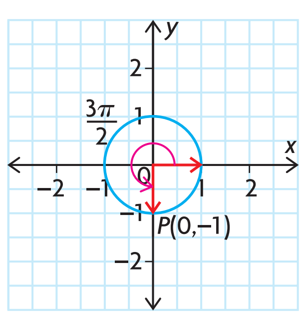
a)



$\frac{\pi}{2}$ is one-quarter of a full revolution, and the point $P(0, 1)$ lies on the unit circle, as shown. Draw the angle in standard position with its terminal arm on the positive y-axis. From the drawing, $x = 0$, $y = 1$, and $r = 1$.

$$\sin\left(\frac{\pi}{2}\right) = \frac{y}{r}$$
$$= \frac{1}{1} = 1$$

b)



$\frac{3\pi}{2}$ is three-quarters of a full revolution, and the point $P(0, -1)$ lies on the unit circle, as shown. Draw the angle in standard position with its terminal arm on the negative y-axis. From the drawing, $x = 0$, $y = -1$ and $r = 1$.

$$\cot\left(\frac{3\pi}{2}\right) = \frac{x}{y}$$
$$= \frac{0}{-1} = 0$$

The relationships between the principal angle, its related acute angle, and the trigonometric ratios for angles in standard position are the same when the angles are measured in radians and degrees.

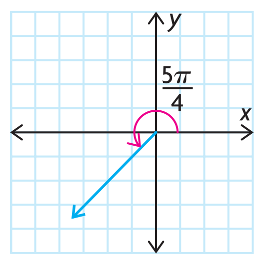
EXAMPLE 3 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of each trigonometric ratio.

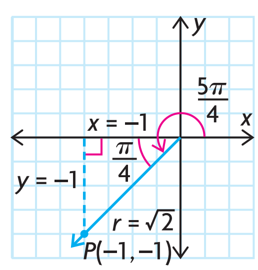
a) $\cos\left(\frac{5\pi}{4}\right)$ **b)** $\csc\left(\frac{11\pi}{6}\right)$

Solution A: Using the special angles

a)



Sketch the angle in standard position. π is a half of a revolution. $\frac{5\pi}{4}$ is halfway between π and $\frac{3\pi}{2}$, and lies in the third quadrant with a related angle of $\frac{5\pi}{4} - \pi$, or $\frac{\pi}{4}$.

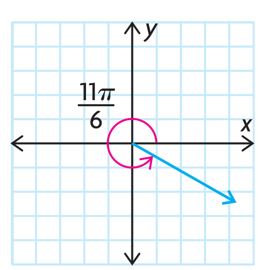


$$\cos\left(\frac{5\pi}{4}\right) = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

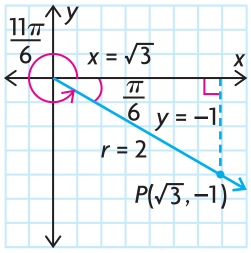
$\frac{\pi}{4}$ is in the 1, 1, $\sqrt{2}$ special triangle. Position this triangle so the right angle lies on the negative x-axis.

Since $(-1, -1)$ lies on the terminal arm, $x = -1$, $y = -1$, and $r = \sqrt{2}$. Therefore, the cosine ratio has a negative value.

b)



Sketch the angle in standard position. $\frac{11\pi}{6}$ is between $\frac{3\pi}{2}$ and 2π, and lies in the fourth quadrant with a related angle of $2\pi - \frac{11\pi}{6}$, or $\frac{\pi}{6}$.



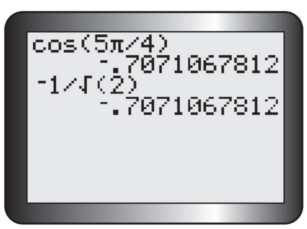
$\frac{\pi}{6}$ is in the 1, $\sqrt{3}$, 2 special triangle. Position it so that the right angle lies on the positive x-axis. Since the point $(\sqrt{3}, -1)$ lies on the terminal arm, $x = \sqrt{3}$, $y = -1$, and $r = 2$. Therefore, the csc ratio has a negative value.

$$\csc\left(\frac{11\pi}{6}\right) = \frac{r}{y}$$

$$= \frac{2}{-1} = -2$$

Solution B: Using a calculator

a)



Set the calculator to radian mode. Enter the expression.

The result is a decimal. Entering $-\frac{1}{\sqrt{2}}$ confirms that the answer is equivalent to this decimal.

$$\cos\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

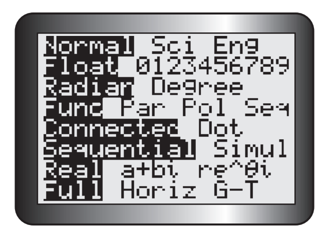
b)

There is no csc key on the calculator. Use the fact that cosecant is the reciprocal of sine.

$$\csc\left(\frac{11\pi}{6}\right) = -2$$

Tech | *Support*

To put a graphing calculator in radian mode, press the MODE key, scroll to Radian, and press ENTER.



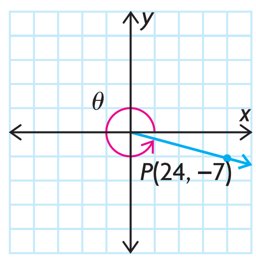
EXAMPLE 4 Solving a trigonometric equation that involves radians

If $\tan\theta = -\frac{7}{24}$, where $0 \leq \theta \leq 2\pi$, evaluate θ to the nearest hundredth.

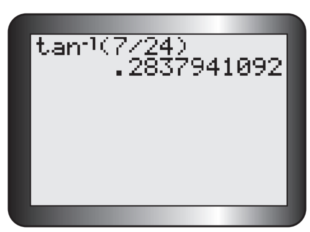
Solution

$\tan\theta = -\frac{7}{24} = \frac{y}{x}$

There are two possibilities to consider:
$x = 24, y = -7$ and
$x = -24, y = 7$.



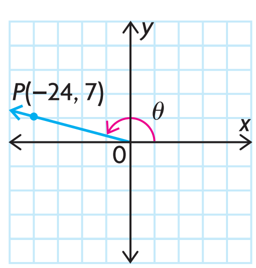
For the ordered pair $(24, -7)$, the terminal arm of the angle θ lies in the fourth quadrant.
$\frac{3\pi}{2} < \theta < 2\pi$



Use a calculator to determine the related acute angle by calculating the inverse tan of $\frac{7}{24}$. The related angle is 0.28, rounded to two decimal places. Subtract 0.28 from 2π to determine one measure of θ.

$2\pi - 0.28 \doteq 6.00$

In the fourth quadrant, θ is about 6.00.



For the ordered pair $(-24, 7)$, the terminal arm of θ lies in the second quadrant, $\frac{\pi}{2} < \theta < \pi$, and also has a related angle of 0.28. Subtract 0.28 from π to determine the other measure of θ.

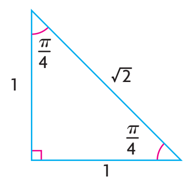
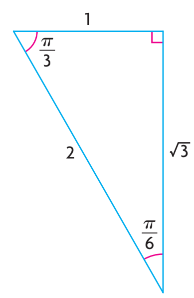
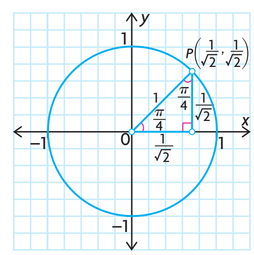
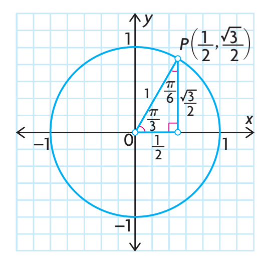
$\pi - 0.28 \doteq 2.86$

In the second quadrant, θ is about 2.86.

In Summary

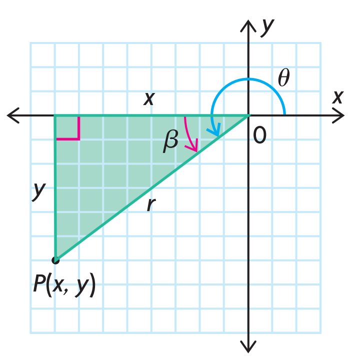
Key Ideas

- The angles in the special triangles can be expressed in radians, as well as in degrees. The radian measures can be used to determine the exact values of the trigonometric ratios for multiples of these angles between 0 and 2π.
- The strategies that are used to determine the values of the trigonometric ratios when an angle is expressed in degrees on the Cartesian plane can also be used when the angle is expressed in radians.

The Special Triangles	The Special Triangles on the Cartesian Plane Using a Circle of Radius 1
 	 

Need to Know

- The trigonometric ratios for any principal angle, θ, in standard position can be determined by finding the related acute angle, β, using coordinates of any point that lies on the terminal arm of the angle.

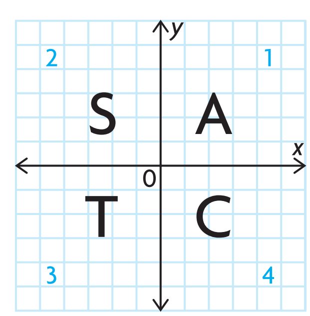


From the Pythagorean theorem, $r^2 = x^2 + y^2$, if $r > 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

- The CAST rule is an easy way to remember which primary trigonometric ratios are positive in which quadrant. Since r is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point.
  - In quadrant 1, **A**ll (A) ratios are positive because both x and y are positive.
  - In quadrant 2, only **S**ine (S) is positive, since x is negative and y is positive.
  - In quadrant 3, only **T**angent (T) is positive because both x and y are negative.
  - In quadrant 4, only **C**osine (C) is positive, since x is positive and y is negative.



CHECK Your Understanding

1. For each trigonometric ratio, use a sketch to determine in which quadrant the terminal arm of the principal angle lies, the value of the related acute angle, and the sign of the ratio.

a) $\sin \frac{3\pi}{4}$

b) $\cos \frac{5\pi}{3}$

c) $\tan \frac{4\pi}{3}$

d) $\sec \frac{5\pi}{6}$

e) $\cos \frac{2\pi}{3}$

f) $\cot \frac{7\pi}{4}$

2. Each of the following points lies on the terminal arm of an angle in standard position.

i) Sketch each angle.

ii) Determine the value of r.

iii) Determine the primary trigonometric ratios for the angle.

iv) Calculate the radian value of θ, to the nearest hundredth, where $0 \leq \theta \leq 2\pi$.

a) $(6, 8)$

b) $(-12, -5)$

c) $(4, -3)$

d) $(0, 5)$

3. Determine the primary trigonometric ratios for each angle.

a) $-\frac{\pi}{2}$

b) $-\pi$

c) $\frac{7\pi}{4}$

d) $-\frac{\pi}{6}$

4. State an equivalent expression in terms of the related acute angle.

a) $\sin \frac{5\pi}{6}$

b) $\cos \frac{5\pi}{3}$

c) $\cot \left(-\frac{\pi}{4}\right)$

d) $\sec \frac{7\pi}{6}$

PRACTISING

5. Determine the exact value of each trigonometric ratio.
K

a) $\sin \frac{2\pi}{3}$

b) $\cos \frac{5\pi}{4}$

c) $\tan \frac{11\pi}{6}$

d) $\sin \frac{7\pi}{4}$

e) $\csc \frac{5\pi}{6}$

f) $\sec \frac{5\pi}{3}$

6. For each of the following values of $\cos\theta$, determine the radian value of θ if $\pi \le \theta \le 2\pi$.

a) $-\frac{1}{2}$ c) $-\frac{\sqrt{2}}{2}$ e) 0

b) $\frac{\sqrt{3}}{2}$ d) $-\frac{\sqrt{3}}{2}$ f) -1

7. The terminal arm of an angle in standard position passes through each of the following points. Find the radian value of the angle in the interval $[0, 2\pi]$, to the nearest hundredth.

a) $(-7, 8)$ c) $(3, 11)$ e) $(9, 10)$

b) $(12, 2)$ d) $(-4, -2)$ f) $(6, -1)$

8. State an equivalent expression in terms of the related acute angle.

a) $\cos\frac{3\pi}{4}$ c) $\csc\left(-\frac{\pi}{3}\right)$ e) $\sin\frac{-\pi}{6}$

b) $\tan\frac{11\pi}{6}$ d) $\cot\frac{2\pi}{3}$ f) $\sec\frac{7\pi}{4}$

9. **A** A leaning flagpole, 5 m long, makes an obtuse angle with the ground. If the distance from the tip of the flagpole to the ground is 3.4 m, determine the radian measure of the obtuse angle, to the nearest hundredth.

10. The needle of a compass makes an angle of 4 radians with the line pointing east from the centre of the compass. The tip of the needle is 4.2 cm below the line pointing west from the centre of the compass. How long is the needle, to the nearest hundredth of a centimetre?

11. **T** A clock is showing the time as exactly 3:00 p.m. and 25 s. Because a full minute has not passed since 3:00, the hour hand is pointing directly at the 3 and the minute hand is pointing directly at the 12. If the tip of the second hand is directly below the tip of the hour hand, and if the length of the second hand is 9 cm, what is the length of the hour hand?

12. **C** If you are given an angle, θ, that lies in the interval $\theta \in \left[\frac{\pi}{2}, 2\pi\right]$, how would you determine the values of the primary trigonometric ratios for this angle?

13. You are given $\cos\theta = -\frac{5}{13}$, where $0 \le \theta \le 2\pi$.

a) In which quadrant(s) could the terminal arm of θ lie?

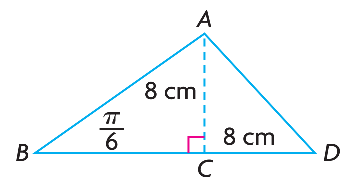
b) Determine all the possible trigonometric ratios for θ.

c) State all the possible radian values of θ, to the nearest hundredth.

14. Use special triangles to show that the equation $\cos\left(\frac{5\pi}{6}\right) = \cos(-150°)$ is true.

15. Show that $2\sin^2\theta - 1 = \sin^2\theta - \cos^2\theta$ for $\frac{11\pi}{6}$.

16. Determine the length of AB. Find the sine, cosine, and tangent ratios of $\angle D$, given $AC = CD = 8$ cm.



17. Given that x is an acute angle, draw a diagram of both angles (in standard position) in each of the following equalities. For each angle, indicate the related acute angle as well as the principal angle. Then, referring to your drawings, explain why each equality is true.

a) $\sin x = \sin(\pi - x)$

b) $\sin x = -\sin(2\pi - x)$

c) $\cos x = -\cos(\pi - x)$

d) $\tan x = \tan(\pi + x)$

Extending

18. Find the sine of the angle formed by two rays that start at the origin of the Cartesian plane if one ray passes through the point $(3\sqrt{3}, 3)$ and the other ray passes through the point $(-4, 4\sqrt{3})$. Round your answer to the nearest hundredth, if necessary.

19. Find the cosine of the angle formed by two rays that start at the origin of the Cartesian plane if one ray passes through the point $(6\sqrt{2}, 6\sqrt{2})$ and the other ray passes through the point $(-7\sqrt{3}, 7)$. Round your answer to the nearest hundredth, if necessary.

20. Julie noticed that the ranges of the sine and cosine functions go from -1 to 1, inclusive. She then began to wonder about the reciprocals of these functions—that is, the cosecant and secant functions. What do you think the ranges of these functions are? Why?

21. The terminal arm of θ is in the fourth quadrant. If $\cot\theta = -\sqrt{3}$, then calculate $\sin\theta\cot\theta - \cos^2\theta$.

6.3 Exploring Graphs of the Primary Trigonometric Functions

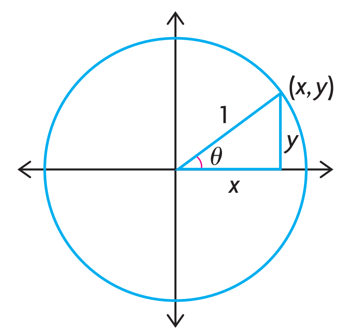
GOAL

Use radians to graph the primary trigonometric functions.

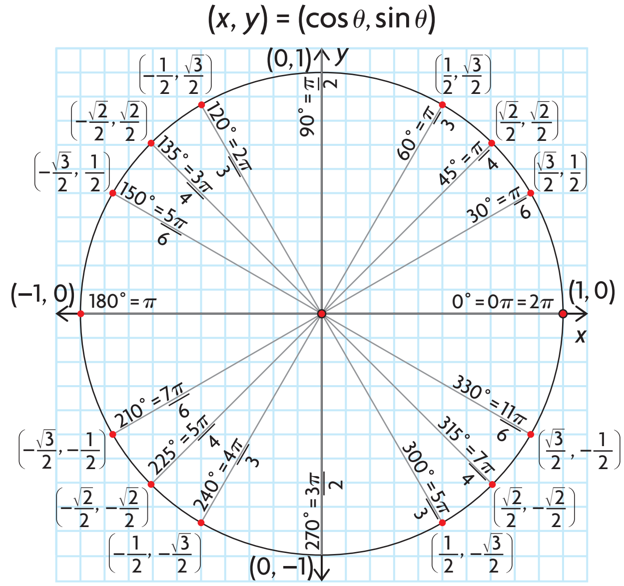
YOU WILL NEED

- graph paper
- graphing calculator

EXPLORE the Math



The unit circle is a circle that is centred at the origin and has a radius of 1 unit. On the unit circle, the sine and cosine functions take a particularly simple form: $\sin\theta = \frac{y}{1} = y$ and $\cos\theta = \frac{x}{1} = x$. The value of $\sin\theta$ is the y-coordinate of each point on the circle, and the value of $\cos\theta$ is the x-coordinate. As a result, each point on the circle can be represented by the ordered pair $(x, y) = (\cos\theta, \sin\theta)$, where θ is the angle formed between the positive x-axis and the terminal arm of the angle that passes through each point. For example, the point $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$ lies on the terminal arm of the angle $\frac{\pi}{6}$. Evaluating each trigonometric expression using the special triangles results in the ordered pair $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Repeating this process for other angles between 0 and 2π results in the following diagram:



? What do the graphs of the primary trigonometric functions look like when θ is expressed in radians?

A. Copy the following table. Complete the table using a calculator and the unit circle shown to approximate each value to two decimal places.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$									
$\cos\theta$									

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$								
$\cos\theta$								

B. Plot the ordered pairs $(\theta, \sin\theta)$, and sketch the graph of the function $y = \sin\theta$. On the same pair of axes, plot the ordered pairs $(\theta, \cos\theta)$ and sketch the graph of the function $y = \cos\theta$.

C. State the domain, range, amplitude, equation of the axis, and period of each function.

D. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Use the values from your table for part A to calculate the value of $\tan\theta$. Use a calculator to confirm your results, to two decimal places.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\frac{\sin\theta}{\cos\theta}$									

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\frac{\sin\theta}{\cos\theta}$								

E. What do you notice about the value of the tangent ratio when $\cos\theta = 0$? What do you notice about its value when $\sin\theta = 0$?

F. Based on your observations in part E, what characteristics does this imply for the graph of $y = \tan\theta$?

G. What do you notice about the value of the tangent ratio when $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}$, and $\pm\frac{7\pi}{4}$? Why does this occur?

H. On a new pair of axes, plot the ordered pairs $(\theta, \tan\theta)$ and sketch the graph of the function $y = \tan\theta$, where $0 \leq \theta \leq 2\pi$.

I. Determine the domain, range, amplitude, equation of the axis, and period of this function, if possible.

Reflecting

J. The tangent function is directly related to the slope of the line segment that joins the origin to each point on the unit circle. Explain why.

K. Where are the vertical asymptotes for the tangent graph located when $0 \leq \theta \leq 2\pi$, and what are their equations? Explain why they are found at these locations.

