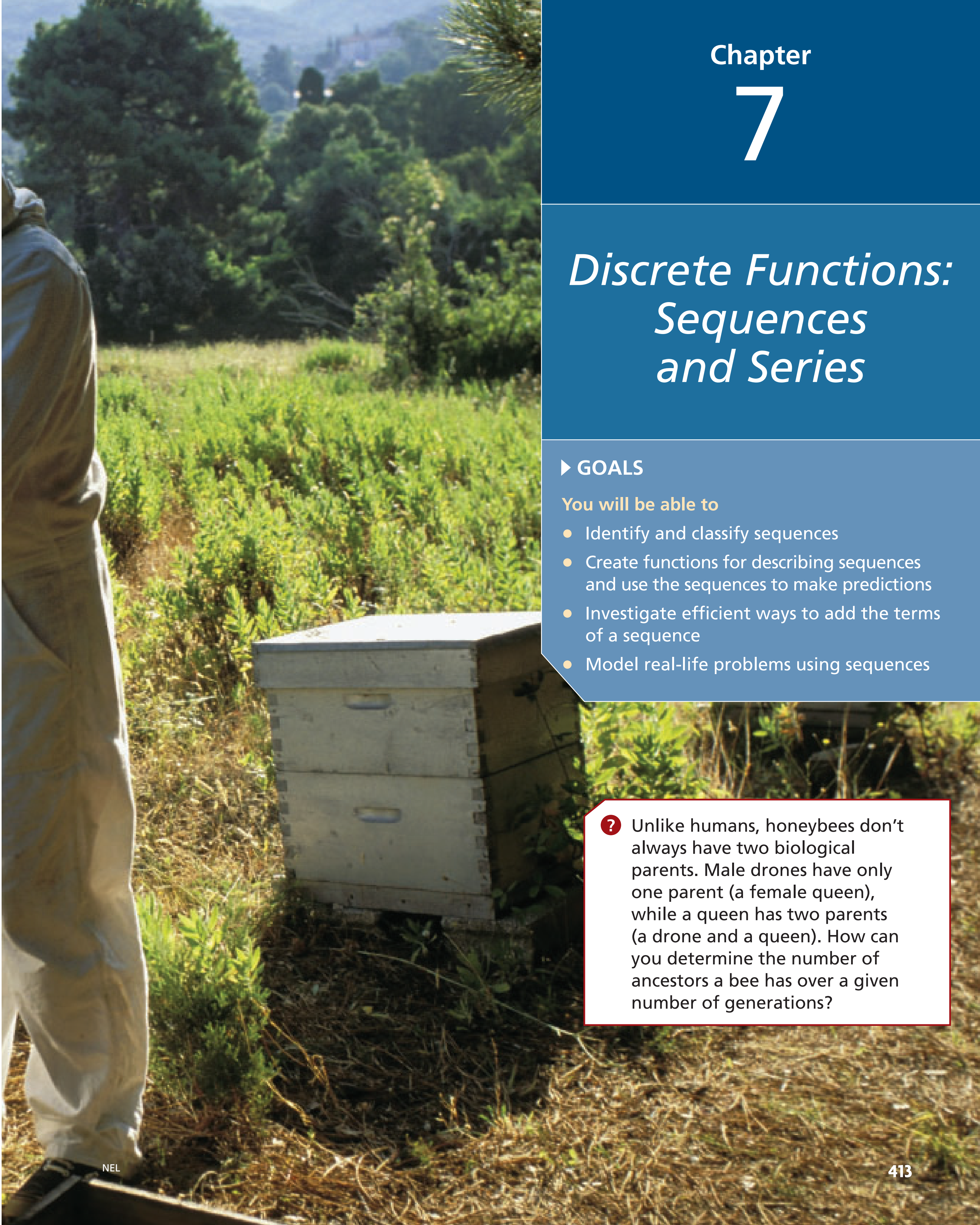

1st generation
(1 ancestor)
2nd generation
(1 ancestor)
3rd generation
(2 ancestors)
4th generation
(3 ancestors)
5th generation
(5 ancestors)
6th generation
(8 ancestors)
drone
queen



Chapter
7

Discrete Functions: Sequences and Series

GOALS

You will be able to

- Identify and classify sequences
- Create functions for describing sequences and use the sequences to make predictions
- Investigate efficient ways to add the terms of a sequence
- Model real-life problems using sequences

? Unlike humans, honeybees don't always have two biological parents. Male drones have only one parent (a female queen), while a queen has two parents (a drone and a queen). How can you determine the number of ancestors a bee has over a given number of generations?

7

Getting Started

SKILLS AND CONCEPTS You Need

1. State the equation of each line.

a) slope $= -\frac{2}{5}$ and y-intercept $= 8$

b) slope $= -9$ and passing through $(5, 4)$

c) x-intercept $= 5$ and y-intercept $= -7$

d) passing through $(-12, 17)$ and $(5, -17)$

Study **Aid**

- For help with Question 2, see Essential Skills Appendix, A-7.

2. Evaluate.

a) $g(-2)$, for $g(x) = 3x^2 + x - 4$

b) $f\left(\frac{3}{4}\right)$, for $f(x) = \frac{4}{5}x + \frac{7}{10}$

c) $g(\sqrt{6})$, for $g(x) = 4x^2 - 24$

d) $h\left(\frac{1}{3}\right)$, for $h(x) = 64^x$

3. Calculate the 1st and 2nd differences to determine whether each relation is linear, quadratic, or neither.

a)

x	$f(x)$
0	18
1	23
2	28
3	33
4	38

b)

x	$f(x)$
0	6
6	12
12	24
18	48
24	96

c)

x	$f(x)$
0	5
1	7
2	13
3	23
4	37

Study **Aid**

- For help with Questions 5 and 6, see Lesson 4.7, Examples 2 and 3.

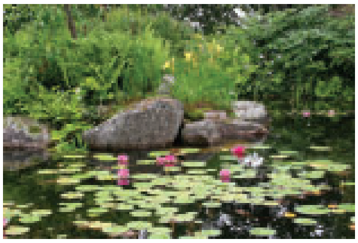
4. Solve each equation.

a) $2x - 3 = 7$

b) $5x + 8 = 2x - 7$

c) $5(3x - 2) + 7x - 4 = 2(4x + 8) - 2x + 3$

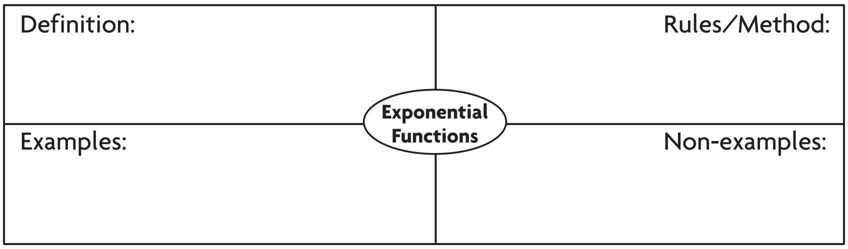
d) $-8x + \frac{3}{4} = \frac{1}{3}x - 12$



5. A radioactive material has a **half-life** of 100 years. If 2.3 kg of the substance is placed in a special disposal container, how much of the radioactive material will remain after 1000 years?

6. 0.1% of a pond is covered by lily pads. Each week the number of lily pads doubles. What percent of the pond will be covered after nine weeks?