L. How does the period of the tangent function compare with the period of the sine and cosine functions?

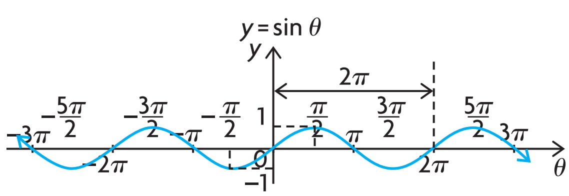
In Summary

Key Idea

- The graphs of the primary trigonometric functions can be summarized as follows:

Key points when $0 \leq \theta \leq 2\pi$

θ	$y = \sin(\theta)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

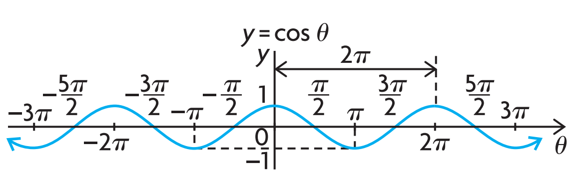


Period $= 2\pi$
Axis: $y = 0$
Amplitude $= 1$
Maximum value $= 1$
Minimum value $= -1$

$D = \{\theta \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

Key points when $0 \leq \theta \leq 2\pi$

θ	$y = \cos(\theta)$
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



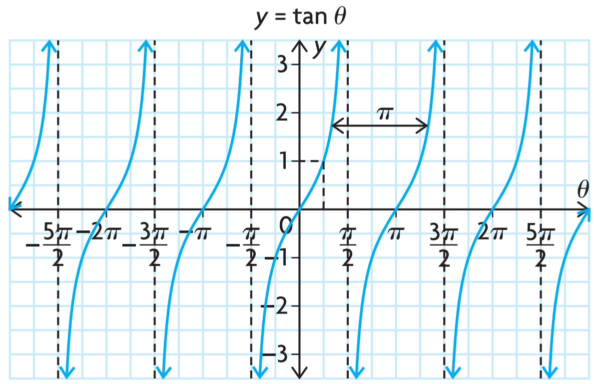
Period $= 2\pi$
Axis: $y = 0$
Amplitude $= 1$
Maximum value $= 1$
Minimum value $= -1$

$D = \{\theta \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

(continued)

Key points:

- y-intercept = 0
- θ-intercepts = $0, \pm\pi, \pm 2\pi, \ldots$



Period = π

Axis: $y = 0$

Amplitude: undefined

No maximum or minimum values

Vertical asymptotes:

$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

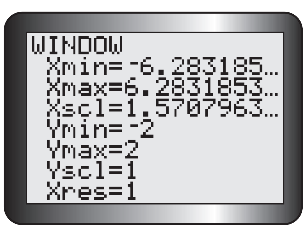
$D = \left\{\theta \in \mathbf{R} \,\middle|\, \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$

$R = \{y \in \mathbf{R}\}$

FURTHER Your Understanding

1. **a)** Examine the graphs of $y = \sin\theta$ and $y = \cos\theta$. Create a table to compare their similarities and differences.

b) Repeat part a) using the graphs of $y = \sin\theta$ and $y = \tan\theta$.



2. **a)** Use a graphing calculator, in radian mode, to create the graphs of the trigonometric functions $y = \sin\theta$ and $y = \cos\theta$ on the interval $-2\pi \leq \theta \leq 2\pi$. To do this, enter the functions $Y1 = \sin\theta$ and $Y2 = \cos\theta$ in the equation editor, and use the window settings shown.

b) Determine the values of θ where the functions intersect.

c) The equation $t_n = a + (n - 1)d$ can be used to represent the general term of any arithmetic sequence, where a is the first term and d is the common difference. Use this equation to find an expression that describes the location of each of the following values for $y = \sin\theta$, where $n \in \mathbf{I}$ and θ is in radians.

i) θ-intercepts

ii) maximum values

iii) minimum values

3. Find an expression that describes the location of each of the following values for $y = \cos\theta$, where $n \in \mathbf{I}$ and θ is in radians.

a) θ-intercepts **b)** maximum values **c)** minimum values

4. Graph $y = \frac{\sin\theta}{\cos\theta}$ using a graphing calculator in radian mode. Compare your graph with the graph of $y = \tan\theta$.

5. Find an expression that describes the location of each of the following values for $y = \tan\theta$, where $n \in \mathbf{I}$ and θ is in radians.

a) θ-intercepts **b)** vertical asymptotes

6.4 Transformations of Trigonometric Functions

GOAL

Use transformations to sketch the graphs of the primary trigonometric functions in radians.

YOU WILL NEED

- graph paper
- graphing calculator

LEARN ABOUT the Math

The following transformations are applied to the graph of $y = \sin x$, where $0 \le x \le 2\pi$:

- a vertical stretch by a factor of 3
- a horizontal compression by a factor of $\frac{1}{2}$
- a horizontal translation $\frac{\pi}{6}$ to the left
- a vertical translation 1 down

? What is the equation of the transformed function, and what does its graph look like?

EXAMPLE 1 Selecting a strategy to apply transformations and graph a sine function

Use the transformations above to sketch the graph of the transformed function in the interval $0 \le x \le 2\pi$.

Solution A: Applying the transformation to the key points of the parent function

$y = \sin x$ is the parent function.

x	$y = \sin(x)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

One cycle of the parent function can be described with five key points. By applying the relevant transformations to these points, a complete cycle of the transformed function can be graphed.

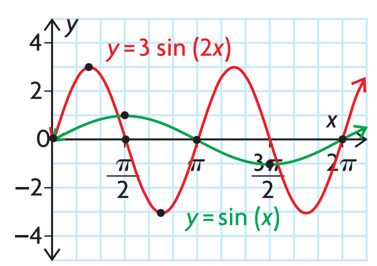
$y = 3 \sin\left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the equation of the transformed function.

Recall that, in the general function $y = af(k(x - d)) + c$, each parameter is associated with a specific transformation. In this case,
$a = 3$ (vertical stretch)
$k = \frac{1}{\frac{1}{2}} = 2$ (horizontal compression)
$d = -\frac{\pi}{6}$ (translation left)
$c = -1$ (translation down)

$$(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$$

Parent Function, $y = \sin x$	Stretched/Compressed Function, $y = 3 \sin (2x)$
$(0, 0)$	$\left(\frac{1}{2}(0), 3(0)\right) = (0, 0)$
$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{1}{2}\left(\frac{\pi}{2}\right), 3(1)\right) = \left(\frac{\pi}{4}, 3\right)$
$(\pi, 0)$	$\left(\frac{1}{2}(\pi), 3(0)\right) = \left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	$\left(\frac{1}{2}\left(\frac{3\pi}{2}\right), 3(-1)\right) = \left(\frac{3\pi}{4}, -3\right)$
$(2\pi, 0)$	$\left(\frac{1}{2}(2\pi), 3(0)\right) = (\pi, 0)$

The parameters k and d affect the x-coordinates of each point on the parent function, and the parameters a and c affect the y-coordinates. All stretches/compressions and reflections must be applied before any translations. In this example, each x-coordinate of the five key points is multiplied by $\frac{1}{2}$, and each y-coordinate is multiplied by 3.



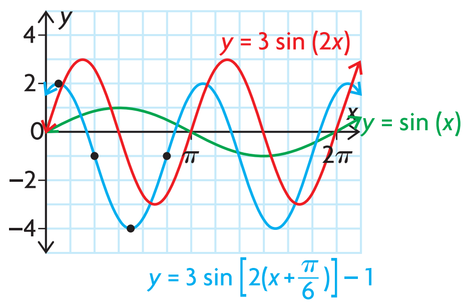
Plot the key points of the parent function and the key points of the transformed function, and draw smooth curves through them. Extend the red curve for one more cycle.

$$\left(\frac{1}{2}x, 3y\right) \rightarrow \left(\frac{1}{2}x - \frac{\pi}{6}, 3y - 1\right)$$

Stretched/Compressed Function, $y = 3 \sin (2x)$	Final Transformed Function, $y = 3 \sin \left(2\left(x + \frac{\pi}{6}\right)\right) - 1$
$(0, 0)$	$\left(0 - \frac{\pi}{6}, 0 - 1\right) = \left(-\frac{\pi}{6}, -1\right)$
$\left(\frac{\pi}{4}, 3\right)$	$\left(\frac{\pi}{4} - \frac{\pi}{6}, 3 - 1\right) = \left(\frac{\pi}{12}, 2\right)$
$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{\pi}{2} - \frac{\pi}{6}, 0 - 1\right) = \left(\frac{\pi}{3}, -1\right)$
$\left(\frac{3\pi}{4}, -3\right)$	$\left(\frac{3\pi}{4} - \frac{\pi}{6}, -3 - 1\right) = \left(\frac{7\pi}{12}, -4\right)$
$(\pi, 0)$	$\left(\pi - \frac{\pi}{6}, 0 - 1\right) = \left(\frac{5\pi}{6}, -1\right)$

Each x-coordinate of the key points on the previous function now has $\frac{\pi}{6}$ subtracted from it, and each y-coordinate has 1 subtracted from it.

These five points represent one complete cycle of the graph. To extend the graph to 2π, copy this cycle by adding the period of π to each x-coordinate in the table of the transformed key points.



Plot the key points of the final transformed function, and draw a smooth curve through them.

Note that the vertical stretch and translation cause corresponding changes in the range of the parent function. The range of the parent function is $-1 \leq y \leq 1$, and the range of the transformed function is $-4 \leq y \leq 2$.

Solution B: Using the features of the transformed function

$y = 3 \sin\left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the equation of the transformed function. It has the following characteristics:

Amplitude $= 3$

Period $= \frac{2\pi}{2} = \pi$

Equation of the axis: $y = -1$

Recall that each parameter in the general function $y = af(k(x - d)) + c$ is associated with a specific transformation. For the transformations applied to $f(x) = \sin x$,

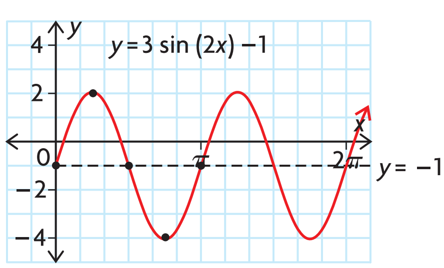
$a = 3$ (vertical stretch)

$k = \frac{1}{\frac{1}{2}} = 2$ (horizontal compression)

$d = -\frac{\pi}{6}$ (translation left)

$c = -1$ (translation down)

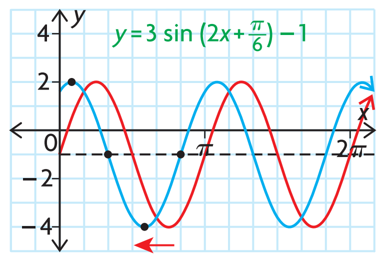
Sketch the graph of $y = 3 \sin(2x) - 1$ by plotting its axis, points on its axis, and maximum and minimum values.



Since the axis is $y = -1$ and the amplitude is 3, the graph has a maximum at 2 and a minimum at -4. Since this is a sine function with a period of π, the maximum occurs at $x = \frac{\pi}{4}$, and the minimum occurs at $x = \frac{3\pi}{4}$. The graph has points on the axis when $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$.

Since the given domain is $0 \leq \theta \leq 2\pi$, add the period π to each point that was plotted for the first cycle and draw a smooth curve.

$y = 3 \sin \left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the function $y = 3 \sin (2x) - 1$ translated $\frac{\pi}{6}$ to the left.



Apply the horizontal translation to the previous graph by shifting the maximum and minimum points and the points on the axis $\frac{\pi}{6}$ to the left.

Reflecting

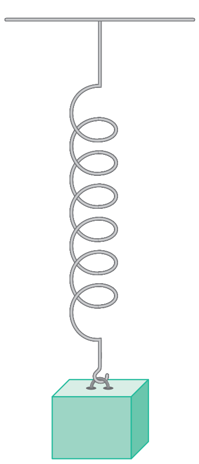
A. What transformations affect each of the following characteristics of a sinusoidal function?

i) period **ii)** amplitude **iii)** equation of the axis

B. In both solutions, it was necessary to extend the graphs after the final transformed points were plotted. Explain how this was done.

C. Which strategy for graphing sinusoidal functions do you prefer? Explain why.

APPLY the Math



argument

the expression on which a function operates; in Example 2, sin is the function and it operates on the expression $2\pi t + 1.5\pi$; so $2\pi t + 1.5\pi$ is the argument

EXAMPLE 2 Using the graph of a sinusoidal function to solve a problem

A mass on a spring is pulled toward the floor and released, causing it to move up and down. Its height, in centimetres, above the floor after t seconds is given by the function $h(t) = 10 \sin (2\pi t + 1.5\pi) + 15, 0 \le t \le 3$. Sketch a graph of height versus time. Then use your graph to predict when the mass will be 18 cm above the floor as it travels in an upward direction.

Solution

$h(t) = 10 \sin (2\pi t + 1.5\pi) + 15$

$h(t) = 10 \sin (2\pi (t + 0.75)) + 15$

For this function, the amplitude is 10 and the period is 1. The equation of the axis is $h = 15$. The function undergoes a horizontal translation 0.75 to the left.

Determine the characteristics that define the graph of this function. To do so, divide out the common factor from the argument. Then determine the values of the parameters a, k, d, and c.

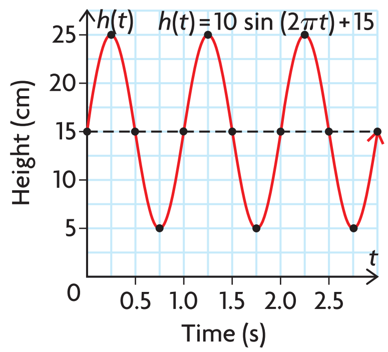
$a = 10$

$k = 2\pi$, so the period is $\frac{2\pi}{2\pi} = 1$

$d = -0.75$

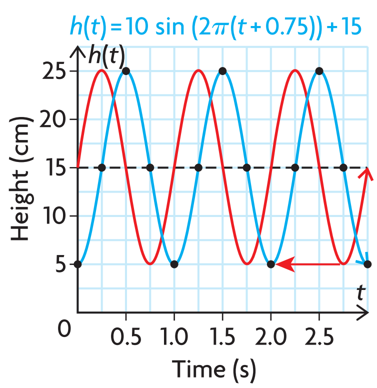
$c = 15$

Sketch the graph of $h(t) = 10 \sin (2\pi t) + 15$ over one cycle using the axis, amplitude, and period.



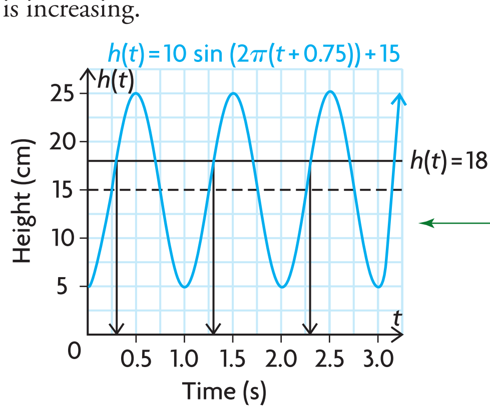
Since the axis is $h(t) = 15$ and the amplitude is 10, the graph will have a maximum at 25 and a minimum at 5. Since this is a sine function with a period of 1, these points will occur at $t = \frac{1}{4}$ and $t = \frac{3}{4}$. The graph has points on the axis when $t = 0$, $t = \frac{1}{2}$, and $t = 1$.

Since the given domain is $0 \le t \le 3$, add the period 1 to each point that was plotted for the first cycle. Repeat using the points on the second cycle to get three complete cycles. Then draw a smooth curve.



Apply the horizontal translation to the previous graph by shifting the maximum and minimum points and the points on the axis 0.75 to the left.

The spring is on its way up on the parts of the graph where the height is increasing.



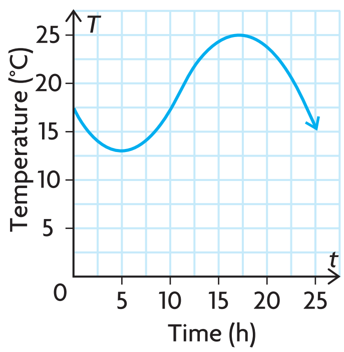
Use the graph to estimate when the spring will be 18 cm above the floor on the intervals $t \in [0, 0.5]$, $[1.0, 1.5]$, and $[2.0, 2.5]$.

On its way up, the spring is at a height of 18 cm at about 0.3 s, 1.3 s, and 2.3 s.

If you are given a graph of a sinusoidal function, then characteristics of its graph can be used to determine the equation of the function.

EXAMPLE 3 Connecting the features of the graph of a sinusoidal function to its equation

The following graph shows the temperature in Nellie's dorm room over a 24 h period.



Determine the equation of this sinusoidal function.

Solution

Use the graph to determine the values of the parameters a, k, d, and c, and write the equation.

The graph resembles the cosine function, so its equation is of the form $y = a\cos(k(x - d)) + c$.

The axis is $c = \frac{13 + 25}{2} = 19$.

The value of c indicates the horizontal axis of the function. The horizontal axis is the mean of the maximum and minimum values.

$$a = \frac{25 - 13}{2} = 6$$

The value of a indicates the amplitude of the function. The amplitude is half the difference between the maximum and minimum values.

Period $= \frac{2\pi}{k}$, so $24 = \frac{2\pi}{k}$

$$24k = 2\pi$$

$$k = \frac{\pi}{12}$$

The value of k is related to the period of the function.

If you assume that this cycle repeats itself over several days, then the period is 1 day, or 24 h.

$d = 17$

The equation is $T(t) = 6\cos\left(\frac{\pi}{12}(t - 17)\right) + 19$.

Let us use a cosine function. The parent function has a maximum value at $t = 0$.

This graph has a maximum value at $t = 17$. Therefore, we translate the function 17 units to the right.

In Summary

Key Idea

- The graphs of functions of the form $f(x) = a \sin(k(x - d)) + c$ and $f(x) = a \cos(k(x - d)) + c$ are transformations of the parent functions $y = \sin(x)$ and $y = \cos(x)$, respectively.

  To sketch these functions, you can use a variety of strategies. Two of these strategies are given below:
  1. Begin with the key points in one cycle of the parent function and apply any stretches/compressions and reflections to these points: $(x, y) \rightarrow \left(\frac{x}{k}, ay\right)$. Take each of the new points, and apply any translations: $\left(\frac{x}{k}, ay\right) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$.
     To graph more cycles, as required by the given domain, add multiples of the period to the x-coordinates of these transformed points and draw a smooth curve.
  2. Using the given equation, determine the equation of the axis, amplitude, and period of the function. Use this information to determine the location of the maximum and minimum points and the points that lie on the axis for one cycle. Plot these points, and then apply the horizontal translation to these points. To graph more cycles, as required by the domain, add multiples of the period to the x-coordinates of these points and draw a smooth curve.

Need to Know

- The parameters in the equations $f(x) = a \sin(k(x - d)) + c$ and $f(x) = a \cos(k(x - d)) + c$ give useful information about transformations and characteristics of the function.

Transformations of the Parent Function	Characteristics of the Transformed Function
$\lvert a\rvert$ gives the vertical stretch/compression factor. If $a < 0$, there is also a reflection in the x-axis.	$\lvert a\rvert$ gives the amplitude.
$\left\lvert\frac{1}{k}\right\rvert$ gives the horizontal stretch/compression factor. If $k < 0$, there is also a reflection in the y-axis.	$\frac{2\pi}{\lvert k\rvert}$ gives the period.
d gives the horizontal translation.	d gives the horizontal translation.
c gives the vertical translation.	$y = c$ gives the equation of the axis.

- If the independent variable has a coefficient other than $+1$, the argument must be factored to separate the values of k and c. For example,
  $y = 3\cos(2x + \pi)$ should be changed to $y = 3\cos\left(2\left(x + \frac{\pi}{2}\right)\right)$.

CHECK Your Understanding

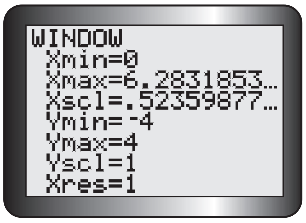
1. State the period, amplitude, horizontal translation, and equation of the axis for each of the following trigonometric functions.

a) $y = 0.5\cos(4x)$

b) $y = \sin\left(x - \frac{\pi}{4}\right) + 3$

c) $y = 2\sin(3x) - 1$

d) $y = 5\cos\left(-2x + \frac{\pi}{3}\right) - 2$



2. Suppose the trigonometric functions in question 1 are graphed using a graphing calculator in radian mode and the window settings shown. Which functions produce a graph that is not cut off on the top or bottom and that displays at least one cycle?

3. Identify the key characteristics of $y = -2\cos(4x + \pi) + 4$, and sketch its graph. Check your graph with a graphing calculator.

PRACTISING

4. The following trigonometric functions have the parent function $f(x) = \sin x$. They have undergone no horizontal translations and no reflections in either axis. Determine the equation of each function.

a) The graph of this trigonometric function has a period of π and an amplitude of 25. The equation of the axis is $y = -4$.

b) The graph of this trigonometric function has a period of 10 and an amplitude of $\frac{2}{5}$. The equation of the axis is $y = \frac{1}{15}$.

c) The graph of this trigonometric function has a period of 6π and an amplitude of 80. The equation of the axis is $y = -\frac{9}{10}$.

d) The graph of this trigonometric function has a period of $\frac{1}{2}$ and an amplitude of 11. The equation of the axis is $y = 0$.

5. **K** State the period, amplitude, and equation of the axis of the trigonometric function that produces each of the following tables of values. Then use this information to write the equation of the function.

a)

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	18	0	−18	0

b)

x	0	π	2π	3π	4π
y	−2	4	−2	−8	−2

c)

x	0	3π	6π	9π	12π
y	4	9	4	9	4

d)

x	0	2π	4π	6π	8π
y	−3	1	−3	1	−3

6. State the transformations that were applied to the parent function $f(x) = \sin x$ to obtain each of the following transformed functions. Then graph the transformed functions.

a) $f(x) = 4 \sin x + 3$

b) $f(x) = -\sin\left(\frac{1}{4}x\right)$

c) $f(x) = \sin(x - \pi) - 1$

d) $f(x) = \sin\left(4x + \frac{2\pi}{3}\right)$

7. The trigonometric function $f(x) = \cos x$ has undergone the following sets of transformations. For each set of transformations, determine the equation of the resulting function and sketch its graph.

a) vertical compression by a factor of $\frac{1}{2}$, vertical translation 3 units up

b) horizontal stretch by a factor of 2, reflection in the y-axis

c) vertical stretch by a factor of 3, horizontal translation $\frac{\pi}{2}$ to the right

d) horizontal compression by a factor of $\frac{1}{2}$, horizontal translation $\frac{\pi}{2}$ to the left

8. Sketch each graph for $0 \le x \le 2\pi$. Verify your sketch using graphing technology.

a) $y = 3 \sin\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$

b) $y = 5 \cos\left(x + \frac{\pi}{4}\right) - 2$

c) $y = -2 \sin\left(2\left(x + \frac{\pi}{4}\right)\right) + 2$

d) $y = -\cos\left(0.5x - \frac{\pi}{6}\right) + 3$

e) $y = 0.5 \sin\left(\frac{x}{4} - \frac{\pi}{16}\right) - 5$

f) $y = \frac{1}{2} \cos\left(\frac{x}{2} - \frac{\pi}{12}\right) - 3$

9. A Each person's blood pressure is different, but there is a range of blood pressure values that is considered healthy. The function

$P(t) = -20 \cos \frac{5\pi}{3}t + 100$ models the blood pressure, p, in

millimetres of mercury, at time t, in seconds, of a person at rest.

a) What is the period of the function? What does the period represent for an individual?

b) How many times does this person's heart beat each minute?

c) Sketch the graph of $y = P(t)$ for $0 \le t \le 6$.

d) What is the range of the function? Explain the meaning of the range in terms of a person's blood pressure.

10. A pendulum swings back and forth 10 times in 8 s. It swings through a total horizontal distance of 40 cm.

a) Sketch a graph of this motion for two cycles, beginning with the pendulum at the end of its swing.

b) Describe the transformations necessary to transform $y = \sin x$ into the function you graphed in part a).

c) Write the equation that models this situation.

11. A rung on a hamster wheel, with a radius of 25 cm, is travelling at a constant speed. It makes one complete revolution in 3 s. The axle of the hamster wheel is 27 cm above the ground.

T

a) Sketch a graph of the height of the rung above the ground during two complete revolutions, beginning when the rung is closest to the ground.

b) Describe the transformations necessary to transform $y = \cos x$ into the function you graphed in part a).

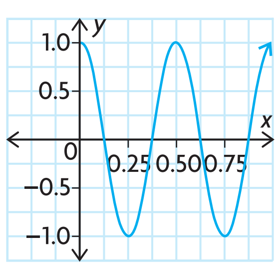
c) Write the equation that models this situation.

12. The graph of a sinusoidal function has been horizontally compressed and horizontally translated to the left. It has maximums at the points $\left(-\frac{5\pi}{7}, 1\right)$ and $\left(-\frac{3\pi}{7}, 1\right)$, and it has a minimum at $\left(-\frac{4\pi}{7}, -1\right)$. If the x-axis is in radians, what is the period of the function?

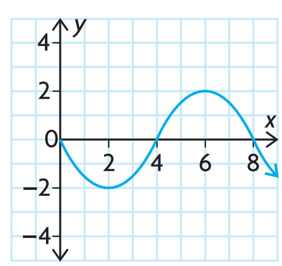
13. The graph of a sinusoidal function has been vertically stretched, vertically translated up, and horizontally translated to the right. The graph has a maximum at $\left(\frac{\pi}{13}, 13\right)$, and the equation of the axis is $y = 9$. If the x-axis is in radians, list one point where the graph has a minimum.

14. Determine a sinusodial equation for each of the following graphs.

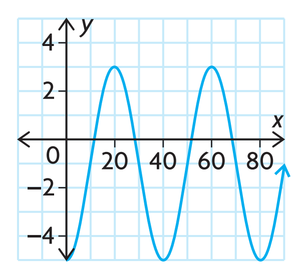
a)

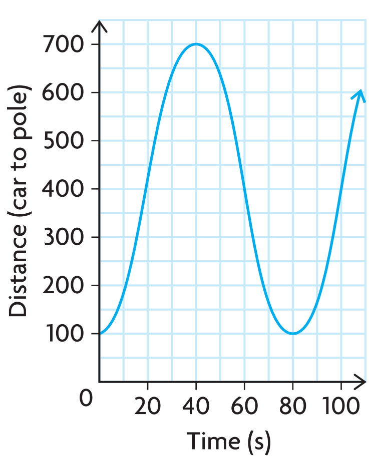


b)



c)





15. Create a flow chart that summarises how you would use transformations to sketch the graph of $f(x) = -2\sin\left(0.5\left(x - \frac{\pi}{4}\right)\right) + 3$.

C

Extending

16. The graph shows the distance from a light pole to a car racing around a circular track. The track is located north of the light pole.

a) Determine the distance from the light pole to the edge of the track.

b) Determine the distance from the light pole to the centre of the track.

c) Determine the radius of the track.

d) Detemine the time that the car takes to complete one lap of the track.

e) Determine the speed of the car in metres per second.

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How are radians and degrees related?

A: Radians are determined by the relationship $\theta = \frac{a}{r}$, where θ is the angle subtended by arc length a in a circle with radius r. One revolution creates an angle of 360°, or 2π radians. Since $360° = 2\pi$ radians, it follows that $180° = \pi$ radians. This relationship can be used to convert between the two measures.

- To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.
- To convert from radians to degrees, either substitute 180° for π or multiply by $\frac{180°}{\pi}$.

Here are three examples:

$$75° = 75° \times \frac{\pi}{180°} = \frac{5\pi}{12}$$

$$\frac{5\pi}{4} = \frac{5(180°)}{4} = 225°$$

$$3 \text{ radians} = \frac{3(180°)}{\pi} \doteq 171.887°$$

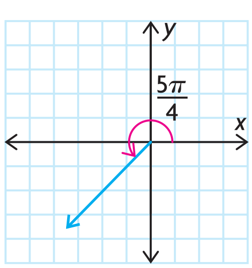
Study *Aid*

- See Lesson 6.1, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: How do you determine exact values of trigonometric ratios for multiples of special angles expressed in radians?

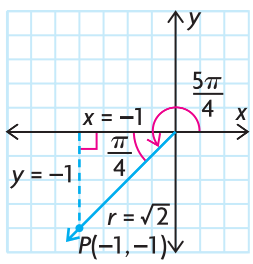
A: An angle on the Cartesian plane is determined by rotating the terminal arm in either a clockwise or counterclockwise direction. The special triangles can be used to determine the coordinates of a point that lies on the terminal arm of the angle. Then, using the x, y, r trigonometric definitions and the related angle, the exact values of the trigonometric ratios can be evaluated for multiples of angles $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.

For example, to determine the exact value of sec $\frac{5\pi}{4}$, sketch the angle in standard position. Determine the related angle. Since the terminal arm of $\frac{5\pi}{4}$ lies in the third quadrant, the related angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$.



Study *Aid*

- See Lesson 6.2, Example 3.
- Try Mid-Chapter Review Questions 4 and 6.



Sketch the 1, 1, $\sqrt{2}$ special triangle by drawing a vertical line from the point $(-1, -1)$ on the terminal arm to the negative x- axis. Use the values of x, y, and r and the appropriate ratio to determine the value.

$$\sec \frac{5\pi}{4} = \frac{r}{x}$$
$$= \frac{\sqrt{2}}{-1}$$
$$= -\sqrt{2}$$

Study Aid

- See Lesson 6.4, Example 3.
- Try Mid-Chapter Review Questions 8 and 9.

Q: How can transformations be used to graph sinusoidal functions?

A: The graphs of functions of the form $f(x) = a \sin (k(x - d)) + c$ and $f(x) = a \cos (k(x - d)) + c$ are transformations of the parent functions $y = \sin (x)$ and $y = \cos (x)$, respectively.

In sinusoidal functions, the parameters a, k, d, and c give the transformations to be applied, as well as the key characteristics of the graph.

- $|a|$ gives the vertical stretch/compression factor and the amplitude of the function.
- $\left|\frac{1}{k}\right|$ determines the horizontal stretch/compression factor, and $\left|\frac{2\pi}{k}\right|$ gives the period of the function.
- When a is negative, the function is reflected in the x-axis. When k is negative, the function is reflected in the y-axis.
- d gives the horizontal translation.
- c gives the vertical translation, and $y = c$ gives the equation of the horizontal axis of the function.

To sketch these functions, begin with the key points of the parent function. Apply any stretches/compressions and reflections first, and then follow them with any translations.

Alternatively, use the equation of the axis, amplitude, and period to sketch a graph of the form $f(x) = a \sin (x) + c$ or $f(x) = a \cos (x) + c$. Then apply the horizontal translation to the points of this graph, if necessary.

PRACTICE Questions

Lesson 6.1

1. Convert each angle from radians to degrees. Express your answer to one decimal place, if necessary.

a) $\frac{\pi}{8}$
b) 4π
c) 5
d) $\frac{11\pi}{12}$

2. Convert each angle from degrees to radians. Express your answer to one decimal place, if necessary.

a) $125°$
b) $450°$
c) $5°$
d) $330°$
e) $215°$
f) $-140°$

3. A tire with a diameter of 38 cm rotates 10 times in 5 s.

a) What is the angle that the tire rotates through, in radians, from 0 s to 5 s?
b) Determine the angular velocity of the tire.
c) Determine the distance travelled by a pebble that is trapped in the tread of the tire.

Lesson 6.2

4. Sketch each angle in standard position, and then determine the exact value of the trigonometric ratio.

a) $\sin \frac{3\pi}{4}$
b) $\sin \frac{11\pi}{6}$
c) $\tan \frac{5\pi}{3}$
d) $\tan \frac{5\pi}{6}$
e) $\cos \frac{3\pi}{2}$
f) $\cos \frac{4\pi}{3}$

5. The terminal arms of angles in standard position pass through the following points. Find the measure of each angle in radians, to the nearest hundredth.

a) $(-3, 14)$
b) $(6, 7)$
c) $(1, 9)$
d) $(-5, -18)$
e) $(2, 3)$
f) $(4, -20)$

6. State an equivalent expression for each of the following expressions, in terms of the related acute angle.

a) $\sin\left(-\frac{7\pi}{6}\right)$
b) $\cot \frac{7\pi}{4}$
c) $\sec\left(-\frac{\pi}{2}\right)$
d) $\cos\left(-\frac{5\pi}{6}\right)$

Lesson 6.3

7. State the x-intercepts and y-intercepts of the graph of each of the following functions.

a) $y = \sin x$
b) $y = \cos x$
c) $y = \tan x$

Lesson 6.4

8. Sketch the graph of each function on the interval $-2\pi \leq x \leq 2\pi$.

a) $y = \tan (x)$
b) $y = 2 \sin (-x) - 1$
c) $y = \frac{5}{2} \cos \left(2\left(x + \frac{\pi}{4}\right)\right) + 3$
d) $y = -\frac{1}{2} \cos \left(\frac{1}{2}x - \frac{\pi}{6}\right)$
e) $y = 2 \sin \left(-3\left(x - \frac{\pi}{2}\right)\right) + 4$
f) $y = 0.4 \sin (\pi - 2x) - 2.5$

9. The graph of the function $y = \sin x$ is transformed by vertically compressing it by a factor of $\frac{1}{3}$, reflecting it in the y-axis, horizontally compressing it by a factor of $\frac{1}{3}$, horizontally translating it $\frac{\pi}{8}$ units to the left, and vertically translating it 23 units down. Write the equation of the resulting graph.

6.5 Exploring Graphs of the Reciprocal Trigonometric Functions

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

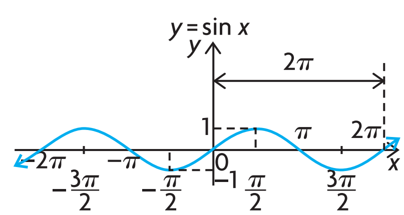
Graph the reciprocal trigonometric functions and determine their key characteristics.

EXPLORE the Math

Recall that the characteristics of the graph of a reciprocal function of a linear or quadratic function are directly related to the characteristics of the original function. Therefore, the key characteristics of the graph of a linear or quadratic function can be used to graph the related reciprocal function. The same strategies can be used to graph the reciprocal of a trigonometric function.

? What do the graphs of the reciprocal trigonometric functions $y = \csc x$, $y = \sec x$, and $y = \cot x$ look like, and what are their key characteristics?

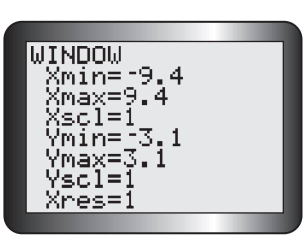
A. Here is the graph of $y = \sin x$.



Use this graph to predict where each of the following characteristics of the graph of $y = \frac{1}{\sin x}$ will occur.

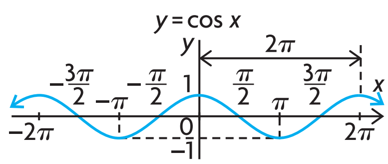
a) vertical asymptotes
b) maximum and minimum values
c) positive and negative intervals
d) intervals of increase and decrease
e) points of intersection for $y = \sin x$ and $y = \frac{1}{\sin x}$

B. Use your predictions in part A to sketch the graph of $y = \frac{1}{\sin x}$ (that is, $y = \csc x$). Verify your sketch by entering $y = \sin x$ into Y1 and $y = \frac{1}{\sin x}$ into Y2 of a graphing calculator, using the window settings shown. Compare the period and amplitude of each function.



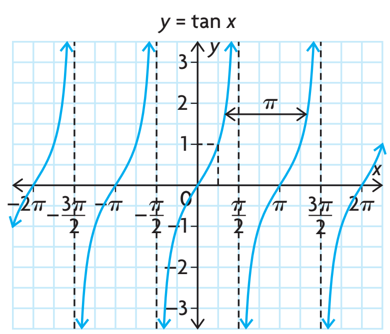
C. Predict what will happen if the period of $y = \sin x$ changes from 2π to π. Change Y1 to $y = \sin(2x)$ and Y2 to $y = \frac{1}{\sin(2x)}$ and discuss the results.

D. Here is the graph of $y = \cos x$.



Repeat parts A to C using the cosine function and its reciprocal $y = \frac{1}{\cos x}$ (that is, $y = \sec x$).

E. Here is the graph of $y = \tan x$. Recall that $\tan x = \frac{\sin x}{\cos x}$.



Repeat parts A to C using this form of the tangent function and its reciprocal $y = \frac{\cos x}{\sin x}$ (that is, $y = \cot x$).

Reflecting

F. Do the primary trigonometric functions and their reciprocal functions have the same kind of relationship that linear and quadratic functions and their reciprocal functions have? Explain.

G. Which x-values of the reciprocal function, in the interval $-2\pi \leq x \leq 2\pi$, result in vertical asymptotes? Why does this happen?

H. What is the relationship between the positive and negative intervals of the primary trigonometric functions and the positive and negative intervals of their reciprocal functions?

I. Where do the points of intersection occur for the primary trigonometric functions and their reciprocal functions?

In Summary

Key Idea

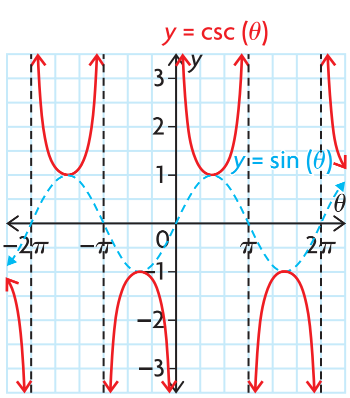
- Each of the primary trigonometric graphs has a corresponding reciprocal function.

Cosecant	**Secant**	**Cotangent**
$y = \csc \theta$	$y = \sec \theta$	$y = \cot \theta$
$y = \dfrac{1}{\sin \theta}$	$y = \dfrac{1}{\cos \theta}$	$y = \dfrac{1}{\tan \theta} = \dfrac{\cos \theta}{\sin \theta}$

Need to Know

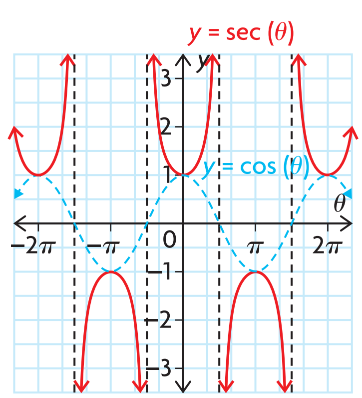
- The graph of a reciprocal trigonometric function is related to the graph of its corresponding primary trigonometric function in the following ways:
  - The graph of the reciprocal function has a vertical asymptote at each zero of the corresponding primary trigonometric function.
  - The reciprocal function has the same positive/negative intervals as the corresponding primary trigonometric function.
  - Intervals of increase on the primary trigonometric function are intervals of decrease on the corresponding reciprocal function. Intervals of decrease on the primary trigonometric function are intervals of increase on the corresponding reciprocal function.
  - The ranges of the primary trigonometric functions include 1 and -1, so a reciprocal function intersects its corresponding primary function at points where the y-coordinate is 1 or -1.
  - If the primary trigonometric function has a local minimum point, the corresponding reciprocal function has a local maximum point at the same θ value. If the primary trigonometric function has a local maximum point, the corresponding reciprocal function has a local minimum point at the same θ value.

Cosecant



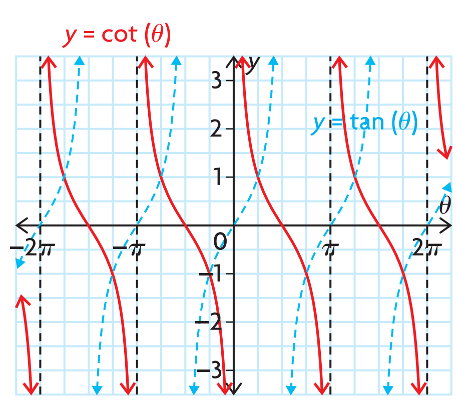
- has vertical asymptotes at the points where $\sin \theta = 0$
- has the same period (2π) as $y = \sin \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Secant



- has vertical asymptotes at the points where $\cos \theta = 0$
- has the same period (2π) as $y = \cos \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq (2n - 1)\frac{\pi}{2}, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Cotangent



- has vertical asymptotes at the points where $\tan \theta = 0$
- has zeros at the points where $y = \tan \theta$ has asymptotes
- has the same period (π) as $y = \tan \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R}\}$

FURTHER Your Understanding

1. The equation $t_n = a + (n - 1)d$ can be used to represent the general term of any arithmetic sequence, where a is the first term and d is the common difference. Use this equation to find an expression that describes the location of each of the following values for $y = \csc x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) maximum values
c) minimum values

2. Find an expression that describes the location of each of the following values for $y = \sec x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) maximum values
c) minimum values

3. Find an expression that describes the location of each of the following values for $y = \cot x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) x-intercepts

4. Use graphing technology to graph $y = \csc x$ and $y = \sec x$. For which values of the independent variable do the graphs intersect? Compare these values with the intersections of $y = \sin x$ and $y = \cos x$. Explain.

5. The graphs of the functions $y = \sin x$ and $y = \cos x$ are congruent, related by a translation of $\frac{\pi}{2}$ where $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. Does this relationship hold for $y = \csc x$ and $y = \sec x$? Verify your conjecture using graphing technology.

6. Two successive transformations can be applied to the graph of $y = \tan x$ to obtain the graph of $y = \cot x$. There is more than one way to apply these transformations, however. Describe one of these compound transformations.