7. Complete the chart to show what you know about exponential functions.



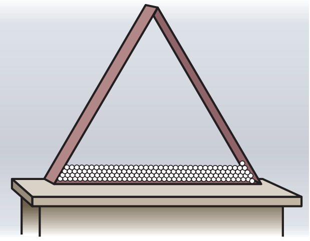
APPLYING *What You Know*

YOU WILL NEED

- counters or coins
- graphing calculator
- spreadsheet software (optional)

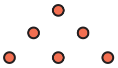
Stacking Golf Balls

George's Golf Garage is having a grand-opening celebration. A display involves constructing a stack of golf balls within a large equilateral triangle frame on one of the walls. The base of the triangle will contain 40 golf balls, with each stacked row using one less ball than the previous row.



? How many golf balls are needed to construct the triangle?

A. Use counters or coins to construct a series of equilateral triangles with side lengths 1, 2, 3, 4, 5, 6, and 7, respectively. Record the total number of counters used to make each triangle in a table.

Side Length	Diagram	Number of Counters Used
1		1
2		3
3		6
4		

B. Create a scatter plot of number of counters versus side length. Then determine the 1st differences between the numbers of counters used. What does this tell you about the type of function that models the number of balls needed to create an equilateral triangle?

C. How is the triangle with side length

- 4 related to the triangle with side length 2?
- 6 related to the triangle with side length 3?
- $2n$ related to the triangle with side length n?

D. Repeat part C for triangles with side lengths of

- 5 and 2
- 7 and 3
- $2n + 1$ and n

E. Use your rules from parts C and D to determine the number of golf balls in a triangle with side length 40.

7.1

Arithmetic Sequences

YOU WILL NEED

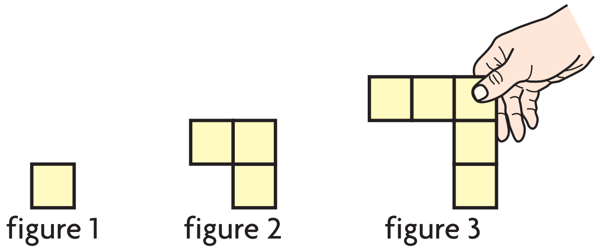
- linking cubes
- graphing calculator or graph paper
- spreadsheet software

GOAL

Recognize the characteristics of arithmetic sequences, and express the general terms in a variety of ways.

INVESTIGATE the Math

Chris used linking cubes to create different shapes. The first three shapes are shown. He wrote the **sequence** that represents the number of cubes in each shape.



? How many linking cubes are there in the 100th figure?

A. Create the next three **terms** of Chris's **arithmetic sequence.**

B. How is each term of this **recursive sequence** related to the previous term?

C. Construct a graph of term (number of cubes) versus figure number. What type of relation is this?

D. Determine a formula for the **general term** of the sequence.

E. Use the general term to calculate the 100th term.

Reflecting

F. Chris's sequence is an arithmetic sequence. Another arithmetic sequence is 635, 630, 625, 620, 615, ... How are the two sequences similar? Different?

G. How does the definition of an arithmetic sequence help you predict the shape of the graph of the sequence?

H. A **recursive formula** for Chris's sequence is $t_1 = 1$, $t_n = t_{n-1} + 2$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of Chris's arithmetic sequence?

sequence

an ordered list of numbers

term

a number in a sequence. Subscripts are usually used to identify the positions of the terms.

arithmetic sequence

a sequence that has the same difference, the **common difference**, between any pair of consecutive terms

recursive sequence

a sequence for which one term (or more) is given and each successive term is determined from the previous term(s)

general term

a formula, labelled t_n, that expresses each term of a sequence as a function of its position. For example, if the general term is $t_n = 2n$, then to calculate the 12th term (t_{12}), substitute $n = 12$.

$$\begin{aligned} t_{12} &= 2(12) \\ &= 24 \end{aligned}$$

recursive formula

a formula relating the general term of a sequence to the previous term(s)

APPLY the Math

EXAMPLE 1 Representing the general term of an arithmetic sequence

a) Determine a formula that defines the arithmetic sequence 3, 12, 21, 30, ...

b) State a formula that defines each term of any arithmetic sequence.

Wanda's Solution: Using Differences

a) $12 - 3 = 9$

I knew that the sequence is arithmetic, so the terms increase by the same amount. I subtracted t_1 from t_2 to determine the common difference.

$$\begin{aligned} t_n &= 3 + (n - 1)(9) \\ &= 3 + 9n - 9 \\ &= 9n - 6 \end{aligned}$$

I wrote this sequence as 3, 3 + 9, 3 + 2(9), 3 + 3(9), ... Each multiple of 9 that I added was one less than the position number. So for the *n*th term, I needed to add $(n - 1)$ 9s.

The general term is $t_n = 9n - 6$.

b) $a, a + d, (a + d) + d, (a + 2d) + d, \ldots$

$= a, a + d, a + 2d, a + 3d, \ldots$

I wrote an arithmetic sequence using a general first term, *a*, and a common difference, *d*. I simplified by collecting like terms.

General term:

$t_n = a + (n - 1)d$

Each multiple of *d* that I added was one less than the position number. So I knew that I had a formula for the general term.

Nathan's Solution: Using Multiples of 9

a) $12 = 3 + 9$

Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

$21 = 12 + 9$

$30 = 21 + 9$

$t_n = 9n$

Since I added 9 each time, I thought about the sequence of multiples of 9 because each term of that sequence goes up by 9s.

9, 18, 27, 36, ...

3, 12, 21, 30, ...

Each term of my sequence is 6 less than the term in the same position in the sequence of multiples of 9.

The general term is $t_n = 9n - 6$.

The general term of the sequence of multiples of 9 is 9*n*, so I subtracted 6 to get the general term of my sequence.

b) $t_n = nd$

$d, 2d, 3d, 4d, \ldots$

Since I added the common difference *d* each time, I thought about the sequence of multiples of *d*.

$d + (a - d), 2d + (a - d), 3d + (a - d), 4d + (a - d), \ldots$

But the first term of my sequence was *a*, not *d*. So to get my sequence, I had to add *a* to, and subtract *d* from, each term of the sequence of multiples of *d*.

General term:

$$\begin{aligned} t_n &= nd + (a - d) \\ &= a + nd - d \\ &= a + (n - 1)d \end{aligned}$$

I simplified to get the general term.

Tina's Solution: Using a Recursive Formula

a) $12 = 3 + 9$

$21 = 12 + 9$

$30 = 21 + 9$

Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

The recursive formula is

$t_1 = 3, t_n = t_{n-1} + 9$, where $n \in \mathbf{N}$ and $n > 1$.

Since I added 9 each time, I expressed the general term of the sequence using a recursive formula.

b) $a, a + d, a + 2d, a + 3d, \ldots$

To get the terms of any arithmetic sequence, I would add *d* to the previous term each time, where *a* is the first term.

Recursive formula:

$t_1 = a, t_n = t_{n-1} + d$, where $n \in \mathbf{N}$ and $n > 1$

Once you know the general term of an arithmetic sequence, you can use it to determine *any* term in the sequence.

EXAMPLE 2 Connecting a specific term to the general term of an arithmetic sequence

What is the 33rd term of the sequence 18, 11, 4, -3, ...?

David's Solution: Using Differences and the General Term

$11 - 18 = -7$

$4 - 11 = -7$

$-3 - 4 = -7$

I subtracted consecutive terms and found that each term is 7 less than the previous term. So the sequence is arithmetic.