7. Use transformations to sketch the graph of each function. Then state the period of the function.

a) $y = \cot\left(\frac{x}{2}\right)$
b) $y = \csc\left(2\left(x + \frac{\pi}{2}\right)\right)$
c) $y = \sec x - 1$
d) $y = \csc(0.5x + \pi)$

6.6 Modelling with Trigonometric Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Model and solve problems that involve trigonometric functions and radian measurement.

LEARN ABOUT the Math

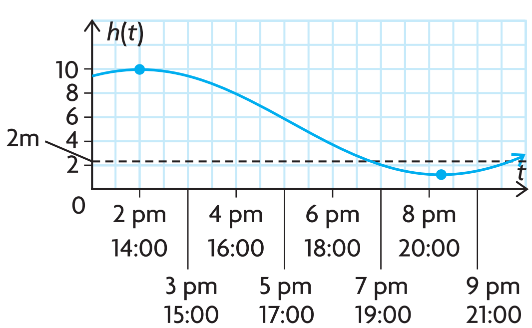
The tides at Cape Capstan, New Brunswick, change the depth of the water in the harbour. On one day in October, the tides have a high point of approximately 10 m at 2 p.m. and a low point of approximately 1.2 m at 8:15 p.m. A particular sailboat has a *draft* of 2 m. This means it can only move in water that is at least 2 m deep. The captain of the sailboat plans to exit the harbour at 6:30 p.m.

? Can the captain exit the harbour safely in the sailboat at 6 p.m.?

EXAMPLE 1 Modelling the problem using a sinusoidal equation

Create a sinusoidal function to model the problem, and use it to determine whether the sailboat can exit the harbour safely at 6 p.m.

Solution



A sinusoidal function can be used to model the height of the water versus time. Draw a sketch to get an idea of when the captain needs to leave. It appears that the captain will have enough depth at 6:30 p.m., but you cannot be sure from a rough sketch.

$$H(t) = a\cos(k(t-d)) + c$$

$$a = \frac{10-1.2}{2}$$

$$a = 4.4$$

Choose the cosine function to model the problem, since the graph starts at a maximum value. The amplitude, period, horizontal translation, and equation of the axis need to be determined.

Use the maximum and minimum measurements of the tides to calculate the amplitude of the function. This gives the value of a in the equation.

$$\text{Period} = \frac{2\pi}{k}$$
$$12.5 = \frac{2\pi}{k}$$
$$12.5k = 2\pi$$
$$k = \frac{2\pi}{12.5} = \frac{4\pi}{25}$$

In a sinusoidal function, the horizontal distance between the maximum and minimum points represents half of one cycle.

Since a maximum tide and a minimum tide occur 6 h 15 min apart, the period must be 12.5 h. The period can be used to determine the value of *k* in the equation.

$$c = \frac{10 + 1.2}{2}$$
$$c = 5.6$$

The equation of the axis is the mean of the maximum and minimum points. This can be used to determine the value of *c* in the equation.

A function that models the tides at Cape Capstan is

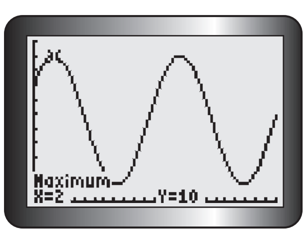
$$H(t) = 4.4\cos\left(\frac{4\pi}{25}(t - 2)\right) + 5.6.$$

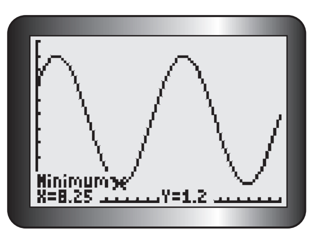
The parent cosine function starts at a maximum point.
If we let $t = 0$ represent noon, then our function needs a maximum at $t = 2$ (or 2 p.m.). We use a horizontal translation right 2 units. Therefore $d = 2$.

$$H(18) = 4.4\cos\left(\frac{4\pi}{25}(6.5 - 2)\right) + 5.6$$
$$= 4.4\cos\left(\frac{18\pi}{25}\right) + 5.6$$
$$\doteq 2.80 \text{ m}$$

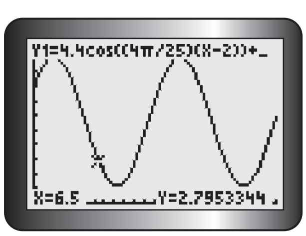
To determine the water level at 6:30 p.m., let $t = 6.5$.

Since the depth of the water is greater than 2 m at 6:30 p.m., the sailboat can safely exit the harbour.





To verify the solution, enter the function in the equation editor as Y1.
Use the value operation to confirm a high tide at 2 p.m. $(t = 2)$ and a low tide at 8:15 p.m. $(t = 8.25)$.



Use the value operation to determine the depth of the water at 6:30 p.m. $(t = 6.5)$.

Reflecting

A. What characteristics of your model would change if you used a sine function to model the problem?

B. What role did the maximum value play in determining the required horizontal translation?

C. If $t = 0$ was set at 2 p.m. instead of noon, how would the equation change? Would this make a difference to your final answer?

APPLY the Math

EXAMPLE 2 Representing a situation described by data using a sinusoidal equation

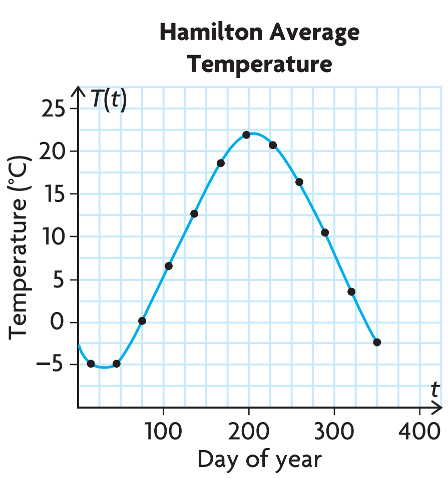
The following table shows the average monthly means of the daily (24 h) temperatures in Hamilton, Ontario. Each month's average temperature is represented by the day in the middle of the month.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Day of Year	15	45	75	106	136	167	197	228	259	289	320	350
°C	-4.8	-4.8	-0.2	6.6	12.7	18.6	21.9	20.7	16.4	10.5	3.6	-2.3

a) Plot the temperature data for Hamilton, and fit a sinusoidal curve to the points.
b) Estimate the average daily temperature in Hamilton on the 200th day of the year.

Solution A: Using the data and reasoning about the characteristics of the graph

a)



Plot the data, and sketch a smooth curve through the points.

The curve appears to be sinusoidal, so use $y = a \sin (k(t - d)) + c$ as the model for this situation.

$$a = \frac{\text{maximum} - \text{minimum}}{2}$$

$$a = \frac{21.9 - (-4.8)}{2}$$

$$a = 13.35$$

Estimate the maximum and minimum temperatures for the year from the graph. Use these temperatures to calculate the values of *a* and *c*. The value of *a* gives the amplitude. The sine function has been stretched vertically by a factor of 13.35.

$$c = \frac{\text{maximum} + \text{minimum}}{2}$$

$$c = \frac{21.9 + (-4.8)}{2}$$

$$c = 8.55$$

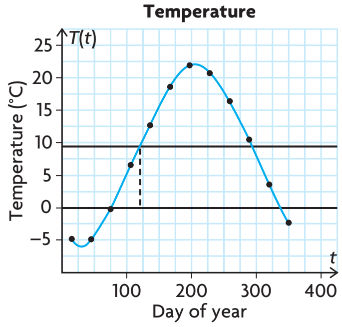
The value of *c* gives the horizontal axis. The sine function has been vertically translated by 8.55 units on the Temperature axis. Lightly draw a horizontal line through your graph at this value.

$$\text{Period} = \frac{2\pi}{k}, \text{ so } k = \frac{2\pi}{\text{period}}$$

$$k = \frac{2\pi}{365}$$

The value of *k* in the equation is determined by the period. Assume that the cycle repeats itself every year (365 days).

Hamilton Average Temperature

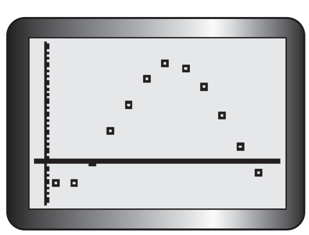


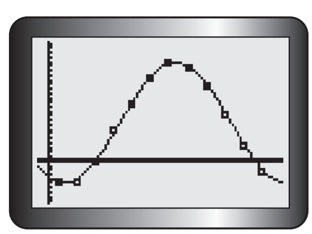
To determine the value of *d*, estimate where the horizontal axis first intersects the curve.

Since this graph appears to have been translated to the right, $d \doteq 116$.

$$T(t) = 13.35 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 8.55$$

Replace the parameters in the general sine equation.





Verify the result by entering the data into L1 and L2 in a graphing calculator and creating a scatter plot. Enter the sine function into Y1 and observe that it matches the data.

b) $T(t) = 13.35 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 8.55$

$$T(200) = 13.35 \sin\left(\frac{2\pi}{365}(200 - 116)\right) + 8.55$$

$$\doteq 21.8\,°\text{C}$$

Let $t = 200$, and evaluate the sine function.

This model predicts that the average daily temperature in Hamilton on the 200th day of the year is about 21.8 °C.

Since sinusoidal functions are periodic, they can be used (where appropriate) to make educated predictions.

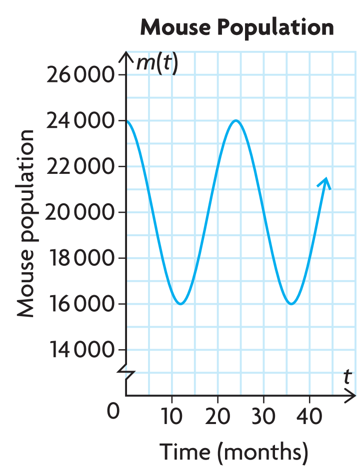
EXAMPLE 3 Analyzing a situation that involves sinusoidal models

The population size, O, of owls (predators) in a certain region can be modelled by the function $O(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$, where t represents the time in months and $t = 0$ represents January. The population size, m, of mice (prey) in the same region is given by the function $m(t) = 20\,000 + 4000 \cos\left(\frac{\pi t}{12}\right)$.

a) Sketch the graphs of these functions.
b) Compare the graphs, and discuss the relationships between the two populations.
c) How does the mice-to-owls ratio change over time?
d) When is there the most food per owl? When is it safest for the mice?

Solution

a) Graph the prey function.



$m(t) = 4000 \cos\left(\frac{\pi t}{12}\right) + 20\,000.$

The mouse population has a maximum of 24 000 and a minimum of 16 000.

$a = 4000$

The amplitude of the curve is 4000.

$c = 20\,000$

The axis is the line $m(t) = 20\,000$.

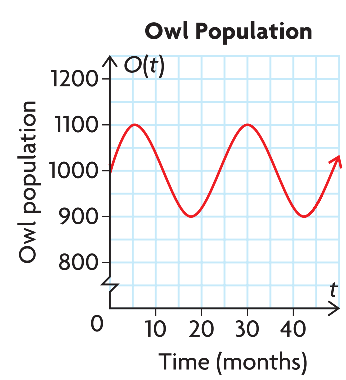
$k = \frac{\pi}{12}$, so the period $= \frac{2\pi}{k}$

$$\text{period} = \frac{2\pi}{\frac{\pi}{12}}$$

$$\text{period} = 2\pi \times \frac{12}{\pi} = 24$$

The period is 24 months.

Graph the predator function.



$O(t) = 100 \sin\left(\frac{\pi t}{12}\right) + 1000.$

The owl population has a maximum of 1100 and a minimum of 900.

$a = 100$

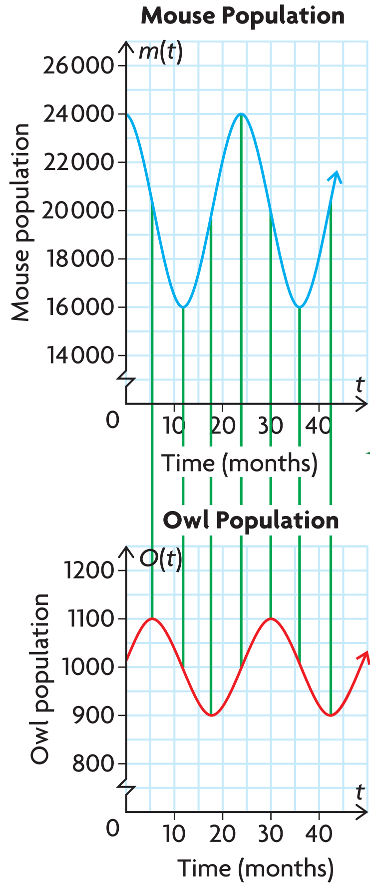
The amplitude of the curve is 100.

$c = 1000$

The axis is the line $O(t) = 1000$.

$k = \frac{\pi}{12}$ as above, so this period is also 24 months.

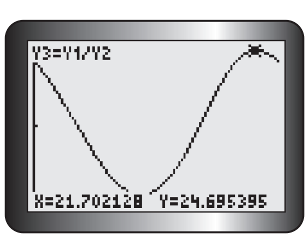
b)



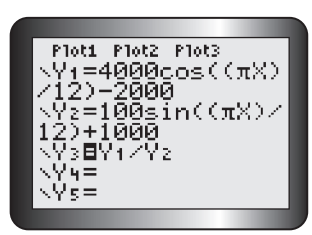
The graphs can be compared, since the same scale was used on both horizontal axes. As the owl population begins to increase, the mouse population begins to decrease. The mouse population continues to decrease, and this has an impact on the owl population, since its food supply dwindles. The owl population peaks and then also starts to decrease. The mouse population reaches a minimum and begins to rise as there are fewer owls to eat the mice. As the mouse population increases, food becomes more plentiful for the owls. So their population begins to rise again. Since both graphs have the same period, this pattern repeats every 24 months.

c) The following table shows the ratio of mice to owls at key points in the first four years.

Time	Mice	Owls	Mice-to-Owl Ratio
0	24 000	1000	24
6	20 000	1100	18.2
12	16 000	1000	16
18	20 000	900	22.2
24	24 000	1000	24



There seems to be a pattern. Enter the mouse function into Y1 of the equation editor of a graphing calculator, and enter the owl function into Y2. Turn off each function, and enter Y3 = Y1/Y2.



The resulting graph is shown. The ratio of mice to owls is also sinusoidal.

d) The most food per owl occurs when the ratio of mice to owls is the highest (there are more mice per owl).

The safest time for the mice occurs at the same time, when the ratio of mice to owls is the highest (there are fewer owls per mouse).

This occurs near the end of the 21st month of the two-year cycle.

In Summary

Key Ideas

- The graphs of $y = \sin x$ and $y = \cos x$ can model periodic phenomena when they are transformed to fit a given situation. The transformed functions are of the form $y = a \sin (k(x - d)) + c$ and $y = a \cos (k(x - d)) + c$, where
  - $|a|$ is the amplitude and $a = \frac{\text{max} - \text{min}}{2}$
  - $|k|$ is the number of cycles in 2π radians, when the period $= \frac{2\pi}{k}$
  - d gives the horizontal translation
  - c is the vertical translation and $y = c$ is the horizontal axis

Need to Know

- Tables of values, graphs, and equations of sinusoidal functions can be used as mathematical models when solving problems. Determining the equation of the appropriate sine or cosine function from the data or graph provided is the most efficient strategy, however, since accurate calculations can be made using the equation.

CHECK Your Understanding

1. A cosine curve has an amplitude of 3 units and a period of 3π radians. The equation of the axis is $y = 2$, and a horizontal shift of $\frac{\pi}{4}$ radians to the left has been applied. Write the equation of this function.

2. Determine the value of the function in question 1 if $x = \frac{\pi}{2}, \frac{3\pi}{4}$, and $\frac{11\pi}{6}$.

3. Sketch a graph of the function in question 1. Use your graph to estimate the x-value(s) in the domain $0 < x < 2$, where $y = 2.5$, to one decimal place.

PRACTISING

4. The height of a patch on a bicycle tire above the ground, as a function of time, is modelled by one sinusoidal function. The height of the patch above the ground, as a function of the total distance it has travelled, is modelled by another sinusoidal function. Which of the following characteristics do the two sinusoidal functions share: amplitude, period, equation of the axis?

5. Mike is waving a sparkler in a circular motion at a constant speed. **K** The tip of the sparkler is moving in a plane that is perpendicular to the ground. The height of the tip of the sparkler above the ground, as a function of time, can be modelled by a sinusoidal function. At $t = 0$, the sparkler is at its highest point above the ground.

a) What does the amplitude of the sinusoidal function represent in this situation?

b) What does the period of the sinusoidal function represent in this situation?

c) What does the equation of the axis of the sinusoidal function represent in this situation?

d) If no horizontal translations are required to model this situation, should a sine or cosine function be used?

6. To test the resistance of a new product to temperature changes, the **A** product is placed in a controlled environment. The temperature in this environment, as a function of time, can be described by a sine function. The maximum temperature is 120 °C, the minimum temperature is -60 °C, and the temperature at $t = 0$ is 30 °C. It takes 12 h for the temperature to change from the maximum to the minimum. If the temperature is initially increasing, what is the equation of the sine function that describes the temperature in this environment?

7. A person who was listening to a siren reported that the frequency of the sound fluctuated with time, measured in seconds. The minimum frequency that the person heard was 500 Hz, and the maximum frequency was 1000 Hz. The maximum frequency occurred at $t = 0$ and $t = 15$. The person also reported that, in 15, she heard the maximum frequency 6 times (including the times at $t = 0$ and $t = 15$). What is the equation of the cosine function that describes the frequency of this siren?

8. A contestant on a game show spins a wheel that is located on a plane perpendicular to the floor. He grabs the only red peg on the circumference of the wheel, which is 1.5 m above the floor, and pushes it downward. The red peg reaches a minimum height of 0.25 m above the floor and a maximum height of 2.75 m above the floor. Sketch two cycles of the graph that represents the height of the red peg above the floor, as a function of the total distance it moved. Then determine the equation of the sine function that describes the graph.

9. At one time, Maple Leaf Village (which no longer exists) had North America's largest Ferris wheel. The Ferris wheel had a diameter of 56 m, and one revolution took 2.5 min to complete. Riders could see Niagara Falls if they were higher than 50 m above the ground. Sketch three cycles of a graph that represents the height of a rider above the ground, as a function of time, if the rider gets on at a height of 0.5 m at $t = 0$ min. Then determine the time intervals when the rider could see Niagara Falls.

10. The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight.

a) Find an equation for $n(t)$, the number of hours of daylight on the nth day of the year.

b) Use your equation to predict the number of hours of daylight in Vancouver on January 30th.

11. **T** The city of Thunder Bay, Ontario, has average monthly temperatures that vary between $-14.8\,°C$ and $17.6\,°C$. The following table gives the average monthly temperatures, averaged over many years. Determine the equation of the sine function that describes the data, and use your equation to determine the times that the temperature is below $0\,°C$.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Average Temperature (°C)	−14.8	−12.7	−5.9	2.5	8.7	13.9	17.6	16.5	11.2	5.6	−2.7	−11.1

12. **C** A nail is stuck in the tire of a car. If a student wanted to graph a sine function to model the height of the nail above the ground during a trip from Kingston, Ontario, to Hamilton, Ontario, should the student graph the distance of the nail above the ground as a function of time or as a function of the total distance travelled by the nail? Explain your reasoning.

Extending

13. A clock is hanging on a wall, with the centre of the clock 3 m above the floor. Both the minute hand and the second hand are 15 cm long. The hour hand is 8 cm long. For each hand, determine the equation of the cosine function that describes the distance of the tip of the hand above the floor as a function of time. Assume that the time, t, is in minutes and that the distance, $D(t)$, is in centimetres. Also assume that $t = 0$ is midnight.

6.7 Rates of Change in Trigonometric Functions

GOAL

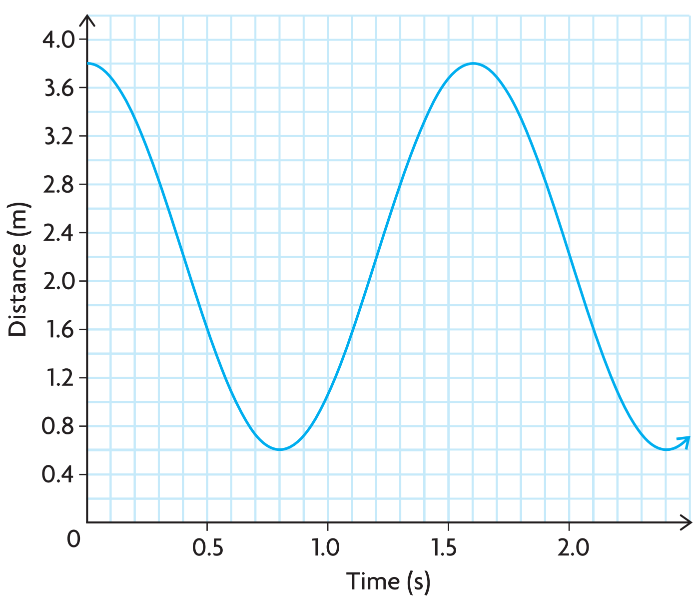
Examine average and instantaneous rates of change in trigonometric functions.

LEARN ABOUT the Math

Melissa used a motion detector to measure the horizontal distance between her and a child on a swing. She stood in front of the child and recorded the distance, $d(t)$, in metres over a period of time, t, in seconds. The data she collected are given in the following tables and are shown on the graph below.

Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Distance (m)	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6	0.72	1.07	1.59

Time (s)	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
Distance (m)	2.2	2.81	3.33	3.68	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6



? How did the speed of the child change as the child swung back and forth?

EXAMPLE 1 Using the data and the graph to analyze the situation

Use the data and the graph to discuss how the speed of the child changed as the child swung back and forth.

Solution

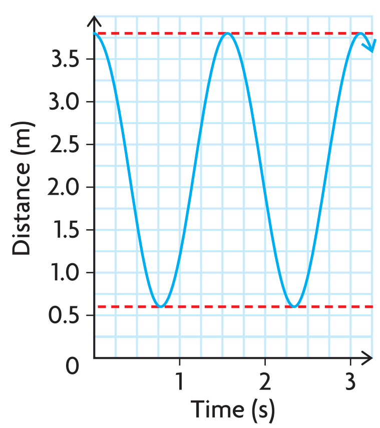
Analyze the motion.

Melissa began recording the motion when the child was the farthest distance from the motion detector, which was 3.8 m. The child's closest distance to the motion detector was 0.6 m and occurred at 0.8 s. The child was moving toward the motion detector between 0 s and 0.8 s and away from the motion detector between 0.8 s and 1.6 s.

Looking at the graph, the maximum value was 3.8 and occurred at 0 s and 1.6 s. It took 1.6 s for the child to swing one complete cycle.

Looking at the data and the graph, the distances between the child and the motion detector were decreasing between 0 s and 0.8 s, and increasing between 0.8 s and 1.6 s. This pattern repeated itself every multiple of 1.6 s.

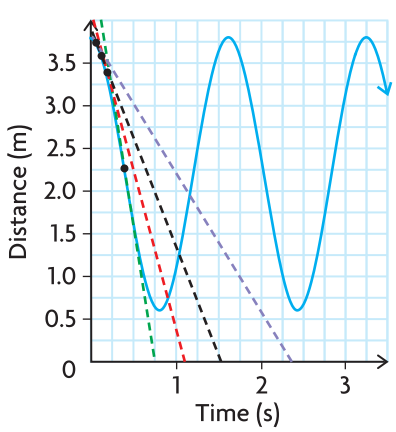
Analyze the instantaneous velocity by drawing tangent lines at various points over one swing cycle.



The slope of a tangent line on any distance versus time graph gives the instantaneous velocity, which is the instantaneous rate of change in distance with respect to time.

When the child was at the farthest point and closest point from the motion detector, the instantaneous velocity was 0.

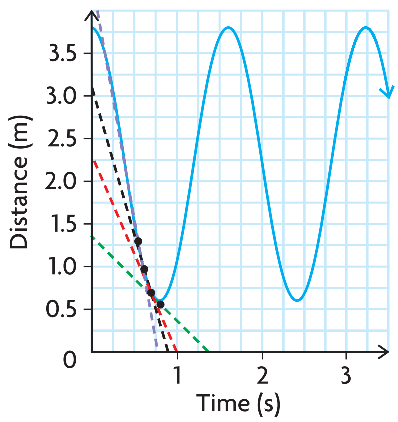
Between 0 s and about 0.4 s, the child's speed was increasing.



On this interval, the tangent lines become steeper as time increases.

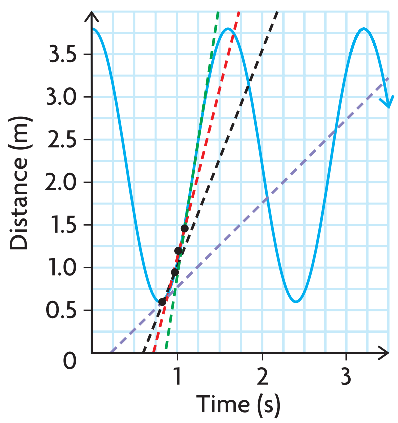
Speed $= |\text{velocity}| = \left|\dfrac{\Delta\text{distance}}{\Delta\text{time}}\right|$. This means the magnitudes of the slopes are increasing. The tangent lines have negative slopes, which means the distance between the child and the motion detector continues to decrease.

Between 0.4 s and about 0.8 s, the child's speed was decreasing.



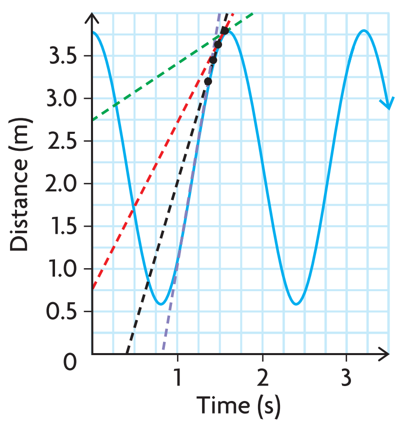
On this interval, the tangent lines are getting less steep as time increases. This means the magnitudes of the slopes are decreasing. The tangent lines still have negative slopes, which means the distance between the child and the motion detector is still decreasing. The child is slowing down as the swing approaches the point where a change in direction occurs. The slopes indicate a change in the child's position from toward the detector to away from the detector.

Between 0.8 s and about 1.2 s, the child's speed was increasing.



On this interval, the tangent lines are getting steeper as time increases. This means the magnitudes of the slopes are increasing. The tangent lines have positive slopes, which means the distance between the child and the motion detector is increasing. Therefore, the motion is away from the detector.

Between 1.2 s and about 1.6 s, the child's speed was decreasing.



On this interval, the tangent lines are getting less steep as time increases. This means the magnitudes of the slopes are decreasing. The tangent lines still have positive slopes, which means the distance between the child and the motion detector is still increasing. The child is slowing down as the swing approaches the point where there is a change in direction from away from the detector to toward the detector.

Reflecting

A. Explain how the data in the table indicates the direction in which the child swung.

B. Explain how the sign of the slope of each tangent line indicates the direction in which the child swung.

C. How can you tell, from the graph, when the speed of the child was 0 m/s?

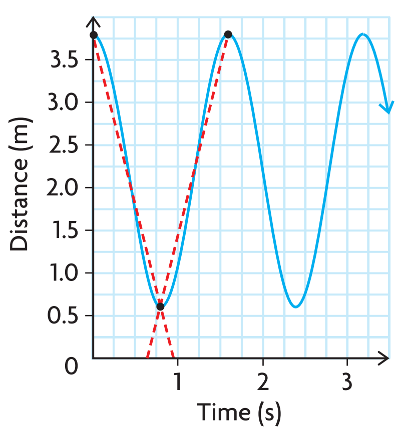
D. If someone began to push the child after 2.4 s, describe what effect this would have on the distance versus time graph.

APPLY the Math

EXAMPLE 2 Using the slopes of secant lines to calculate average rate of change

Calculate the child's average speed over the intervals of time as the child swung toward and away from the motion detector on the first swing.

Solution



The absolute value of the slope of a secant line on any distance versus time graph gives the average rate of change in distance, with respect to time or average speed.

The secant line that is decreasing has a negative slope, indicating that the distance between the child and the motion detector was decreasing between 0 s and 0.8 s.

The secant line that is increasing has a positive slope, indicating that the distance between the child and the motion detector was increasing between 0.8 s and 1.6 s.

Interval	$\frac{\Delta \text{ Distance}}{\Delta \text{ Time}}$	Average Speed (m/s)
$0 \le t \le 0.8$	$\left\lvert \frac{0.6 - 3.8}{0.8 - 0} \right\rvert$	$\lvert -4 \rvert = 4$
$0.8 \le t \le 1.6$	$\left\lvert \frac{3.8 - 0.6}{1.6 - 0.8} \right\rvert$	$\lvert 4 \rvert = 4$

Use the data in the table and the relationship $\left\lvert \frac{\Delta \text{ distance}}{\Delta \text{ time}} \right\rvert$ to calculate the average speed.

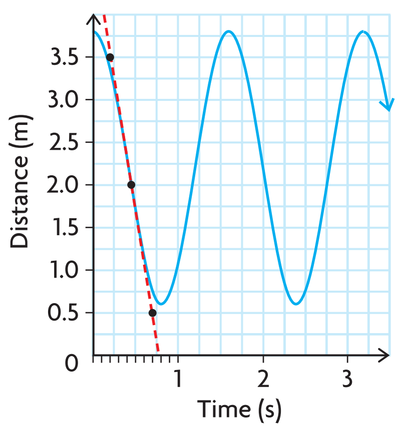
The child's average speed was the same in both directions as the child swung back and forth.

EXAMPLE 3 Using the difference quotient to estimate instantaneous rates of change

To model the motion of the child on the swing, Melissa determined that she could use the equation $d(t) = 1.6\cos\left(\frac{\pi}{0.8}t\right) + 2.2$, where $d(t)$ is the distance from the child to the motion detector, in metres, and t is the time, in seconds. Use this equation to estimate when the child was moving the fastest and what speed the child was moving at this time.

Solution

The child must have been moving the fastest at around 0.4 s. Drawing a tangent line at $t = 0.4$ supports this, since the tangent line appears to be steepest here.



The child moved slowest near the ends of the swing, approaching the points where a change in direction occurred. At these points, the tangent lines are horizontal so their slopes are 0. The child's speed was 0 m/s at 0 s, 0.8 s, and 1.6 s. The child's speed increased between 0 s and 0.4 s, and then decreased between 0.4 s and 0.8 s.

Estimate the coordinates of two points on the tangent line to estimate its slope. Use $(0.2, 3.5)$ and $(0.7, 0.5)$.

$$\text{Slope} = \frac{0.5 - 3.5}{0.7 - 0.2} = -6$$

The child was moving at about 6 m/s.

$$\text{Speed} = \left|\frac{d(0.4 + h) - d(0.4)}{h}\right|$$

To get a better estimate of the child's speed at this time, use the difference quotient $\frac{d(a + h) - d(a)}{h}$, where $a = 0.4$. Use a very small value for h.

Let $h = 0.001$.

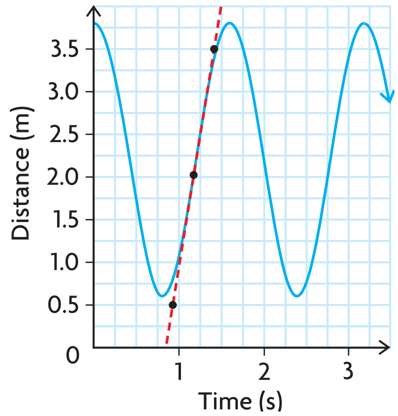
$$\begin{aligned}\text{Speed} &= \left|\frac{d(0.4 + 0.001) - d(0.4)}{0.001}\right| \\ &= \left|\frac{d(0.401) - d(0.4)}{0.001}\right| \\ &= \left|\frac{\left[1.6\cos\left(\frac{\pi}{0.8}(0.401)\right) + 2.2\right] - \left[1.6\cos\left(\frac{\pi}{0.8}(0.4)\right) + 2.2\right]}{0.001}\right| \\ &\doteq \left|\frac{2.19372 - 2.2}{0.001}\right| \\ &\doteq |-6.28| \text{ or } 6.28\end{aligned}$$

This speed is about 23 km/h.

The child's fastest speed was about 6.3 m/s.

The child was also travelling the fastest at around 1.2 s. Drawing a tangent line at $t = 1.2$ supports this, since the tangent line appears to be steepest here.

The child's speed increased between 0.8 s and 1.2 s, and then decreased between 1.2 s and 1.6 s.



Estimate the coordinates of two points on the tangent line to estimate the slope of the line.

Use $(0.9, 0.5)$ and $(1.4, 3.5)$.

$$\text{Slope} = \frac{3.5 - 0.5}{1.4 - 0.9} = 6$$

The child was moving at about 6 m/s.

$$\text{Speed} = \left|\frac{d(1.2 + h) - d(1.2)}{h}\right|$$

Let $h = 0.001$.

$$\text{Speed} = \left|\frac{d(1.2 + 0.001) - d(1.2)}{0.001}\right|$$

To improve the estimate of the child's speed at this time, use the difference quotient $\frac{d(a + h) - d(a)}{h}$, where $a = 1.2$. Use a very small value for h.

$$= \left|\frac{d(1.201) - d(1.2)}{0.001}\right|$$

$$= \left|\frac{\left[1.6 \cos\left(\frac{\pi}{0.8}(1.201)\right) + 2.2\right] - \left[1.6 \cos\left(\frac{\pi}{0.8}(1.2)\right) + 2.2\right]}{0.001}\right|$$

$$= \left|\frac{2.20628 - 2.2}{0.001}\right|$$

$$\doteq |6.28| \text{ or } 6.28$$

The child's fastest speed was about 6.3 m/s.

In Summary

Key Idea

- The average and instantaneous rates of change of a sinusoidal function can be determined using the same strategies that were used for other types of functions.

Need to Know

- The tangent lines at the maximum and minimum values of a sinusoidal function are horizontal. Since the slope of a horizontal line is zero, the instantaneous rate of change at these points is zero.
- In a sinusoidal function, the slope of a tangent line is the least at the point that lies halfway between the maximum and minimum values. The slope is the greatest at the point that lies halfway between the minimum and maximum values. As a result, the instantaneous rate of change at these points is the least and greatest, respectively. The approximate value of the instantaneous rate of change can be determined using one of the strategies below:
  - sketching an approximate tangent line on the graph and estimating its slope using two points that lie on the secant line
  - using two points in the table of values (preferably two points that lie on either side and/or as close as possible to the tangent point) to calculate the slope of the corresponding secant line
  - using the defining equation of the trigonometric function and a very small interval near the point of tangency to calculate the slope of the corresponding secant line

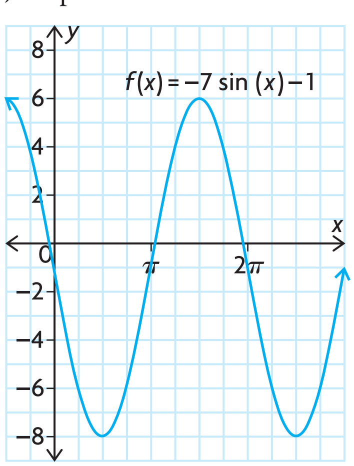
CHECK Your Understanding

1. For the following graph of a function, state two intervals in which the function has an average rate of change in $f(x)$ that is

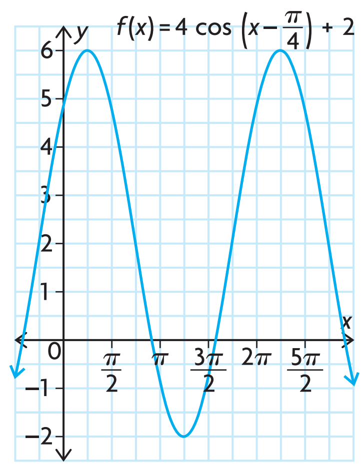
a) zero

b) a negative value

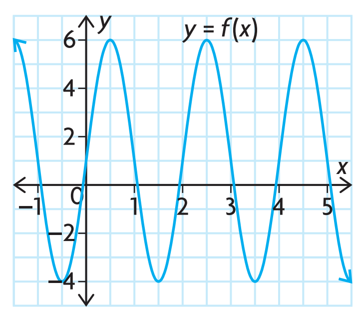
c) a positive value



2. For this graph of a function, state two points where the function has an instantaneous rate of change in $f(x)$ that is
   a) zero
   b) a negative value
   c) a positive value



3. Use the graph to calculate the average rate of change in $f(x)$ on the interval $2 \leq x \leq 5$.



4. Determine the average rate of change of the function $y = 2\cos\left(x - \frac{\pi}{3}\right) + 1$ for each interval.
   a) $0 \leq x \leq \frac{\pi}{2}$
   b) $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$
   c) $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$
   d) $\frac{\pi}{2} \leq x \leq \frac{5\pi}{4}$

PRACTISING

5. **K** State two intervals where the function $y = 3\cos(4x) - 4$ has an average rate of change that is
   a) zero
   b) a negative value
   c) a positive value

6. State two points where the function $y = -2 \sin (2\pi x) + 7$ has an instantaneous rate of change that is

a) zero

b) a negative value

c) a positive value

7. State the average rate of change of each of the following functions over the interval $\frac{\pi}{4} \le x \le \pi$.

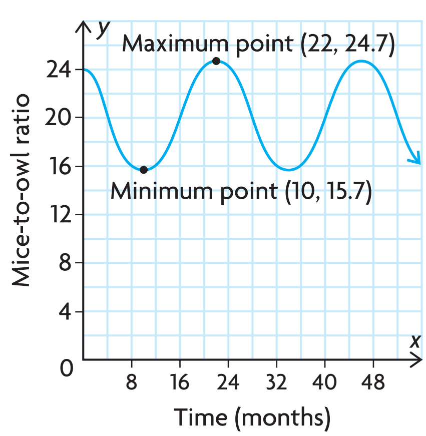
a) $y = 6 \cos (3x) + 2$

b) $y = -5 \sin \left(\frac{1}{2}x\right) - 9$

c) $y = \frac{1}{4} \cos (8x) + 6$

8. T The height of the tip of an airplane propeller above the ground once the airplane reaches full speed can be modelled by a sine function. At full speed, the propeller makes 200 revolutions per second. At $t = 0$, the tip of the propeller is at its minimum height above the ground. Determine whether the instantaneous rate of change in height at $t = \frac{1}{300}$ is a negative value, a positive value, or zero.

9. Recall in Section 6.6, Example 3, the situation that modelled the populations of mice and owls in a particular area.



a) Determine an equation for the curve that models the ratio of mice per owl.

b) Use the curve to determine when the ratio of mice per owl has its fastest and slowest instantaneous rates of change.

c) Use the equation you determined in part a) to estimate the instantaneous rate of change in mice per owl when this rate is at its maximum. Use a centred interval of 1 month before to 1 month after the time when the instantaneous rate of change is at its maximum to make your estimate.

10. The number of tons of paper waiting to be recycled at a 24 h recycling plant can be modelled by the equation $P(t) = 0.5 \sin\left(\frac{\pi}{6}t\right) + 4$, where t is the time, in hours, and $P(t)$ is the number of tons waiting to be recycled.

a) Use the equation to estimate the instantaneous rate of change in tons of paper waiting to be recycled when this rate is at its maximum. To make your estimate, use each of the following centred intervals:

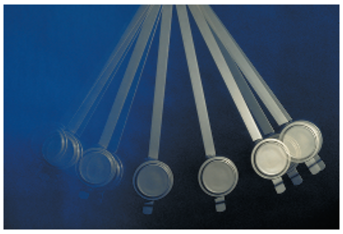
i) 1 h before to 1 h after the time when the instantaneous rate of change is at its maximum

ii) 0.5 h before to 0.5 h after the time when the instantaneous rate of change is at its maximum

iii) 0.25 h before to 0.25 h after the time when the instantaneous rate of change is at its maximum

b) Which estimate is the most accurate? What is the relationship between the size of the interval and the accuracy of the estimate?

11. A strobe photography camera takes photos at regular intervals to capture the motion of a pendulum as it swings from right to left. A student takes measurements from the photo below to analyze the motion.



Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Horizontal Distance from Rest Position* (cm)	7.2	6.85	5.8	4.25	2.2	0.0	−2.2	−4.25	−5.8	−6.85	−7.2

*negative is left of rest position

a) Plot the data, and draw a smooth curve through the points.

b) What portion of one cycle is represented by the curve?

c) Select the endpoints, and determine the average rate of change in horizontal distance on this interval of time.

d) Can you tell, from the photo, when the pendulum bob is moving the fastest? Explain.

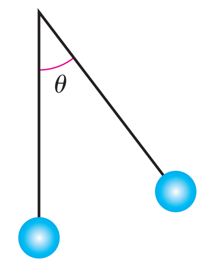
e) Explain how your answer to part d) relates to the rate of change as it is represented on the graph.