$a = 18, d = -7$

$t_n = a + (n - 1)d$

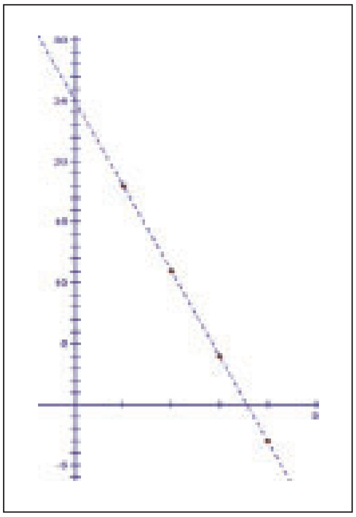
The first term of the sequence is 18. Since the terms are decreasing, the common difference is -7.

$$t_{33} = 18 + (33 - 1)(-7)$$
$$= -206$$

I substituted these numbers into the formula for the general term of an arithmetic sequence. To get the 33rd term, I let $n = 33$.

The 33rd term is -206.

Leila's Solution: Using a Graph and Function Notation



I represented the sequence as a function using ordered pairs with the term number (n) as the x-coordinate and the term (t_n) as the y-coordinate.

n	t_n	1st Differences
1	18	-7
2	11	-7
3	4	-7
4	-3	

The 1st differences are constant so these points lie on a line.

Since $n \in \mathbf{N}$ and the terms lie on this line, I used a dashed line to connect the points.

Communication | ***Tip***

A dashed line on a graph indicates that the x-coordinates of the points on the line are natural numbers.

$f(x) = -7x + 25$

The slope of the line is $m = -7$ and the y-intercept is $b = 25$. I used this information to write the function that describes the line.

The y-intercept corresponds to the term t_0 but it is *not* a term of the sequence since $x \in \mathbf{N}$.

$$f(33) = -7(33) + 25$$
$$= -206$$

To get the 33rd term of the sequence, I substituted $x = 33$ into the equation $f(x) = -7x + 25$.

The 33rd term is -206.

Arithmetic sequences can be used to model problems that involve increases or decreases that occur at a constant rate.

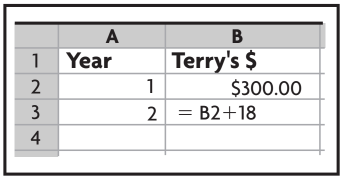
EXAMPLE 3 Representing an arithmetic sequence

Terry invests $300 in a GIC (guaranteed investment certificate) that pays 6% simple interest per year. When will his investment be worth $732?

Philip's Solution: Using a Spreadsheet

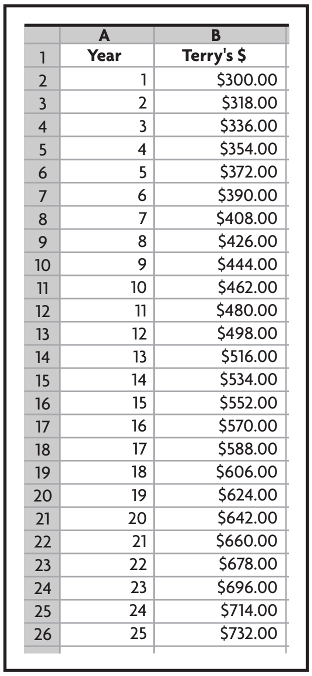
6% of $300 = $18

Terry earns 6% of $300, which is $18 interest per year.



	A	B
1	Year	Terry's $
2	1	$300.00
3	2	= B2+18
4		

I set up a spreadsheet. In one column I entered the year number, and in the other column, I entered the amount. I set up a formula to increase the amount by $18 per year.



	A	B
1	Year	Terry's $
2	1	$300.00
3	2	$318.00
4	3	$336.00
5	4	$354.00
6	5	$372.00
7	6	$390.00
8	7	$408.00
9	8	$426.00
10	9	$444.00
11	10	$462.00
12	11	$480.00
13	12	$498.00
14	13	$516.00
15	14	$534.00
16	15	$552.00
17	16	$570.00
18	17	$588.00
19	18	$606.00
20	19	$624.00
21	20	$642.00
22	21	$660.00
23	22	$678.00
24	23	$696.00
25	24	$714.00
26	25	$732.00

I used the spreadsheet to continue the pattern until the amount reached $732.

Tech Support

For help using a spreadsheet to enter values and formulas, fill down, and fill right, see Technical Appendix, B-21.

From the spreadsheet, Terry's investment will be worth $732 at the beginning of the 25th year.

Jamie's Solution: Using the General Term

$$t_n = a + (n - 1)d$$

$$t_n = 300 + (n - 1)(18)$$

Terry earns 6% of $300, or $18 interest, per year. So his investment increases by $18/year. This is an arithmetic sequence, where $a = 300$ and $d = 18$.

$$732 = 300 + (n - 1)(18)$$

I needed to determine when $t_n = 732$.

$$\begin{aligned} 732 &= 300 + 18n - 18 \\ 732 - 300 + 18 &= 18n \\ 450 &= 18n \\ 25 &= n \end{aligned}$$

I solved for n.

Terry's investment will be worth $732 in the 25th year.

Suzie's Solution: Using Reasoning

$$a = 300, d = 18$$

Terry earns 6% of $300 = $18 interest per year. So the amount at the start of each year will form an arithmetic sequence.

$$732 - 300 = 432$$

I calculated the difference between the starting and ending values to know how much interest was earned.

$$432 \div 18 = 24$$

I divided by the amount of interest paid per year to determine how many interest payments were made.

The investment will be worth $732 at the beginning of the 25th year.

Since interest was paid every year except the first year, $732 must occur in the 25th year.

If you know two terms of an arithmetic sequence, you can determine *any* term in the sequence.

EXAMPLE 4 Solving an arithmetic sequence problem

The 7th term of an arithmetic sequence is 53 and the 11th term is 97. Determine the 100th term.

Tanya's Solution: Using Reasoning

$$\begin{aligned} t_{11} - t_7 &= 97 - 53 \\ &= 44 \end{aligned}$$

I knew that the sequence is arithmetic, so the terms increase by the same amount each time.

$$\begin{aligned} 4d &= 44 \\ d &= 11 \end{aligned}$$

There are four differences to go from t_7 to t_{11}. So I divided 44 by 4 to get the common difference.

$$\begin{aligned} t_{100} &= 97 + 89 \times 11 \\ &= 97 + 979 \\ &= 1076 \end{aligned}$$

Since the common difference is 11, I knew that to get the 100th term, I would have to add it to t_{11} 89 times.

The 100th term is 1076.

Deepak's Solution: Using Algebra

$$t_n = a + (n - 1)d$$

I knew that the sequence is arithmetic, so I wrote the formula for the general term.

$$\begin{array}{ll} t_7 & t_{11} \\ 53 = a + (7 - 1)d & 97 = a + (11 - 1)d \\ 53 = a + 6d & 97 = a + 10d \end{array}$$

For the 7th term, I substituted $t_7 = 53$ and $n = 7$ into the general term. For the 11th term, I substituted $t_{11} = 97$ and $n = 11$. Since both equations describe terms of the same arithmetic sequence, a and d are the same in both equations.

$$\begin{array}{r} 97 = a + 10d \\ -53 = -(a + 6d) \\ \hline 44 = 4d \\ 11 = d \end{array}$$

The equations for t_7 and t_{11} represent a linear system. To solve for d, I subtracted the equation for t_7 from the equation for t_{11}.

$$53 = a + 6(11)$$
$$53 = a + 66$$
$$-13 = a$$

To solve for a, I substituted $d = 11$ into the equation for t_7.

$$t_n = a + (n - 1)d$$
$$t_{100} = -13 + (100 - 1)(11)$$
$$= 1076$$

To get the 100th term, I substituted $a = -13$, $d = 11$, and $n = 100$ into the formula for the general term.

The 100th term is 1076.

In Summary

Key Ideas

- Every sequence is a discrete function. Since each term is identified by its position in the list (1st, 2nd, and so on), the domain is the set of natural numbers, **N** = {1, 2, 3, ...}. The range is the set of all the terms of the sequence.
  For example, 4, 12, 20, 28, ...
  ↗ 1st term ↖ 2nd term

  Domain: {1, 2, 3, 4, ...}
  Range: {4, 12, 20, 28, ...}

- An arithmetic sequence is a recursive sequence in which new terms are created by adding the same value (the common difference) each time.
  For example, 2, 6, 10, 14, ... is increasing with a common difference of 4,
  +4 +4 +4

  $$t_2 - t_1 = 6 - 2 = 4$$
  $$t_3 - t_2 = 10 - 6 = 4$$
  $$t_4 - t_3 = 14 - 10 = 4$$
  $$\vdots$$

  and 9, 6, 3, 0, ... is decreasing with a common difference of −3.
  −3 −3 −3

  $$t_2 - t_1 = 6 - 9 = -3$$
  $$t_3 - t_2 = 3 - 6 = -3$$
  $$t_4 - t_3 = 0 - 3 = -3$$
  $$\vdots$$

Need to Know

- An arithmetic sequence can be defined
  - by the general term $t_n = a + (n - 1)d$,
  - recursively by $t_1 = a$, $t_n = t_{n-1} + d$, where $n > 1$, or
  - by a discrete linear function $f(n) = dn + b$, where $b = t_0 = a - d$.

  In all cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference.

CHECK Your Understanding

1. Determine which sequences are arithmetic. For those that are, state the common difference.

a) 1, 5, 9, 13, 17, ...
b) 3, 7, 13, 17, 23, 27, ...
c) 3, 6, 12, 24, ...
d) 59, 48, 37, 26, 15, ...

2. State the general term and the recursive formula for each arithmetic sequence.

a) 28, 42, 56, ... **b)** 53, 49, 45, ... **c)** $-1, -111, -221, \ldots$

3. The 10th term of an arithmetic sequence is 29 and the 11th term is 41. What is the 12th term?

4. What is the 15th term of the arithmetic sequence 85, 102, 119, ...?

PRACTISING

5. **i)** Determine whether each sequence is arithmetic.
ii) If a sequence is arithmetic, state the general term and the recursive formula.

a) 8, 11, 14, 17, ...
b) 15, 16, 18, 19, ...
c) 13, 31, 13, 31, ...
d) 3, 6, 12, 24, ...
e) 23, 34, 45, 56, ...
f) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$

6. Determine the recursive formula and the general term for the arithmetic sequence in which

a) the first term is 19 and consecutive terms increase by 8
b) $t_1 = 4$ and consecutive terms decrease by 5
c) the first term is 21 and the second term is 26
d) $t_4 = 35$ and consecutive terms decrease by 12

7. **i)** Determine whether each recursive formula defines an arithmetic sequence, where $n \in \mathbf{N}$ and $n > 1$.
ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_1 = 13, t_n = 14 + t_{n-1}$
b) $t_1 = 5, t_n = 3t_{n-1}$
c) $t_1 = 4, t_n = t_{n-1} + n - 1$
d) $t_1 = 1, t_n = 2t_{n-1} - n + 2$

8. For each arithmetic sequence, determine

K **i)** the general term **ii)** the recursive formula **iii)** t_{11}

a) 35, 40, 45, ...
b) 31, 20, 9, ...
c) $-29, -41, -53, \ldots$
d) 11, 11, 11, ...
e) $1, \frac{6}{5}, \frac{7}{5}, \ldots$
f) 0.4, 0.57, 0.74, ...

9. **i)** Determine whether each general term defines an arithmetic sequence.
ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_n = 8 - 2n$

b) $t_n = n^2 - 3n + 7$

c) $f(n) = \frac{1}{4}n + \frac{1}{2}$

d) $f(n) = \frac{2n + 5}{7 - 3n}$

10. **A** An opera house has 27 seats in the first row, 34 seats in the second row, 41 seats in the third row, and so on. The last row has 181 seats.

a) How many seats are in the 10th row?
b) How many rows of seats are in the opera house?

11. Janice gets a job and starts out earning \$9.25/h. Her boss promises her a raise of \$0.15/h after each month of work. When will Janice start earning at least twice her starting wage?

12. Phil invests \$5000 in a high-interest savings account and earns 3.5% simple interest per year. How long will he have to leave his money in the account if he wants to have \$7800?

13. Determine the number of terms in each arithmetic sequence.

a) 7, 9, 11, 13, ... , 63
b) $-20, -25, -30, -35, \ldots, -205$
c) $31, 27, 23, 19, \ldots, -25$
d) 9, 16, 23, 30, ... , 100
e) $-33, -26, -19, -12, \ldots, 86$
f) $28, 19, 10, 1, \ldots, -44$

14. **T** You are given the 4th and 8th terms of a sequence. Describe how to determine the 100th term *without* finding the general term.

15. The 50th term of an arithmetic sequence is 238 and the 93rd term is 539. State the general term.

16. **C** Two terms of an arithmetic sequence are 20 and 50.

a) Create three different arithmetic sequences given these terms. Each of the three sequences should have a different first term and a different common difference.
b) How are the common differences related to the terms 20 and 50?

Extending

17. The first term of an arithmetic sequence is 13. Two other terms of the sequence are 37 and 73. The common difference between consecutive terms is an integer. Determine all possible values for the 100th term.

18. Create an arithmetic sequence that has $t_1 > 0$ and in which each term is greater than the previous term. Create a new sequence by picking, from the original sequence, the terms described by the sequence. (For example, for the sequence 3, 7, 11, 15, ... , you would choose the 3rd, 7th, 11th, 15th, ... terms of the original sequence as $t_1, t_2, t_3, t_4, \ldots$ of your new sequence.) Is this new sequence always arithmetic?

7.2

Geometric Sequences

YOU WILL NEED

- graphing calculator
- graph paper

GOAL

Recognize the characteristics of geometric sequences and express the general terms in a variety of ways.

INVESTIGATE the Math

A local conservation group set up a challenge to get trees planted in a community. The challenge involves each person planting a tree and signing up seven other people to each do the same. Denise and Lise both initially accepted the challenge.

? If the pattern continues, how many trees will be planted at the 10th stage?

geometric sequence

a sequence that has the same ratio, the **common ratio**, between any pair of consecutive terms

A. Create the first five terms of the geometric sequence that represents the number of trees planted at each stage.

B. How is each term of this recursive sequence related to the previous term?

C. Use a graphing calculator to graph the term (number of trees planted) versus stage number. What type of relation is this?

D. Determine a formula for the general term of the sequence.

E. Use the general term to calculate the 10th term.

Reflecting

F. The tree-planting sequence is a geometric sequence. Another geometric sequence is 1 000 000, 500 000, 250 000, 125 000, ... How are the two sequences similar? Different?

G. How is the general term of a geometric sequence related to the equation of its graph?

H. A recursive formula for the tree-planting sequence is $t_1 = 2, t_n = 7t_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of this geometric sequence?

APPLY the Math

EXAMPLE 1 Connecting a specific term to the general term of a geometric sequence

a) Determine the 13th term of a geometric sequence if the first term is 9 and the common ratio is 2.

b) State a formula that defines each term of any geometric sequence.