12. **A** A ship that is docked in a harbour rises and falls with the waves. The function $h(t) = \sin\left(\frac{\pi}{5}t\right)$ models the vertical movement of the ship, h in metres, at t seconds.

a) Determine the average rate of change in the height of the ship over the first 5 s.

b) Estimate the instantaneous rate of change in the height of the ship at $t = 6$.

13. For a certain pendulum, the angle θ shown is given by the equation $\theta = \frac{1}{5}\sin\left(\frac{1}{2}\pi t\right)$ where t is in seconds and θ is in radians.



a) Sketch a graph of the function given by the equation.

b) Calculate the average rate of change in the angle the pendulum swings through in the interval $t \in [0, 1]$.

c) Estimate the instantaneous rate of change in the angle the pendulum swings through at $t = 1.5$ s.

d) On the interval $t \in [0, 8]$, estimate the times when the pendulum's speed is greatest.

14. **C** Compare the instantaneous rates of change of $f(x) = \sin x$ and $f(x) = 3\sin x$ for the same values of x. What can you conclude? Are there values of x for which the instantaneous rates of change of the two functions are the same?

Extending

15. In calculus, the derivative of a function is a function that yields the instantaneous rate of change of a function at any given point.

a) Estimate the instantaneous rate of change of the function $f(x) = \sin x$ for the following values of x: $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, and π.

b) Plot the points that represent the instantaneous rate of change, and draw a sinusoidal curve through them. What function have you graphed? Based on this information, what is the derivative of $f(x) = \sin x$?

16. **a)** Estimate the instantaneous rate of change of the function $f(x) = \cos x$ for the following values of x: $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, and π.

b) Plot the points that represent the instantaneous rate of change, and draw a sinusoidal curve through them. What function have you graphed? Based on this information, what is the derivative of $f(x) = \cos x$?

6 Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 6.5.
- Try Chapter Review Question 13.

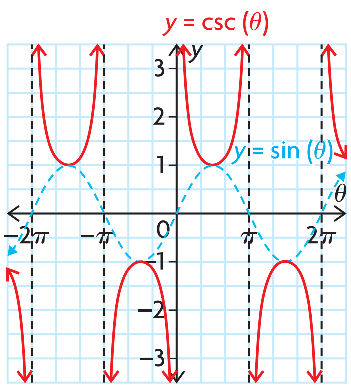
Q: What do the graphs of the reciprocal trigonometric functions look like, and what are their defining characteristics?

A: Each of the primary trigonometric graphs has a corresponding reciprocal function:

Cosecant

$y = \csc x$

$y = \dfrac{1}{\sin x}$

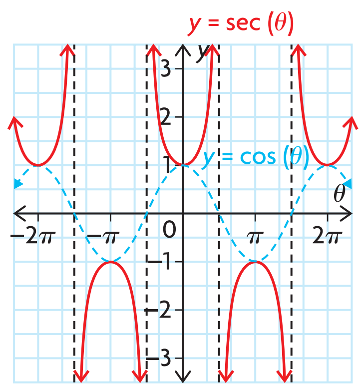


- has vertical asymptotes at the points where $\sin x = 0$
- has a period of 2π radians, the same period as $y = \sin x$
- has the domain $\{x \in \mathbf{R} \mid x \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Secant

$y = \sec x$

$y = \dfrac{1}{\cos x}$



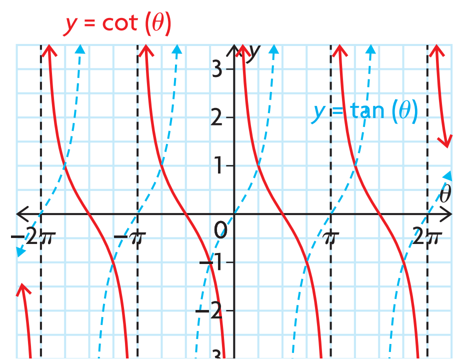
- has vertical asymptotes at the points where $\cos x = 0$
- has a period of 2π radians, the same period as $y = \cos x$
- has the domain $\{x \in \mathbf{R} \mid x \neq (2n - 1)\frac{\pi}{2}, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Cotangent

$y = \cot x$

$y = \dfrac{1}{\tan x}$

$y = \cot(x)$



- has vertical asymptotes at the points where $y = \tan x$ crosses the x-axis
- has zeros at the points where $y = \tan x$ has asymptotes
- has a period of π, the same period as $y = \tan x$
- has the domain $\{x \in \mathbf{R} \mid x \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R}\}$

Study *Aid*

- See Lesson 6.6, Example 1.
- Try Chapter Review Questions 14, 15, and 16.

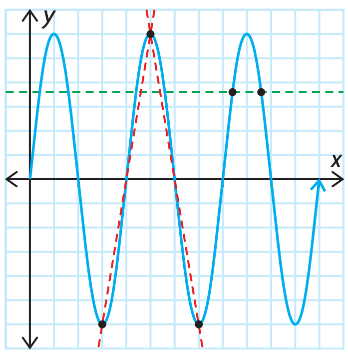
Q: How can you use a sinusoidal function to model a periodic situation?

A: If you are given a description of a periodic situation, draw a rough sketch of one cycle. If you are given data, create a scatter plot. Based on the graph, decide whether you will use a sine model or a cosine model. Use these graphs to determine the equation of the axis, the vertical translation, c, and the amplitude, a, of the function.

Use the period to help you determine k. Determine the horizontal translation, d, that must be applied to a key point on the parent function to map its corresponding location on the model. Use the parameters you found to write the equation in the form $y = a \sin (k(x - d)) + c$ or $y = a \cos (k(x - d)) + c$.

Q: Does the average rate of change of a sinusoidal function have any unique characteristics?

A:



For a sinusoidal function,

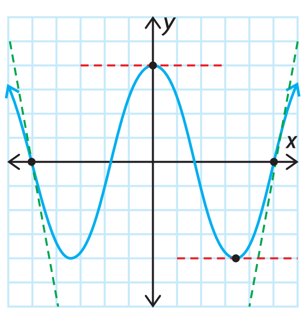
- the average rate of change is zero on any interval where the values of the function are the same
- the absolute value of the average rate of change on the intervals between a maximum and a minimum and between a minimum and a maximum are equal

Study Aid

- See Lesson 6.7, Example 2.
- Try Chapter Review Questions 17 and 19.

Q: Do the instantaneous rates of change of a sinusoidal function have any unique characteristics?

A:



For a sinusoidal function, the instantaneous rate of change is

- zero at any maximum or minimum
- at its least value halfway between a maximum and a minimum
- at its greatest value halfway between a minimum and a maximum

Study Aid

- See Lesson 6.7, Examples 1 and 3.
- Try Chapter Review Question 18.

PRACTICE Questions

Lesson 6.1

1. An arc 33 m long subtends a central angle of a circle with a radius of 16 m. Determine the measure of the central angle in radians.

2. A circle has a radius of 75 cm and a central angle of $\frac{14\pi}{15}$. Determine the arc length.

3. Convert each of the following to exact radian measure and then evaluate to one decimal.

a) $20°$ **c)** $160°$

b) $-50°$ **d)** $420°$

4. Convert each of the following to degree measure.

a) $\frac{\pi}{4}$ **c)** $\frac{8\pi}{3}$

b) $-\frac{5\pi}{4}$ **d)** $-\frac{2\pi}{3}$

Lesson 6.2

5. For each of the following values of $\sin\theta$, determine the measure of θ if $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

a) $\frac{1}{2}$ **c)** $\frac{\sqrt{2}}{2}$

b) $-\frac{\sqrt{3}}{2}$ **d)** $-\frac{1}{2}$

6. If $\cos\theta = \frac{-5}{13}$ and $0 \leq \theta \leq 2\pi$, determine

a) $\tan\theta$

b) $\sec\theta$

c) the possible values of θ to the nearest tenth

7. A tower that is 65 m high makes an obtuse angle with the ground. The vertical distance from the top of the tower to the ground is 59 m. What obtuse angle does the tower make with the ground, to the nearest hundredth of a radian?

Lesson 6.3

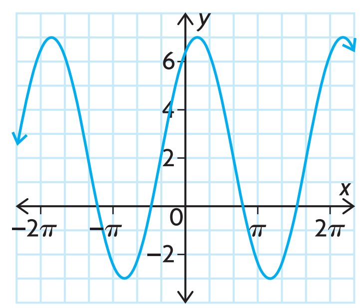
8. State the period of the graph of each function, in radians.

a) $y = \sin x$ **c)** $y = \tan x$

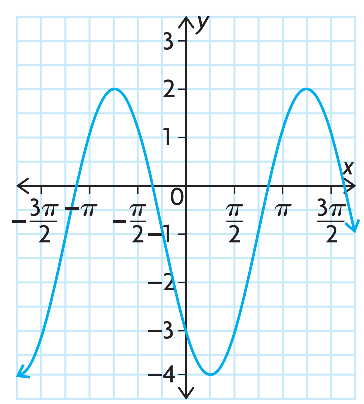
b) $y = \cos x$

Lesson 6.4

9. The following graph is a sine curve. Determine the equation of the graph.



10. The following graph is a cosine curve. Determine the equation of the graph.



11. State the transformations that have been applied to $f(x) = \cos x$ to obtain each of the following functions.

a) $f(x) = -19\cos x - 9$

b) $f(x) = \cos\left(10\left(x + \frac{\pi}{12}\right)\right)$

c) $f(x) = \frac{10}{11}\cos\left(x - \frac{\pi}{9}\right) + 3$

d) $f(x) = -\cos(-x + \pi)$

12. The current, I, in amperes, of an electric circuit is given by the function $I(t) = 4.5 \sin (120\pi t)$, where t is the time in seconds.
- **a)** Draw a graph that shows one cycle.
- **b)** What is the singular period?
- **c)** At what value of t is the current a maximum in the first cycle?
- **d)** When is the current a minimum in the first cycle?

Lesson 6.5

13. State the period of the graph of each function, in radians.
- **a)** $y = \csc x$
- **b)** $y = \sec x$
- **c)** $y = \cot x$

Lesson 6.6

14. A bumblebee is flying in a circular motion within a vertical plane, at a constant speed. The height of the bumblebee above the ground, as a function of time, can be modelled by a sinusoidal function. At $t = 0$, the bumblebee is at its lowest point above the ground.
- **a)** What does the amplitude of the sinusoidal function represent in this situation?
- **b)** What does the period of the sinusoidal function represent in this situation?
- **c)** What does the equation of the axis of the sinusoidal function represent in this situation?
- **d)** If a reflection in the horizontal axis was applied to the sinusoidal function, was the sine function or the cosine function used?

15. The population of a ski-resort town, as a function of the number of months into the year, can be described by a cosine function. The maximum population of the town is about 15 000 people, and the minimum population is about 500 people. At the beginning of the year, the population is at its greatest. After six months, the population reaches its lowest number of people. What is the equation of the cosine function that describes the population of this town?

16. A weight is bobbing up and down on a spring attached to a ceiling. The data in the following table give the height of the weight above the floor as it bobs. Determine the sine function that models this situation.

t (s)	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
$h(t)$ (cm)	120	136	165	180	166	133	120	135	164	179	165	133

Lesson 6.7

17. State two intervals in which the function $y = 7 \sin \left(\frac{1}{5}x\right) + 2$ has an average rate of change that is
- **a)** zero
- **b)** a negative value
- **c)** a positive value

18. State two points where the function $y = \frac{1}{4} \cos (4\pi x) - 3$ has an instantaneous rate of change that is
- **a)** zero
- **b)** a negative value
- **c)** a positive value

19. A person's blood pressure, $P(t)$, in millimetres of mercury (mm Hg), is modelled by the function $P(t) = 100 - 20 \cos \left(\frac{8\pi}{3}t\right)$, where t is the time in seconds.
- **a)** What is the period of the function?
- **b)** What does the value of the period mean in this situation?
- **c)** Calculate the average rate of change in a person's blood pressure on the interval $t \in [0.2, 0.3]$.
- **d)** Estimate the instantaneous rate of change in a person's blood pressure at $t = 0.5$.

6 Chapter Self-Test

1. Which trigonometric function has an asymptote at $x = \frac{5\pi}{2}$?

2. Which expression does not have the same value as all the other expressions?
$\sin \frac{3\pi}{2}$, $\cos \pi$, $\tan \frac{7\pi}{4}$, $\csc \frac{3\pi}{2}$, $\sec 2\pi$, $\cot \frac{3\pi}{4}$

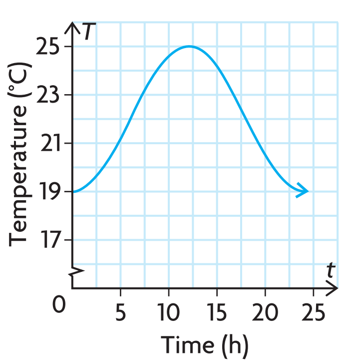
3. The function $y = \cos x$ is reflected in the x-axis, vertically stretched by a factor of 12, horizontally compressed by a factor of $\frac{3}{5}$, horizontally translated $\frac{\pi}{6}$ units to the left, and vertically translated 100 units up. Determine the value of the new function, to the nearest tenth, when $x = \frac{5\pi}{4}$.

4. The daily high temperature of a city, in degrees Celsius, as a function of the number of days into the year, can be described by the function $T(d) = -20 \cos \left(\frac{2\pi}{365}(d - 10)\right) + 25$. What is the average rate of change, in degrees Celsius per day, of the daily high temperature of the city from February 21 to May 8?

5. Arrange the following angles in order, from smallest to largest:
$\frac{5\pi}{8}$, $113°$, $\frac{2\pi}{3}$, $110°$, $\frac{3\pi}{5}$

6. Write an equivalent sine function for $y = \cos\left(x + \frac{\pi}{8}\right)$.

7. The point $(5, y)$ lies on the terminal arm of an angle in standard position. If the angle measures 4.8775 radians, what is the value of y to the nearest unit?



8. The temperature, T, in degrees Celsius, of the surface water in a swimming pool varies according to the following graph, where t is the number of hours since sunrise at 6 a.m.
   a) Find a possible equation for the temperature of the surface water as a function of time.
   b) Calculate the average rate of change in water temperature from sunrise to noon.
   c) Estimate the instantaneous rate of change in water temperature at 6 p.m.

6 Chapter Task

Investigating Changes in Temperature

The following table gives the mean monthly temperatures for Sudbury and Windsor, two cities in Ontario. Each month is represented by the day of the year in the middle of the month.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Day of Year	15	45	75	106	136	167	197	228	259	289	320	350
Temperature for Sudbury (°C)	−13.7	−11.9	−5.9	3.0	10.6	15.8	18.9	17.4	12.2	6.2	−1.2	−10.1
Temperature for Windsor (°C)	−4.7	−3.8	2.3	8.7	14.6	20.2	22.6	22.0	17.9	11.5	4.8	−1.2

? Which city has the greatest rate of increase in mean daily temperature, and when does this occur?

A. Make a conjecture about which city has the greatest rate of increase in mean daily temperature. Provide reasons for your conjecture.

B. Create a scatter plot of mean monthly temperature versus day of the year for each city.

C. Draw the curve of best fit for each graph.

D. Use your graphs to estimate when the mean daily temperature increases the fastest in both cities. Explain how you determined these values.

E. Use your graphs to estimate the rate at which the mean daily temperature is increasing at the times you estimated in part D.

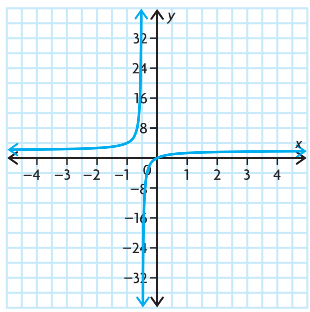
F. Determine an equation of a sinusoidal function to model the data for each city.

G. Use the equations you found in part F to estimate the fastest rate at which the mean daily temperature is increasing.

Task *Checklist*

- ✔ Did you provide reasons for your conjecture?
- ✔ Did you draw and label your graphs accurately?
- ✔ Did you determine when the mean daily temperature is increasing the fastest in both cities?
- ✔ Did you show all the steps in your calculations of rates of change and clearly explain your thinking?

d) $x = -0.5$; vertical asymptote: $x = -0.5$; $D = \{x \in \mathbf{R} \mid x \neq -0.5\}$; x-intercept $= 0$; y-intercept $= 0$; horizontal asymptote $= 2$; $R = \{y \in \mathbf{R} \mid x \neq 2\}$; positive on $x < -0.5$ and $x > 0$; negative on $-0.5 < x < 0$



The function is never decreasing and is increasing on $(-\infty, -0.5)$ and $(-0.5, \infty)$.

6. Answers may vary. For example, consider the function $f(x) = \frac{1}{x-6}$. You know that the vertical asymptote would be $x = 6$. If you were to find the value of the function very close to $x = 6$ (say $f(5.99)$ or $f(6.01)$) you would be able to determine the behaviour of the function on either side of the asymptote.

$$f(5.99) = \frac{1}{(5.99) - 6} = -100$$

$$f(6.01) = \frac{1}{(6.01) - 6} = 100$$

To the left of the vertical asymptote, the function moves toward $-\infty$. To the right of the vertical asymptote, the function moves toward ∞.

7. **a)** $x = 6$
b) $x = 0.2$ and $x = -\frac{2}{3}$
c) $x = -6$ or $x = 2$
d) $x = -1$ and $x = 3$

8. about 12 min

9. $x = 1.82$ days and 3.297 days

10. **a)** $x < -3$ and $-2.873 < x < 4.873$
b) $-16 < x < -11$ and $-5 < x$
c) $-2 < x < -1.33$ and $-1 < x < 0$
d) $0 < x < 1.5$

11. $-0.7261 < t < 0$ and $t > 64.73$

12. **a)** -6; $x = 3$
b) 0.2; $x = -2$ and $x = -1$

13. **a)** 0.455 mg/L/h
b) -0.04 mg/L/h
c) The concentration of the drug in the blood stream appears to be increasing most rapidly during the first hour and a half; the graph is steep and increasing during this time.

14. $x = 5$ and $x = 8$; $x = 6.5$

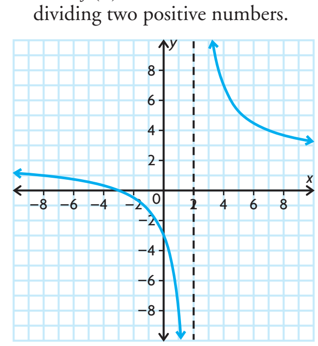
15. **a)** As the x-coordinate approaches the vertical asymptote of a rational function, the line tangent to graph will get closer and closer to being a vertical line. This means that the slope of the line tangent to the graph will get larger and larger, approaching positive or negative infinity depending on the function, as x gets closer to the vertical asymptote.
b) As the x-coordinate grows larger and larger in either direction, the line tangent to the graph will get closer and closer to being a horizontal line. This means that the slope of the line tangent to the graph will always approach zero as x gets larger and larger.

Chapter Self-Test, p. 310

1. **a)** B
b) A

2. **a)** If $f(n)$ is very large, then that would make $\frac{1}{f(n)}$ a very small fraction.
b) If $f(n)$ is very small (less than 1), then that would make $\frac{1}{f(n)}$ very large.
c) If $f(n) = 0$, then that would make $\frac{1}{f(n)}$ undefined at that point because you cannot divide by 0.
d) If $f(n)$ is positive, then that would make $\frac{1}{f(n)}$ also positive because you are dividing two positive numbers.

3.



4. 4326 kg; \$0.52/kg

5. **a)** Algebraic; $x = -1$ and $x = -3$
b) Algebraic with factor table
The inequality is true on $(-10, -5.5)$ and on $(-5, 1.2)$.

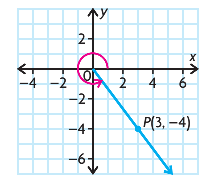
6. **a)** To find the vertical asymptotes of the function, find the zeros of the expression in the denominator. To find the equation of the horizontal asymptotes, divide the first two terms of the expressions in the numerator and denominator.
b) This type of function will have a hole when both the numerator and the denominator share the same factor $(x + a)$.