Leo's Solution: Using a Recursive Formula

a)

n	1	2	3	4	5	6	7
t_n	9	18	36	72	144	288	576

n	8	9	10	11	12	13
t_n	1152	2304	4608	9216	18 432	36 864

I knew that the sequence is geometric so the terms increase by the same multiple each time. I made a table starting with the first term, and I multiplied each term by 2 to get the next term until I got the 13th term.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

To get the terms of any geometric sequence, I would multiply the previous term by *r* each time, where *a* is the first term.

Recursive formula:

$t_1 = a, t_n = rt_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$

Tamara's Solution: Using Powers of *r*

a) $a = 9$

$r = 2$

I knew that the sequence is geometric with first term 9 and common ratio 2.

$$t_{13} = 9 \times 2^{12}$$
$$= 36\ 864$$

To get the 13th term, I started with the first term. Then I multiplied the common ratio 12 times.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

$= a, ar, ar^2, ar^3, \ldots$

I wrote a geometric sequence using a general first term, *a*, and a common ratio, *r*. I simplified the terms.

General term:

$t_n = ar^{n-1}$ or $f(n) = ar^{n-1}$

Each time I multiplied by *r*, the result was one less than the position number. I recognized this as an exponential function, so I knew that I had a formula for the general term.

Geometric sequences can be used to model problems that involve increases or decreases that change exponentially.

EXAMPLE 2 Solving a problem by using a geometric sequence

A company has 3 kg of radioactive material that must be stored until it becomes safe to the environment. After one year, 95% of the radioactive material remains. How much radioactive material will be left after 100 years?

Jacob's Solution

$$3, 3 \times 0.95, (3 \times 0.95) \times 0.95, (3 \times 0.95^2) \times 0.95, \ldots$$
$$= 3, 3 \times 0.95, 3 \times 0.95^2, 3 \times 0.95^3, \ldots$$

Every year, 95% of the radioactive material remains. I represented the amount of radioactive material as a sequence. The terms show the amounts in each year.

$$a = 3$$
$$r = 0.95$$

The sequence is geometric with first term 3 and common ratio 0.95.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term.

$$\begin{aligned} f(100) &= 3 \times 0.95^{100-1} \\ &= 3 \times 0.95^{99} \\ &\doteq 0.019 \end{aligned}$$

I needed to determine the value of $f(n)$ when $n = 100$. So I substituted $a = 3$, $r = 0.95$, and $n = 100$ into the formula.

After the 100th year, there will be about 19 g of radioactive material left.

EXAMPLE 3 Selecting a strategy to determine the number of terms in a geometric sequence

How many terms are in the geometric sequence 52 612 659, 17 537 553, ... , 11?

Suzie's Solution

$$a = 52\,612\,659$$
$$r = \frac{17\,537\,553}{52\,612\,659} = \frac{1}{3}$$

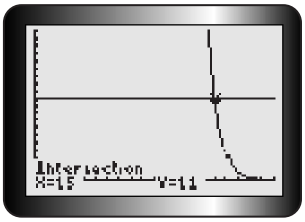
I knew that the sequence is geometric with first term 52 612 659. I calculated the common ratio by dividing t_2 by t_1.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term of a geometric sequence.

$$11 = 52\,612\,659 \times \left(\frac{1}{3}\right)^{n-1}$$

The last term of the sequence is 11, so its position number will be equal to the number of terms in the sequence. I determined the value of n when $f(n) = 11$ by substituting $a = 52\,612\,659$, $r = \frac{1}{3}$, and $f(n) = 11$ into the formula.



Intersection
X=15 Y=11

Instead of using guess and check to determine n, I graphed the functions Y1 = 52 612 659(1/3)^(X−1) and Y2 = 11 using my graphing calculator. Then I found the point of intersection. The x-coordinate represents the number of terms in the sequence.

There are 15 terms in the geometric sequence.

Tech | *Support*

For help using a graphing calculator to determine the point of intersection of two functions, see Technical Appendix, B-12.

In Summary

Key Idea

- A geometric sequence is a recursive sequence in which new terms are created by multiplying the previous term by the same value (the common ratio) each time. For example, 2, 6, 18, 54, ... is increasing with a common ratio of 3,

  × 3 × 3 × 3

$$\frac{t_2}{t_1} = \frac{6}{2} = 3$$
$$\frac{t_3}{t_2} = \frac{18}{6} = 3$$
$$\frac{t_4}{t_3} = \frac{54}{18} = 3$$
$$\vdots$$

and 144, 72, 36, 18, ... is decreasing with a common ratio of $\frac{1}{2}$.

$\times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$$\frac{t_2}{t_1} = \frac{72}{144} = \frac{1}{2}$$
$$\frac{t_3}{t_2} = \frac{36}{72} = \frac{1}{2}$$
$$\frac{t_4}{t_3} = \frac{18}{36} = \frac{1}{2}$$
$$\vdots$$

If the common ratio is negative, the sequence has terms that alternate from positive to negative. For example, 5, −20, 80, −320, ... has a common ratio of −4.

× (−4) × (−4) × (−4)

Need to Know

- A geometric sequence can be defined
  - by the general term $t_n = ar^{n-1}$,
  - recursively by $t_1 = a$, $t_n = rt_{n-1}$, where $n > 1$, or
  - by a discrete exponential function $f(n) = ar^{n-1}$.

  In all cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.

CHECK Your Understanding

1. Determine which sequences are geometric. For those that are, state the common ratio.

a) 15, 26, 37, 48, ...
b) 5, 15, 45, 135, ...
c) 3, 9, 81, 6561, ...
d) 6000, 3000, 1500, 750, 375, ...

2. State the general term and the recursive formula for each geometric sequence.

a) 9, 36, 144, ...
b) 625, 1250, 2500, ...
c) 10 125, 6750, 4500, ...

3. The 31st term of a geometric sequence is 123 and the 32nd term is 1107. What is the 33rd term?

4. What is the 10th term of the geometric sequence 1 813 985 280, 302 330 880, 50 388 480, ...?

PRACTISING

5. **i)** Determine whether each sequence is geometric.
ii) If a sequence is geometric, state the general term and the recursive formula.

a) 12, 24, 48, 96, ...
b) 1, 3, 7, 15, ...
c) 3, 6, 9, 12, ...
d) 5, -15, 45, -135, ...
e) 6, 7, 14, 15, ...
f) 125, 50, 20, 8, ...

6. For each geometric sequence, determine
K **i)** the general term **ii)** the recursive formula **iii)** t_6

a) 4, 20, 100, ...
b) -11, -22, -44, ...
c) 15, -60, 240, ...
d) 896, 448, 224, ...
e) 6, 2, $\frac{2}{3}$, ...
f) 1, 0.2, 0.04, ...

7. **i)** Determine whether each sequence is arithmetic, geometric, or neither.
ii) If a sequence is arithmetic or geometric, state the general term.

a) 9, 13, 17, 21, ...
b) 7, -21, 63, -189, ...
c) 18, -18, 18, -18, ...
d) 31, 32, 34, 37, ...
e) 29, 19, 9, -1, ...
f) 128, 96, 72, 54, ...