Chapter 6

Getting Started, p. 314

1. **a)** 28°
b) 332°

2. **a)**



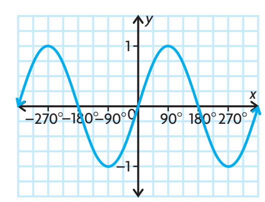
$\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

b) 307°

3. **a)** $\frac{\sqrt{3}}{2}$ **c)** $\frac{\sqrt{3}}{2}$ **e)** $-\sqrt{2}$
b) 0 **d)** $\frac{1}{2}$ **f)** -1

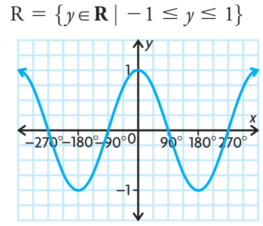
4. **a)** 60°, 300°
b) 30°, 210°
c) 45°, 225°
d) 180°
e) 135°, 315°
f) 90°

5. **a)**



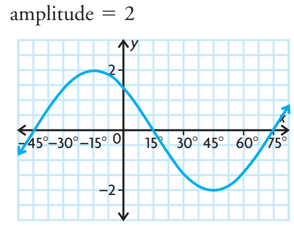
period $= 360°$; amplitude $= 1$; $y = 0$; $R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

b)



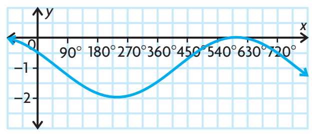
period $= 360°$; amplitude $= 1$; $y = 0$; $R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

6. **a)** period $= 120°$; $y = 0$; 45° to the left; amplitude $= 2$



Answers

b) period $= 720°$; $y = -1$; $60°$ to the right; amplitude $= 1$



7. a is the amplitude, which determines how far above and below the axis of the curve of the function rises and falls; k defines the period of the function, which is how often the function repeats itself; d is the horizontal shift, which shifts the function to the right or the left; and c is the vertical shift of the function.

Lesson 6.1, pp. 320–322

1. a) π radians; $180°$

b) $\frac{\pi}{2}$ radians; $90°$

c) $-\pi$ radians; $-180°$

d) $-\frac{3\pi}{2}$ radians; $-270°$

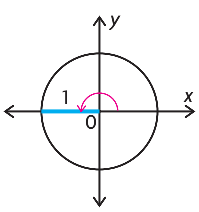
e) -2π radians; $-360°$

f) $\frac{3\pi}{2}$ radians; $270°$

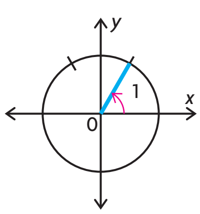
g) $-\frac{4\pi}{3}$ radians $= -240°$

h) $\frac{2\pi}{3}$ radians; $120°$

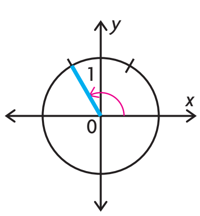
2. a)



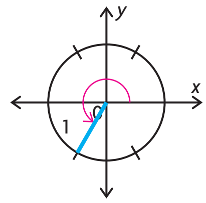
b)



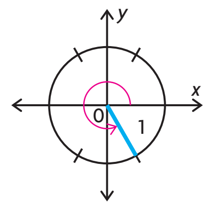
c)



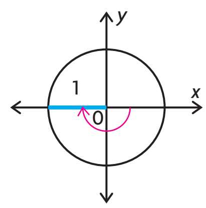
d)



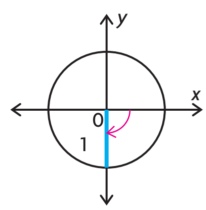
e)



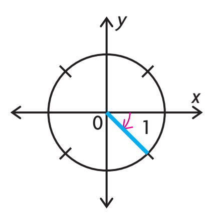
f)



g)



h)



3. a) $\frac{5\pi}{12}$ radians c) $\frac{20\pi}{9}$ radians

b) $\frac{10\pi}{9}$ radians d) $= \frac{16\pi}{9}$ radians

4. a) $300°$ c) $171.89°$

b) $54°$ d) $495°$

5. a) 2 radians; $114.6°$

b) $\frac{25\pi}{9}$ cm

6. a) 28 cm

b) $\frac{40\pi}{3}$ cm

7. a) $\frac{\pi}{2}$ radians e) $\frac{5\pi}{4}$ radians

b) $\frac{3\pi}{2}$ radians f) $\frac{\pi}{3}$ radians

c) π radians g) $\frac{4\pi}{3}$ radians

d) $\frac{\pi}{4}$ radians h) $\frac{4\pi}{3}$ radians

8. a) $120°$ e) $210°$

b) $60°$ f) $90°$

c) $45°$ g) $330°$

d) $225°$ h) $270°$ – convert to pos angle by adding period

9. a) $\frac{247\pi}{4}$ m

b) ~~162.5 m~~ 81.25

c) $\frac{325\pi}{6}$ cm

10. $4.50\sqrt{2}$ cm

11. a) $\doteq 0.418\ 88$ radians/s

b) $\doteq 377.0$ m

12. a) 36

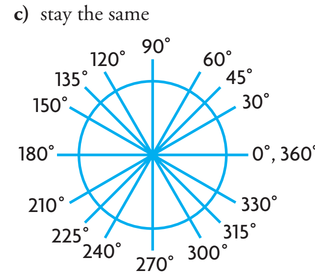
b) 0.8 m

13. a) equal to

b) greater than

c) stay the same

14.



$0° = 0$ radians; $30° = \frac{\pi}{6}$ radians;

$45° = \frac{\pi}{4}$ radians; $60° = \frac{\pi}{3}$ radians;

$90° = \frac{\pi}{2}$ radians; $120° = \frac{2\pi}{3}$ radians;

$135° = \frac{3\pi}{4}$ radians; $150° = \frac{5\pi}{6}$ radians;

$180° = \pi$ radians; $210° = \frac{7\pi}{6}$ radians;

$225° = \frac{5\pi}{4}$ radians; $240° = \frac{4\pi}{3}$ radians;

$270° = \frac{3\pi}{2}$ radians; $300° = \frac{5\pi}{3}$ radians;

$315° = \frac{7\pi}{4}$ radians; $330° = \frac{11\pi}{6}$ radians;

$360° = 2\pi$ radians

15. Circle B, Circle A, and Circle C

16. about ~~144.5~~ radians/s
86.8

Lesson 6.2, pp. 330–332

1. a) second quadrant; $\frac{\pi}{4}$; positive

b) fourth quadrant; $\frac{\pi}{3}$; positive

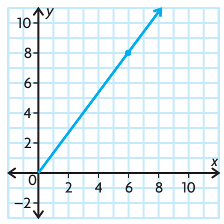
c) third quadrant; $\frac{\pi}{3}$; positive

d) second quadrant; $\frac{\pi}{6}$; negative

e) second quadrant; $\frac{\pi}{3}$; negative

f) fourth quadrant; $\frac{\pi}{4}$; negative

2. **a)** **i)**



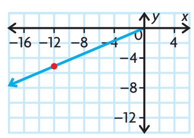
ii) $r = 10$

iii) $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3},$

$\csc\theta = \frac{5}{4}, \sec\theta = \frac{5}{3}, \cot\theta = \frac{3}{4}$

iv) $\theta \doteq 0.93$

b) **i)**



ii) $r = 13$

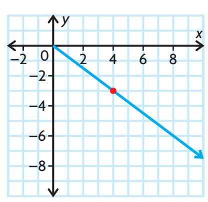
iii) $\sin\theta = -\frac{5}{13}, \cos\theta = -\frac{12}{13},$

$\tan\theta = \frac{5}{12}, \csc\theta = -\frac{13}{5},$

$\sec\theta = -\frac{13}{12}, \cot\theta = \frac{12}{5}$

iv) $\theta \doteq 3.54$

c) **i)**



ii) $r = 5$

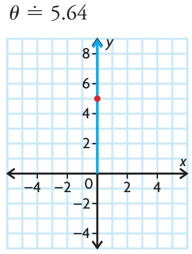
iii) $\sin\theta = -\frac{3}{5}, \cos\theta = \frac{4}{5},$

$\tan\theta = -\frac{3}{4}, \csc\theta = -\frac{5}{3},$

$\sec\theta = \frac{5}{4}, \cot\theta = -\frac{4}{3}$

iv) $\theta \doteq 5.64$

d) **i)**



ii) $r = 5$

iii) $\sin\theta = \frac{5}{5} = 1,$

$\cos\theta = \frac{0}{5} = 0,$

$\tan\theta = \frac{5}{0} = \text{undefined},$

$\csc\theta = \frac{5}{5} = 1,$

$\sec\theta = \frac{5}{0} = \text{undefined},$

$\cot\theta = \frac{0}{5} = 0$

iv) $\theta \doteq \frac{\pi}{2}$

3. **a)** $\sin\left(-\frac{\pi}{2}\right) = -1,$

$\cos\left(-\frac{\pi}{2}\right) = 0,$

$\tan\left(-\frac{\pi}{2}\right) = \text{undefined},$

$\csc\left(-\frac{\pi}{2}\right) = -1,$

$\sec\left(-\frac{\pi}{2}\right) = \text{undefined},$

$\cot\left(-\frac{\pi}{2}\right) = 0$

b) $\sin(-\pi) = 0,$
$\cos(-\pi) = -1,$
$\tan(-\pi) = 0,$
$\csc(-\pi) = \text{undefined},$
$\sec(-\pi) = -1,$
$\cot(-\pi) = \text{undefined}$

c) $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2},$

$\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},$

$\tan\left(\frac{7\pi}{4}\right) = -1,$

$\csc\left(\frac{7\pi}{4}\right) = -\sqrt{2},$

$\sec\left(\frac{7\pi}{4}\right) = \sqrt{2},$

$\cot\left(\frac{7\pi}{4}\right) = -1$

d) $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2},$

$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$

$\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3},$

$\csc\left(-\frac{\pi}{6}\right) = -2,$

$\sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3},$

$\cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$

4. **a)** $\sin\frac{\pi}{6}$ **c)** $\cot\frac{3\pi}{4}$

b) $\cos\frac{\pi}{3}$ **d)** $\sec\frac{5\pi}{6}$

5. **a)** $\frac{\sqrt{3}}{2}$ **d)** $-\frac{\sqrt{2}}{2}$

b) $-\frac{\sqrt{2}}{2}$ **e)** 2

c) $-\frac{\sqrt{3}}{3}$ **f)** 2

6. **a)** $\frac{4\pi}{3}$ **d)** $\frac{7\pi}{6}$

b) $\frac{11\pi}{6}$ **e)** $\frac{3\pi}{2}$

c) $\frac{5\pi}{4}$ **f)** π

7. **a)** $\theta \doteq 2.29$ **d)** $\theta \doteq 3.61$
b) $\theta \doteq 0.17$ **e)** $\theta \doteq 0.84$
c) $\theta \doteq 1.30$ **f)** $\theta \doteq 6.12$

8. **a)** $\cos\frac{5\pi}{4}$ **d)** $\cot\frac{5\pi}{3}$

b) $\tan\frac{5\pi}{6}$ **e)** $\sin\frac{7\pi}{6}$

c) $\csc\frac{4\pi}{3}$ **f)** $\sec\frac{\pi}{4}$

9. $\pi - 0.748 \doteq 2.39$

10. $x \doteq 5.55$ cm

11. $x \doteq 4.5$ cm

12. Draw the angle and determine the measure of the reference angle. Use the CAST rule to determine the sign of each of the ratios in the quadrant in which the angle terminates. Use this sign and the value of the ratios of the reference angle to determine the values of the primary trigonometric ratios for the given angle.

13. **a)** second or third quadrant

b) $\sin\theta = \frac{12}{13}$ or $-\frac{12}{13},$

$\tan\theta = \frac{12}{5}$ or $-\frac{12}{5},$

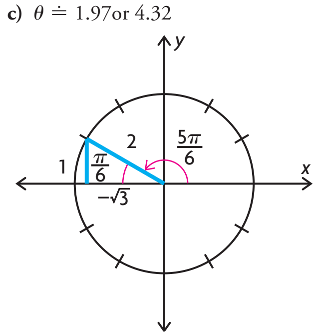
$\sec\theta = -\frac{13}{5},$

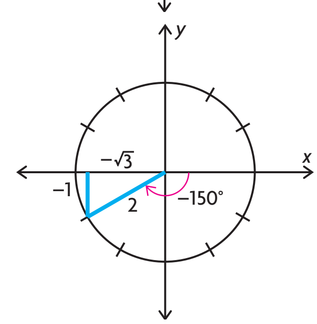
$\csc\theta = \frac{13}{12}$ or $-\frac{13}{12},$

$\cot\theta = \frac{5}{12}$ or $-\frac{5}{12}$

c) $\theta \doteq 1.97$or 4.32

14.





By examining the special triangles, we see

$$\cos\left(\frac{5\pi}{6}\right) = \cos(-150°) = -\frac{\sqrt{3}}{2}$$

15. $$2\left(\sin^2\left(\frac{11\pi}{6}\right)\right) - 1 = 2\left(-\frac{1}{2}\right)^2 - 1$$
$$= 2\left(\frac{1}{4}\right) - 1$$
$$= -\frac{1}{2}$$

$$\left(\sin^2\frac{11\pi}{6}\right) - \left(\cos^2\frac{11\pi}{6}\right)$$
$$= \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4} - \frac{3}{4}$$
$$= -\frac{1}{2}$$

$$2\left(\sin^2\left(\frac{11\pi}{6}\right)\right) - 1$$
$$= \left(\sin^2\frac{11\pi}{6}\right) - \left(\cos^2\frac{11\pi}{6}\right)$$

16. $AB = 16$;

$\sin D = \frac{8}{8\sqrt{2}} = \frac{\sqrt{2}}{2}$;

$\cos D = \frac{8}{8\sqrt{2}} = \frac{\sqrt{2}}{2}$;

$\tan D = \frac{8}{8} = 1$

17. **a)** The first and second quadrants both have a positive y-value.

b) The first quadrant has a positive y-value, and the fourth quadrant has a negative y-value.

c) The first quadrant has a positive x-value, and the second quadrant has a negative x-value.

d) The first quadrant has a positive x-value and a positive y-value, and the third quadrant has a negative x-value and a negative y-value.

18. 1

19. $\cos 150° \doteq -0.26$

20. The ranges of the cosecant and secant functions are both $\{y \in \mathbf{R} \mid -1 \geq y$ or $y \geq 1\}$. In other words, the values of these functions can never be between -1 and 1. For the values of these functions to be between -1 and 1, the values of the sine and cosine functions would have to be greater than 1 and less than -1, which is never the case.

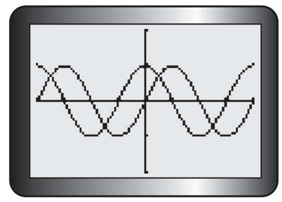
21. $\frac{2\sqrt{3} - 3}{4}$

Lesson 6.3, p. 336

1. **a)** $y = \sin\theta$ and $y = \cos\theta$ have the same period, axis, amplitude, maximum value, minimum value, domain, and range. They have different y- and θ-intercepts.

b) $y = \sin\theta$ and $y = \tan\theta$ have no characteristics in common except for their y-intercept and zeros.

2. **a)**



b) $\theta = -5.50, \theta = -2.36, \theta = 0.79, \theta = 3.93$

c) **i)** $t_n = n\pi, n \in \mathbf{I}$

ii) $t_n = \frac{\pi}{2} + 2n\pi, n \in \mathbf{I}$

iii) $t_n = \frac{3\pi}{2} + 2n\pi, n \in \mathbf{I}$

3. **a)** $t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

b) $t_n = 2n\pi, n \in \mathbf{I}$

c) $t_n = -\pi + 2n\pi, n \in \mathbf{I}$

4. The two graphs appear to be identical.

5. **a)** $t_n = n\pi, n \in \mathbf{I}$

b) $t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

Lesson 6.4, pp. 343–346

1. **a)** period: $\frac{\pi}{2}$
amplitude: 0.5
horizontal translation: 0
equation of the axis: $y = 0$

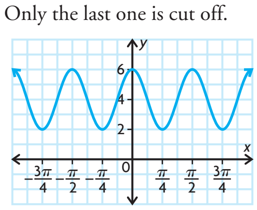
b) period: 2π
amplitude: 1
horizontal translation: $\frac{\pi}{4}$
equation of the axis: $y = 3$

c) period: $\frac{2\pi}{3}$
amplitude: 2
horizontal translation: 0
equation of the axis: $y = -1$

d) period: π
amplitude: 5
horizontal translation: $\frac{\pi}{6}$
equation of the axis: $y = -2$

2. Only the last one is cut off.

3.



period: $\frac{\pi}{2}$
amplitude: 2
horizontal translation: $\frac{\pi}{4}$ to the left
equation of the axis: $y = 4$

4. **a)** $f(x) = 25\sin(2x) - 4$

b) $f(x) = \frac{2}{5}\sin\left(\frac{\pi}{5}x\right) + \frac{1}{15}$

c) $f(x) = 80\sin\left(\frac{1}{3}x\right) - \frac{9}{10}$

d) $f(x) = 11\sin(4\pi x)$

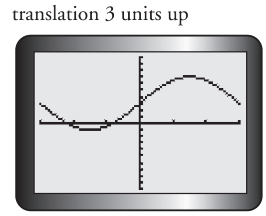
5. **a)** period $= 2\pi$, amplitude $= 18$, equation of the axis is $y = 0$;
$y = 18\sin x$

b) period $= 4\pi$, amplitude $= 6$, equation of the axis is $y = -2$;
$y = +6\sin(0.5x) - 2$

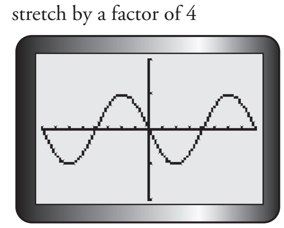
c) period $= 6\pi$, amplitude $= 2.5$, equation of the axis is $y = 6.5$;
$y = -2.5\cos\left(\frac{1}{3}x\right) + 6.5$

d) period $= 4\pi$, amplitude $= 2$, equation of the axis is $y = -1$;
$y = -2\cos\left(\frac{1}{2}x\right) - 1$

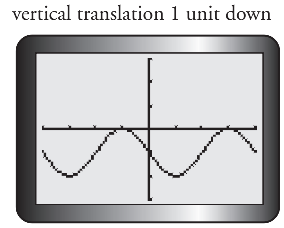
6. **a)** vertical stretch by a factor of 4, vertical translation 3 units up



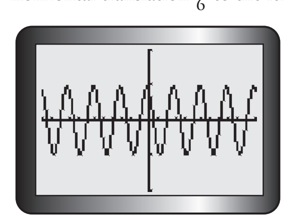
b) reflection in the x–axis, horizontal stretch by a factor of 4



c) horizontal translation π to the right, vertical translation 1 unit down



d) horizontal compression by a factor of $\frac{1}{4}$, horizontal translation $\frac{\pi}{6}$ to the left



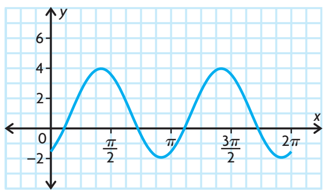
7. a) $f(x) = \frac{1}{2}\cos x + 3$

b) $f(x) = \cos\left(-\frac{1}{2}x\right)$

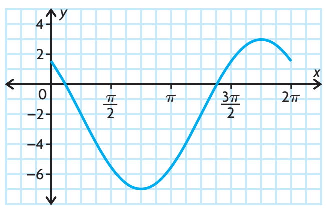
c) $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$

d) $f(x) = \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$

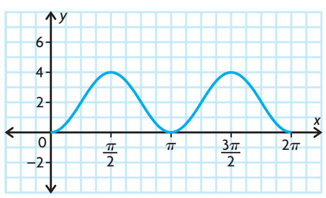
8. a)



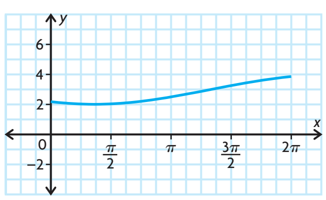
b)



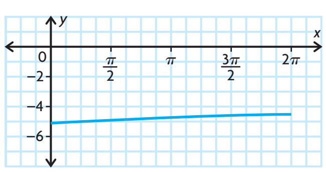
c)



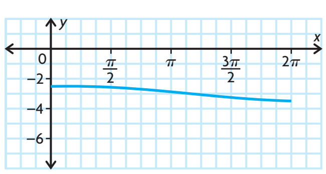
d)



e)



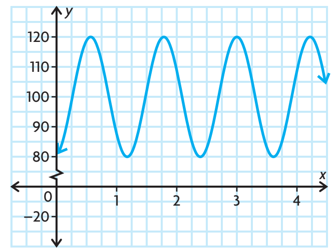
f)



9. a) The period of the function is $\frac{6}{5}$. This represents the time between one beat of a person's heart and the next beat.