8. Determine the recursive formula and the general term for the geometric sequence in which

a) the first term is 19 and the common ratio is 5
b) $t_1 = -9$ and $r = -4$
c) the first term is 144 and the second term is 36
d) $t_1 = 900$ and $r = \frac{1}{6}$

9. **i)** Determine whether each recursive formula defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_1 = 18, t_n = \left(\frac{2}{3}\right)^{n-1} t_{n-1}$, where $n > 1$

b) $t_1 = -8, t_n = -3t_{n-1}$, where $n > 1$

c) $t_1 = 123, t_n = \frac{t_{n-1}}{3}$, where $n > 1$

d) $t_1 = 10, t_2 = 20, t_n = 4t_{n-2}$, where $n > 2$

10. **i)** Determine whether each general term defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_n = 4^n$

b) $t_n = 3^n + 5$

c) $f(n) = n^2 - 13n + 8$

d) $t_n = 7 \times (-5)^{n-4}$

e) $f(n) = \frac{2}{3n + 1}$

f) $f(n) = \frac{11}{13^n}$

11. The 5th term of a geometric sequence is 45 and the 8th term is 360. Determine the 20th term.

12. **A** A doctor makes observations of a bacterial culture at fixed time intervals. The table below shows his first four observations. If the pattern continues, how many bacteria will be present at the 9th observation?

Observation	Number of Bacteria
1	5 120
2	7 680
3	11 520
4	17 280

13. Sam invested $5000 in a GIC earning 8% compound interest per year. The interest gets added to the amount invested, so the next year Sam gets interest on the interest already earned, as well as on the original amount. How much will Sam's investment be worth at the end of 10 years?

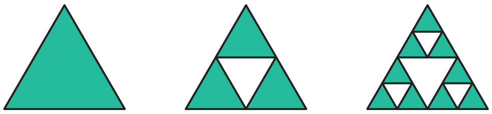
14. A certain antibiotic reduces the number of bacteria in your body by 10% each dose.

a) If four doses of the antibiotic are taken, what percent of the original bacterial population is left?

b) Biologists have determined that when a person has a bacterial infection, if the bacterial level can be reduced to 5% of its initial population, the person can fight off the infection. How many doses must be administered to reduce the bacterial population to the desired level?

15. You are given the 5th and 7th terms of a geometric sequence. Is it possible to determine the 29th term *without* finding the general term? If so, describe how you would do it.

16. **T** The Sierpinski gasket is a fractal created from an equilateral triangle. At each stage, the "middle" is cut out of each remaining equilateral triangle. The first three stages are shown.



a) If the process continues indefinitely, the stages get closer to the Sierpinski gasket. How many shaded triangles would be present in the sixth stage?
b) If the triangle in the first stage has an area of 80 cm^2, what is the area of the shaded portion of the 20th stage?

17. **C** In what ways are arithmetic and geometric sequences similar? Different?

Extending

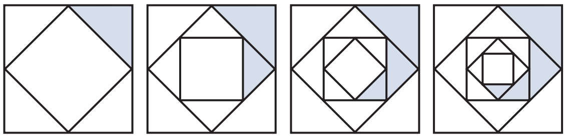
18. Given the geometric sequence with $t_1 = 1$ and $r = \frac{1}{2}$, calculate the sum of the first 1, 2, 3, and 4 terms. What would happen to the sum if you added more and more terms?

19. Determine the 10th term of the sequence 3, 10, 28, 72, 176, ... State the general term.

20. Is it possible for the first three terms of an arithmetic sequence to be equal to the first three terms of a geometric sequence? If so, provide an example.

21. Create an arithmetic sequence such that some of its terms form a geometric sequence. How is the geometric sequence related to the arithmetic sequence?

22. A square has a side length of 12 cm. The midpoints of the square are joined creating a smaller square and four triangles. If you continue this process, what will be the total area of the shaded region in stage 6?



7.3 Creating Rules to Define Sequences

GOAL

Create rules for generating sequences that are neither arithmetic nor geometric.

YOU WILL NEED

- circles (or squares) of paper of increasing size

LEARN ABOUT the Math

The Tower of Hanoi is a game played with three pegs and 10 discs of increasing size. At the start of the game, all of the discs are arranged in order of size on one of the pegs, with the smallest on top. The object of the game is to stack all the discs on a different peg in the same order of size as you started with in the fewest number of moves. The rules for moving discs are:

- You may move only one disc at a time.
- You may move only a disc that is alone on a peg, or one that is on top of a pile.
- You may place a disc only on an open peg, or on top of another disc that is larger than it.

start

end

? What is the minimum number of moves required to complete the game?

EXAMPLE 1 Using a pattern to represent the moves

Determine the minimum number of moves needed to move 10 discs to another peg.

Mario's Solution

Number of Discs	Number of Moves
1	1
2	3
3	7

I started with a simpler problem by counting the moves needed with 1, 2, and 3 discs, respectively. I noticed that for 3 discs, I first had to move the top two discs to another peg, then move the third disc to the open peg, and finally move the top two discs on top of the third disc.

$t_1 = 1, t_n = 2t_{n-1} + 1$

$t_2 = 3, t_2 = 2 \times 1 + 1$

$t_3 = 7, t_3 = 2 \times 3 + 1$

I noticed that each term was double the previous term plus 1. I wrote my pattern rule as a recursive formula, and it worked for the first three cases.

$t_4 = 2 \times 7 + 1$

$= 15$

I assumed that this pattern was correct. Then I used my formula to calculate the number of moves needed for 4 discs.

$t_5 = 2 \times 15 + 1 = 31$

$t_6 = 2 \times 31 + 1 = 63$

$t_7 = 2 \times 63 + 1 = 127$

$t_8 = 2 \times 127 + 1 = 255$

$t_9 = 2 \times 255 + 1 = 511$

$t_{10} = 2 \times 511 + 1 = 1023$

I then used the formula to calculate the number of moves needed for 10 discs.

To move 10 discs to a new peg requires 1023 moves.

Reflecting

A. How is Mario's recursive formula useful for understanding this sequence?

B. Why would it be difficult to use a recursive formula to figure out the number of moves if there were 1000 discs?

C. Add 1 to each term in the sequence. Use these new numbers to help you determine the general term of the sequence.

APPLY the Math

If a sequence is neither arithmetic nor geometric, identify the type of pattern (if one exists) that relates the terms to each other to get the general term.

EXAMPLE 2 Using reasoning to determine the next terms of a sequence

Given the sequence 1, 8, 16, 26, 39, 56, 78, ... , determine the next three terms. Explain your reasoning.

Tina's Solution

Term	1st Difference
1	7
8	8
16	10
26	13
39	

I calculated the 1st differences of the first five terms to determine whether the sequence is arithmetic. The sequence is not arithmetic since the 1st differences were not constant.

$$\frac{t_2}{t_1} = \frac{8}{1} = 8$$

$$\frac{t_3}{t_2} = \frac{16}{8} = 2$$

I checked to see if the sequence is geometric. There was no common ratio so the sequence is not geometric.

Term	1st Difference	2nd Difference
1	7	
8	8	1
16	10	2
26	13	3
39		

I calculated the 2nd differences. The 2nd differences go up by 1. If this pattern continues, I could determine the next terms of the sequence. I first checked whether this pattern was valid by calculating the next two terms.

Term	1st Difference	2nd Difference
1	7	
8	8	1
16	10	2
26	13	3
39	13 + 4 = **17**	3 + 1 = **4**
39 + 17 = **56**	17 + 5 = **22**	4 + 1 = **5**
56 + 22 = **78**		

I calculated the next two 2nd differences and worked backward to get the next two 1st differences. I worked backward again to calculate the next two terms. My values of t_6 and t_7 matched those in the given sequence, so the pattern rule seemed to be valid.

Term	1st Difference	2nd Difference
1	7	
8	8	1
16	10	2
26	13	3
39	**17**	**4**
56	**22**	**5**
78	22 + 6 = **28**	5 + 1 = **6**
78 + 28 = **106**	28 + 7 = **35**	6 + 1 = **7**
106 + 35 = **141**	35 + 8 = **43**	7 + 1 = **8**
141 + 43 = **184**		

I applied the rule to determine the next three terms.

The next three terms of the sequence are 106, 141, and 184.

Sometimes the pattern between terms in a sequence that is neither arithmetic nor geometric can be best described using a recursive formula.

EXAMPLE 3 Using reasoning to determine the recursive formula of a sequence

Given the sequence 5, 14, 41, 122, 365, 1094, 3281, ... , determine the recursive formula. Explain your reasoning.

Ali's Solution

$$t_2 - t_1 = 14 - 5 = 9$$
$$t_3 - t_2 = 41 - 14 = 27$$

I calculated some 1st differences and found they were not the same. The sequence is not arithmetic.

$$\frac{t_2}{t_1} = \frac{14}{5} = 2.8$$

I calculated a few ratios and found they were not the same. The sequence is not geometric.

$$\frac{t_3}{t_2} = \frac{41}{14} \doteq 2.93$$
$$\frac{t_4}{t_3} = \frac{122}{41} \doteq 2.98$$
$$\frac{t_5}{t_4} = \frac{365}{122} \doteq 2.99$$

The ratios seemed to be getting closer to 3.

n	t_n	$3t_{n-1}$
1	5	—
2	14	15
3	41	42
4	122	123
5	365	366
6	1094	1095
7	3281	3282

Since the ratios were almost 3, I pretended that they were actually 3. I created a table to compare each term t_n in the given sequence with 3 times the previous term, $3t_{n-1}$. I noticed that the value of t_n was one less than the value of $3t_{n-1}$.

$t_1 = 5$, $t_n = 3t_{n-1} - 1$, where $n \in \mathbf{N}$ and $n > 1$

I wrote a recursive formula for the sequence based on the pattern in my table.

Assuming that the pattern continues, the recursive formula for the sequence 5, 14, 41, 122, 365, 1094, 3281, ... is $t_1 = 5$, $t_n = 3t_{n-1} - 1$, where $n \in \mathbf{N}$ and $n > 1$.

If the terms of a sequence are rational numbers, you can sometimes find a pattern between terms if you look at the numerators and the denominators on their own.

EXAMPLE 4 Using reasoning to determine the general term of a sequence

Given the sequence $\frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \frac{11}{36}, \frac{13}{49}, \frac{15}{64}, \ldots$, determine the general term. Explain your reasoning.

Monica's Solution

3, 5, 7, 9, 11, 13, 15, ...

I looked at just the numerators to see if they formed a sequence. There was a common difference of 2 between terms, so the numerators formed an arithmetic sequence.

$$\begin{aligned} N_n &= a + (n - 1)d \\ &= 3 + (n - 1)(2) \\ &= 2n + 1 \end{aligned}$$

I wrote the numerator in terms of n by substituting $a = 3$ and $d = 2$ into the general formula for an arithmetic sequence.

$4, 9, 16, 25, 36, 49, 64, \ldots$

$2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, \ldots$

Next, I looked at the denominators to see if they formed a sequence. The terms were perfect squares but of the next position's value, not the position value for the current term.

$D_n = (n + 1)^2$

I wrote the denominator in terms of n by using a general expression for perfect squares.

$$t_n = \frac{N_n}{D_n}$$

The general term of the given sequence is $t_n = \frac{N_n}{D_n}$.

$$= \frac{2n + 1}{(n + 1)^2}$$

I substituted the expressions for N_n and D_n into t_n.

Assuming that the pattern continues, the general term of the given sequence is $t_n = \frac{2n + 1}{(n + 1)^2}$, where $n \in \mathbf{N}$.

In Summary

Key Idea

- A sequence is an ordered list of numbers that may or may not follow a predictable pattern. For example, the sequence of primes, 2, 3, 5, 7, 11, ... , is well understood, but no function or recursive formula has ever been discovered to generate them.

Need to Know

- A sequence has a general term if an algebraic rule using the term number, n, can be found to generate each term.
- If a sequence is arithmetic or geometric, a general term can always be found because arithmetic and geometric sequences follow a predictable pattern. For any other type of sequence, it is not always possible to find a general term.

CHECK *Your Understanding*

1. Sam wrote a solution to determine the 10th term of the sequence 1, 5, 4, -1, -5, -4, ...

Sam's Solution

$$t_n = t_{n-1} - t_{n-2}$$

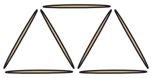
$$\therefore t_{10} = -1$$

Do you think Sam is right? Explain.

2. Determine a rule for calculating the terms of the sequence $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, ... Explain your reasoning.

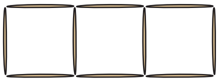
PRACTISING

3. Leila used toothpicks to make a row of triangles.



a) Determine a rule for calculating t_n, the number of toothpicks needed for n triangles. Explain your reasoning.

b) How will your rule change if the row of triangles is replaced with a row of squares? Explain your reasoning.



c) Determine a rule for calculating t_n, the number of toothpicks needed to create an $n \times n$ grid of squares. Explain your reasoning.

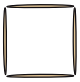


figure 1

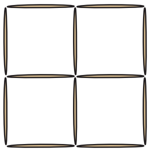


figure 2

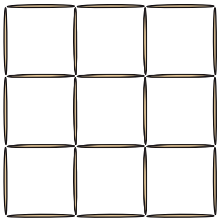


figure 3

d) Show that both your rules work for $n = 4$.

4. You are given the sequence 0, 1, -1, 2, -2, 3, -3, ...

a) Determine a rule for calculating the general term. Explain your reasoning.

b) Compare your rule with that of a classmate. Did you come up with the same rule? Which rule is "better"? Why?

c) Determine t_{12345}. Did you have to modify your rule to do this? If so, what is your new rule?

5. Determine an expression for the general term of the sequence $x + \frac{1}{y}$, $2x + \frac{1}{y^2}$, $3x + \frac{1}{y^3}$, ...

6. Determine a rule for calculating the terms of the sequence $\frac{3}{5}, \frac{21}{55}, \frac{147}{555}, \frac{1029}{5555}, \frac{7203}{55\,555}, \frac{50\,421}{555\,555}, \ldots$. Explain your reasoning.

7. Determine the next three terms of each sequence. Explain your reasoning.

K

- **a)** 4, 9, 19, 39, 79, ...
- **b)** 100, 99, 97, 94, 90, ...
- **c)** 1, 1, 2, 3, 5, 8, 13, 21, ...
- **d)** 3, 5, 10, 12, 24, 26, 52, ...
- **e)** 1, −8, 27, −64, 125, ...
- **f)** 6, 13, 27, 55, ...

8. In computer science, a bubble sort is an algorithm used to sort numbers from lowest to highest.

A

- The algorithm compares the first two numbers in a list to see if they are in the correct order. If they are not, the numbers switch places. Otherwise, they are left alone.
- The process continues with the 2nd and 3rd numbers, and then the 3rd and 4th, all the way through the list until the last two numbers.
- The algorithm starts at the beginning and repeats the whole process.
- The algorithm stops after it goes through the complete list and makes no switches. For example, a bubble sort of the numbers 3, 1, 5, 4, 2 would look like this:

Compare **3, 1**, 5, 4, 2. ⇒ Switch to give 1, 3, 5, 4, 2.
Compare 1, **3, 5**, 4, 2. ⇒ Leave as is.
Compare 1, 3, **5, 4**, 2. ⇒ Switch to give 1, 3, 4, 5, 2.
Compare 1, 3, 4, **5, 2**. ⇒ Switch to give 1, 3, 4, 2, 5.
Compare **1, 3**, 4, 2, 5. ⇒ Leave as is.
Compare 1, **3, 4**, 2, 5. ⇒ Leave as is.
Compare 1, 3, **4, 2**, 5. ⇒ Switch to give 1, 3, 2, 4, 5.
Compare 1, 3, 2, **4, 5**. ⇒ Leave as is.
Compare **1, 3**, 2, 4, 5. ⇒ Leave as is.
Compare 1, **3, 2**, 4, 5. ⇒ Switch to give 1, 2, 3, 4, 5.