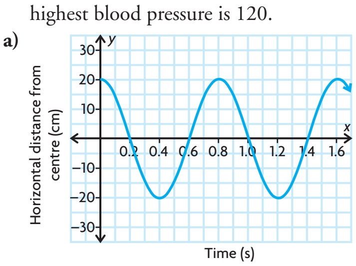
b) ~~80~~ 50

c)



d) The range for the function is between 80 and 120. The range means the lowest blood pressure is 80 and the highest blood pressure is 120.

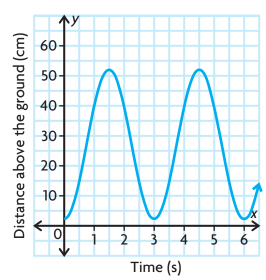
10. a)



b) There is a vertical stretch by a factor of 20, followed by a horizontal compression by a of factor of $\frac{2}{5\pi}$, and then a horizontal translation 0.2 to the left.

c) $y = 20\sin\left(\frac{5\pi}{2}(x + 0.2)\right)$

11. a)



b) vertical stretch by a factor of 25, reflection in the x-axis, vertical translation 27 units up, horizontal compression by a factor of $\frac{1}{|k|} = \frac{3}{2\pi}$

c) $y = -25\cos\left(\frac{2\pi}{3}x\right) + 27$

12. $\frac{2\pi}{7}$

13. Answers may vary. For example, $\left(\frac{14\pi}{13}, 5\right)$.

14. a) $y = \cos(4\pi x)$

b) $y = -2\sin\left(\frac{\pi}{4}x\right)$

c) $y = 4\sin\left(\frac{\pi}{20}(x - 10)\right) - 1$

15. Start with graph of $y = \sin x$.

↓

Reflect in the x-axis and stretch vertically by a factor of 2 to produce graph of $y = -2\sin x$.

↓

Stretch horizontally by a factor of 2 to produce graph of $y = -2\sin(0.5x)$.

↓

Translate $\frac{\pi}{4}$ units to the right to produce graph of $y = -2\sin\left(0.5\left(x - \frac{\pi}{4}\right)\right)$.

↓

Translate 3 units up to produce graph of $y = -2\sin\left(0.5\left(x - \frac{\pi}{4}\right)\right) + 3$.

16. a) 100 m
b) 400 m
c) 300 m
d) 80 s
e) about 23.561 94 m/s

Mid-Chapter Review, p. 349

1. a) 22.5°
b) 720°
c) 286.5°
d) 165°

2. a) 125° ≐ 2.2 radians
b) 450° ≐ 7.9 radians
c) 5° ≐ 0.1 radians
d) 330° ≐ 5.8 radians
e) 215° ≐ 3.8 radians
f) −140° ≐ −2.4 radians

3. a) 20π
b) 4π radians/s
c) 380π cm

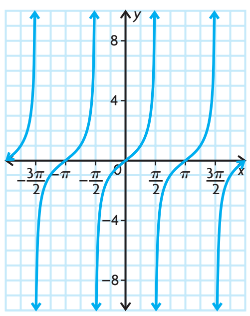
4. a) $\frac{\sqrt{2}}{2}$
b) $-\frac{1}{2}$
c) $-\sqrt{3}$
d) $-\frac{\sqrt{3}}{3}$
e) 0
f) $-\frac{1}{2}$

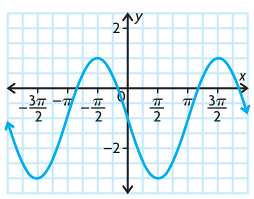
5. a) about 1.78
b) about 0.86
c) about 1.46
d) about 4.44
e) about 0.98
f) about 4.91

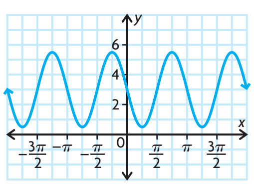
Answers

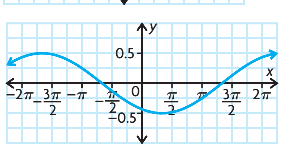
6. a) $\sin \frac{\pi}{6}$
b) $\cot \frac{3\pi}{4}$
c) $\sec \frac{\pi}{2}$
d) $\cos \frac{5\pi}{6}$

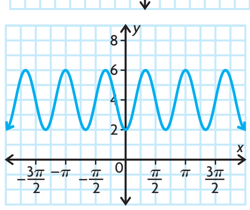
7. a) $x = 0, \pm\pi, \pm 2\pi, \ldots; y = 0$
b) $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots; y = 1$
c) $x = 0, \pm\pi, \pm 2\pi, \ldots; y = 0$

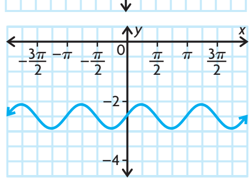
8. a)


b)


c)


d)


e)


f)


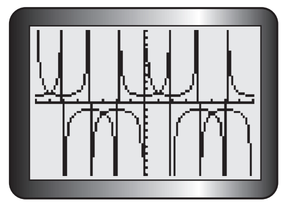
9. $y = \frac{1}{3}\sin\left(-3\left(x + \frac{\pi}{8}\right)\right) - 23$

Lesson 6.5, p. 353

1. a) $t_n = n\pi, n \in \mathbf{I}$
b) no maximum value
c) no minimum value

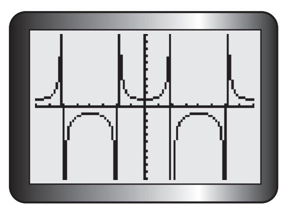
2. a) $t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$
b) no maximum value
c) no minimum value

3. a) $t_n = n\pi, n \in \mathbf{I}$
b) $t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

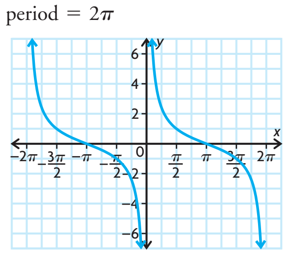
4.


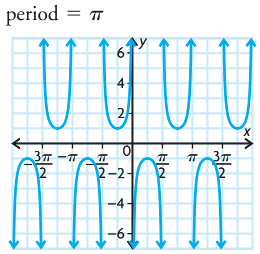
$-5.50, -2.35, 0.79, 3.93$

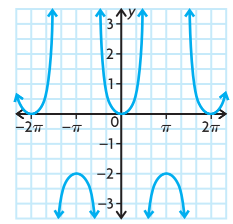
5. Yes, the graphs of $y = \csc\left(x + \frac{\pi}{2}\right)$ and $y = \sec x$ are identical.

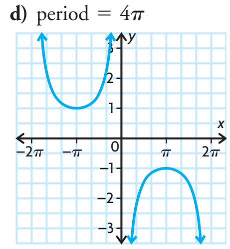


6. Answers may vary. For example, reflect the graph of $y = \tan x$ across the y-axis and then translate the graph $\frac{\pi}{2}$ units to the left.

7. a) period $= 2\pi$


b) period $= \pi$


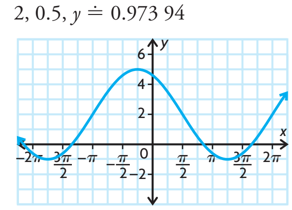
c) period $= 2\pi$


d) period $= 4\pi$


Lesson 6.6, pp. 360–362

1. $y = 3\cos\left(\frac{2}{3}\left(x + \frac{\pi}{4}\right)\right) + 2$

2. $2, 0.5, y \doteq 0.973\ 94$

3.


$x = 1.3$

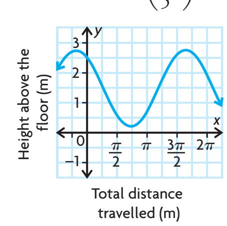
4. amplitude and equation of the axis

5. a) the radius of the circle in which the tip of the sparkler is moving
b) the time it takes Mike to make one complete circle with the sparkler
c) the height above the ground of the centre of the circle in which the tip of the sparkler is moving
d) cosine function

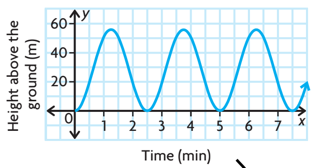
6. $y = 90\sin\left(\frac{\pi}{12}x\right) + 30$

7. $y = 250\cos\left(\frac{2\pi}{3}x\right) + 750$

8. $y = -1.25\sin\left(\frac{4}{5}x\right) + 1.5$



9. 0.98 min $< t <$ 1.52 min, 3.48 min $< t <$ 4.02 min, 5.98 min $< t <$ 6.52 min



10. a) $y = 3.7\ \cancel{\sin}\ \cos\left(\frac{2\pi}{365}(x) - 172\right) + 12$

b) $y \doteq$ ~~13.87 hours~~ 9.16

11. $T(t) = 16.2 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 1.4,$ $0 < t < 111$ and $304 < t < 365$

$y = 16.2\cos\left[\frac{2\pi}{12}(x-7)\right] + 1.4$
using
$x = 1$ for
Jan

12. The student should graph the height of the nail above the ground as a function of the total distance travelled by the nail, because the nail would not be travelling at a constant speed. If the student graphed the height of the nail above the ground as a function of time, the graph would not be sinusoidal.

13. minute hand:

$D(t) = 15 \cos\left(\frac{\pi}{30}t\right) + 300;$

second hand:

$D(t) = 15 \cos(2\pi t) + 300;$

hour hand:

$D(t) = 8 \cos\left(\frac{\pi}{360}t\right) + 300$

Lesson 6.7, pp. 369–373

1. a) $0 < x < \pi, \pi < x < 2\pi$

b) $-\frac{\pi}{2} < x < \frac{\pi}{2}, \frac{3\pi}{2} < x < \frac{5\pi}{2}$

c) $\frac{\pi}{2} < x < \frac{3\pi}{2}, \frac{5\pi}{2} < x < 3\pi$

2. a) $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

b) $x = \frac{\pi}{2}, x = \frac{5\pi}{2}$

c) $x = 0, x = 2\pi$

3. 0

4. a) about 0.465
b) 0
c) about -0.5157
d) about -1.554

5. a) $0 < x < \frac{\pi}{2}, \pi < x < \frac{3\pi}{2}$

b) $0 < x < \frac{\pi}{4}, \pi < x < \frac{5\pi}{4}$

c) $\frac{\pi}{4} < x < \frac{\pi}{2}, \frac{5\pi}{4} < x < \frac{3\pi}{2}$

6. a) $x = \frac{1}{4}, x = \frac{3}{4}$

b) $x = 0, x = 1$

c) $x = \frac{1}{2}, x = \frac{3}{2}$

7. a) about -0.7459
b) about -1.310
c) 0

8. negative

9. a) $R(t) = 4.5 \cos\left(\frac{\pi}{12}(t - 12)\right) + 20.2$

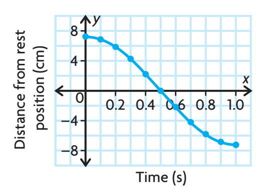
b) fastest: $t = 4$ months, $t = 16$ months, $t = 28$ months, $t = 40$ months; slowest: $t = 10$ months, $t = 22$ months, $t = 34$ months, $t = 46$ months, $t = 58$ months

c) about 1.164 mice per owl/s

10. a) i) 0.25 t/h
ii) about 0.2588 t/h
iii) 0.2612 t/h

b) The estimate calculated in part iii) is the most accurate. The smaller the interval, the more accurate the estimate.

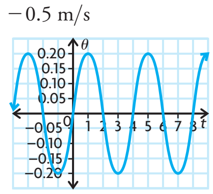
11. a)



b) half of one cycle
c) -14.4 cm/s
d) The bob is moving the fastest when it passes through its rest position. You can tell because the images of the balls are farthest apart at this point.
e) The pendulum's rest position is halfway between the maximum and minimum values on the graph. Therefore, at this point, the pendulum's instantaneous rate of change is at its maximum.

12. a) 0
b) -0.5 m/s

13. a)

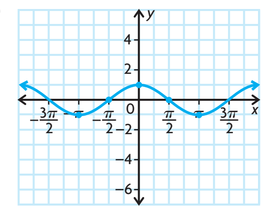


b) 0.2 radians/s
c) Answers may vary. For example, about $-\frac{2}{3}$ radians/s.
d) $t = 0, 2, 4, 6,$ and 8

14. Answers may vary. For example, for $x = 0$, the instantaneous rate of change of $f(x) = \sin x$ is approximately 0.9003, while the instantaneous rate of change of $f(x) = 3\sin x$ is approximately 2.7009. (The interval $-\frac{\pi}{4} < x < \frac{\pi}{4}$ was used.) Therefore, the instantaneous rate of change of $f(x) = 3\sin x$ is at its maximum three times more than the instantaneous rate of change of $f(x) = \sin x$. However, there are points where the instantaneous rate of change is the same for the two functions. For example, at $x = \frac{\pi}{2}$, it is 0 for both functions.

15. a) $-1, 0, 1, 0,$ and -1

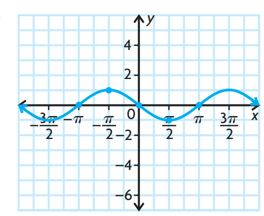
b)



The function is $f(x) = \cos x$. Based on this information, the derivative of $f(x) = \sin x$ is $\cos x$.

16. a) $0, 1, 0, -1,$ and 0

b)



The function is $f(x) = -\sin x$. Based on this information, the derivative of $f(x) = \cos x$ is $-\sin x$.

Chapter Review, pp. 376–377

1. $\frac{33}{16}$

2. 70π

3. a) $\frac{\pi}{9}$ radians

b) $\frac{-5\pi}{18}$ radians

c) $\frac{8\pi}{9}$ radians

d) $\frac{7\pi}{3}$ radians

4. a) 45°
b) $-225°$
c) 480°
d) $-120°$

5. a) $\frac{5\pi}{6}$
b) $\frac{4\pi}{3}$
c) $\frac{3\pi}{4}$
d) $\frac{7\pi}{6}$

6. a) $\tan\theta = \cancel{\frac{12}{5}}\ \pm 12/5$

b) $\sec\theta = -\frac{13}{5}$

c) about ~~[illegible]~~ 1.97 or 5.11

7. 2.00

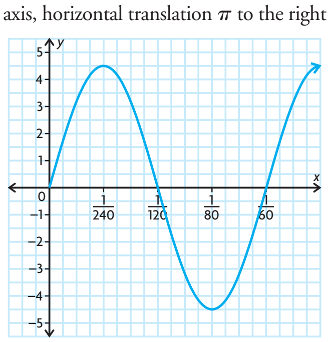
8. **a)** 2π radians
b) 2π radians
c) π radians

9. $y = 5\sin\left(x + \frac{\pi}{3}\right) + 2$

10. $y = -3\cos\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$

11. **a)** reflection in the x-axis, vertical stretch by a factor of 19, vertical translation 9 units down
b) horizontal compression by a factor of $\frac{1}{10}$, horizontal translation $\frac{\pi}{12}$ to the left
c) vertical compression by a factor of $\frac{10}{11}$, horizontal translation $\frac{\pi}{9}$ to the right, vertical translation 3 units up
d) reflection in the x-axis, reflection in the y-axis, horizontal translation π to the right

12. **a)**



b) $\frac{1}{60}$
c) $\frac{1}{240}$
d) $\frac{1}{80}$

13. **a)** 2π radians
b) 2π radians
c) π radians

14. **a)** the radius of the circle in which the bumblebee is flying
b) the time that the bumblebee takes to fly one complete circle
c) the height, above the ground, of the centre of the circle in which the bumblebee is flying
d) cosine function

15. $P(m) = 7250\cos\left(\frac{\pi}{6}m\right) + 7750$

16. $h(t) = 30\sin\left(\frac{5\pi}{3}t - \frac{\pi}{2}\right) + 150$

17. **a)** $0 < x < 5\pi$, $10\pi < x < 15\pi$
b) $2.5\pi < x < 7.5\pi$, $12.5\pi < x < 17.5\pi$
c) $0 < x < 2.5\pi$, $7.5\pi < x < 12.5\pi$

18. **a)** $x = 0$, $x = \frac{1}{2}$
b) $x = \frac{1}{8}$, $x = \frac{5}{8}$
c) $x = \frac{3}{8}$, $x = \frac{7}{8}$

19. **a)** $x = \frac{3}{4}$ s
b) the time between one beat of a person's heart and the next beat
c) 140
d) -129

Chapter Self-Test, p. 378

1. $y = \sec x$
2. $\sec 2\pi$
3. $y \doteq 108.5$
4. about 0.31 °C per day
5. $\frac{3\pi}{5}$, 110°, $\frac{5\pi}{8}$, 113°, and $\frac{2\pi}{3}$
6. $y = \sin\left(x + \frac{5\pi}{8}\right)$
7. $y \doteq -30$
8. **a)** $-3\cos\left(\frac{\pi}{12}x\right) + 22$
b) about 0.5 °C per hour
c) about 0 °C per hour

Cumulative Review Chapters 4–6, pp. 380–383

1.	(d)	**9.**	(c)	**17.**	(d)	**25.**	(b)
2.	(b)	**10.**	(c)	**18.**	(b)	**26.**	(d)
3.	(a)	**11.**	(d)	**19.**	(b)	**27.**	(a)
4.	(c)	**12.**	(a)	**20.**	(b)	**28.**	(c)
5.	(a)	**13.**	(d)	**21.**	(d)	**29.**	(b)
6.	(b)	**14.**	(c)	**22.**	(c)		
7.	(a)	**15.**	(d)	**23.**	(a)		
8.	(c)	**16.**	(a)	**24.**	(d)		

30. **a)** If x is the length in centimetres of a side of one of the corners that have been cut out, the volume of the box is $(50 - 2x)(40 - 2x)x$ cm^3.
b) 5 cm or 10 cm
c) $x \doteq 7.4$ cm
d) $3 < x < 12.8$

31. **a)** The zeros of $f(x)$ are $x = 2$ or $x = 3$. The zero of $g(x)$ is $x = 3$. The zero of $\frac{f(x)}{g(x)}$ is $x = 2$. $\frac{g(x)}{f(x)}$ does not have any zeros.
b) $\frac{f(x)}{g(x)}$ has a hole at $x = 3$; no asymptotes. $\frac{g(x)}{f(x)}$ has an asymptote at $x = 2$ and $y = 0$.
c) $x = 1$; $\frac{f(x)}{g(x)}$: $y = x - 2$, $\frac{g(x)}{f(x)}$: $y = -x$

32. **a)** Vertical compressions and stretches do not affect location of zeros; maximum and minimum values are multiplied by the scale factor, but locations are unchanged; instantaneous rates of change are multiplied by the scale factor.
Horizontal compressions and stretches move locations of zeros, maximums, and minimums toward or away from the y-axis by the reciprocal of the scale factor; instantaneous rates of change are multiplied by the reciprocal of scale factor.
Vertical translations change location of zeros or remove them; maximum and minimum values are increased or decreased by the amount of the translation, but locations are unchanged; instantaneous rates of change are unchanged.
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b) For $y = \cos x$, the answer is the same as in part **a)**, except that a horizontal reflection does not affect instantaneous rates of change. For $y = \tan x$, the answer is also the same as in part **a)**, except that nothing affects the maximum and minimum values, since there are no maximum or minimum values for $y = \tan x$.

Chapter 7

Getting Started, p. 386

1. **a)** 1
b) $-\frac{22}{7}$
c) 8 or -3
d) $\frac{2}{3}$ or $-\frac{5}{2}$
e) $-1 \pm \sqrt{2}$
f) $\frac{3 \pm \sqrt{21}}{6}$

2. To do this, you must show that the two distances are equal:

$$D_{AB} = \sqrt{(2-1)^2 + \left(\tfrac{1}{2} - 0\right)^2} = \frac{\sqrt{5}}{2};$$

$$D_{CD} = \sqrt{\left(0 - \tfrac{1}{2}\right)^2 + (6-5)^2} = \frac{\sqrt{5}}{2}.$$

Since the distances are equal, the line segments are the same length.

3. **a)** $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, $\csc A = \frac{17}{8}$, $\sec A = \frac{17}{15}$, $\cot A = \frac{15}{8}$
b) 0.5 radians
c) 61.9°

8. **a)** 2π radians
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c) π radians

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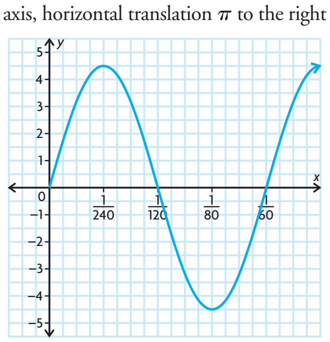

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