The algorithm would then make 6 more comparisons, with no changes, and stop.

Suppose you had the numbers 100, 99, 98, 97, ... , 3, 2, 1. How many comparisons would the algorithm have to make to arrange these numbers from lowest to highest?

9. Determine the next three terms of the sequence 2, 11, 54, 271, 1354, 6771, 33 854, ... Explain your reasoning.

T

10. A sequence is defined by

$$t_1 = 1$$

$$t_n = \begin{cases} \frac{1}{2}t_{n-1}, & \text{if } t_{n-1} \text{ is even} \\ \frac{5}{2}(t_{n-1} + 1), & \text{if } t_{n-1} \text{ is odd} \end{cases}$$

Determine t_{1000}. Explain your reasoning.

11. Create your own sequence that is neither arithmetic nor geometric. State a rule for generating the sequence.

C

7.4 Exploring Recursive Sequences

GOAL

Explore patterns in sequences in which a term is related to the previous two terms.

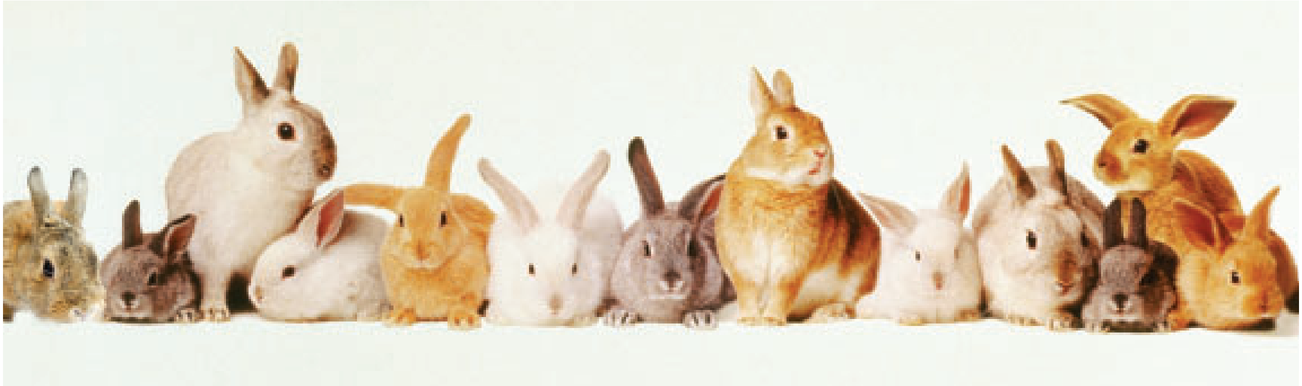
YOU WILL NEED

- graph paper

EXPLORE the Math

In his book *Liber Abaci* (*The Book of Calculation*), Italian mathematician Leonardo Pisano (1170–1250), nicknamed Fibonacci, described a situation like this:

A man put a pair of newborn rabbits (one male and one female) in an area surrounded on all sides by a wall. When the rabbits are in their second month of life, they produce a new pair of rabbits every month (one male and one female), which eventually mate. If the cycle continues, how many pairs of rabbits are there every month?



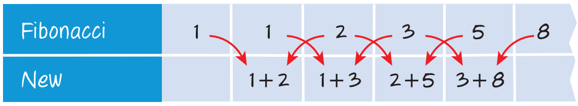
The sequence that represents the number of pairs of rabbits each month is called the Fibonacci sequence in Pisano's honour.

? What relationships can you determine in the Fibonacci sequence?

A. The first five terms of the Fibonacci sequence are 1, 1, 2, 3, and 5. Explain how these terms are related and generate the next five terms. Determine an expression for generating any term, F_n, in the sequence.

B. French mathematician Edouard Lucas (1842–91) named the sequence in the rabbit problem "the Fibonacci sequence." He studied the related sequence 1, 3, 4, ... , whose terms are generated in the same way as the Fibonacci sequence. Generate the next five terms of the Lucas sequence.

C. Starting with the Fibonacci sequence, create a new sequence by adding terms that are two apart. The first four terms are shown.



Fibonacci	1	1	2	3	5	8
New		1+2	1+3	2+5	3+8	

Repeat this process with the Lucas sequence. How are these new sequences related to the Fibonacci and Lucas sequences?

D. Determine the ratios of consecutive terms in the Fibonacci sequence. The first three ratios are shown.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1, \qquad \frac{F_3}{F_2} = \frac{2}{1} = 2, \qquad \frac{F_4}{F_3} = \frac{3}{2} = 1.5$$

What happens to the ratios if you continue the process? What happens if you repeat this process with the Lucas sequence? Based on your answers, how are the Fibonacci and Lucas sequences related to a geometric sequence?

E. Starting with the Fibonacci sequence, create two new sequences as shown.

Fibonacci	1	1	2	3	
New 1	1 x 1	1 x 1	2 x 2	3 x 3	
New 2	1 x 2	1 x 3	2 x 5	3 x 8	

The first new sequence is the squares of the Fibonacci terms. The second is the products of Fibonacci terms that are two apart. How are these two sequences related? What happens if you repeat this process with the Lucas sequence?

F. Create a new sequence by multiplying a Fibonacci number by a Lucas number from the same position. How is this new sequence related to the Fibonacci sequence?

Reflecting

G. How are the Fibonacci and Lucas sequences similar? different?

H. Although the Fibonacci and Lucas sequences have different starting values, they share the same relationship between consecutive terms, and they have many similar properties. What properties do you think different sequences with the same relationship between consecutive terms have? How would you check your conjecture?

I. From part D, the Fibonacci and Lucas sequences are closely related to a geometric sequence. How are these sequences similar? different?

In Summary

Key Ideas

- The Fibonacci sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$. This sequence models the number of petals on many kinds of flowers, the number of spirals on a pineapple, and the number of spirals of seeds on a sunflower head, among other naturally occurring phenomena.
- The Lucas sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 3$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$, and has many of the properties of the Fibonacci sequence.

Need to Know

- In a recursive sequence, the terms depend on one or more of the previous terms.
- Two different sequences with the same relationship between consecutive terms have similar properties.

FURTHER Your Understanding

1. Pick any two numbers and use the same relationship between consecutive terms as the Fibonacci and Lucas sequences to generate a new sequence. What properties does this new sequence share with the Fibonacci and Lucas sequences?

2. Since the ratios of consecutive terms of the Fibonacci and Lucas sequences are *almost* constant, these sequences are similar to a geometric sequence. Substitute the general term for a geometric sequence, $t_n = ar^{n-1}$, into the recursive formulas for the Fibonacci and Lucas sequences, and solve for r. How does this value of r relate to what you found in part D?

3. A sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 5$, $t_n = t_{n-1} + 2t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$.

a) Generate the first 10 terms.

b) Calculate the ratios of consecutive terms. What happens to the ratios?

c) Develop a formula for the general term.

Tech | *Support*

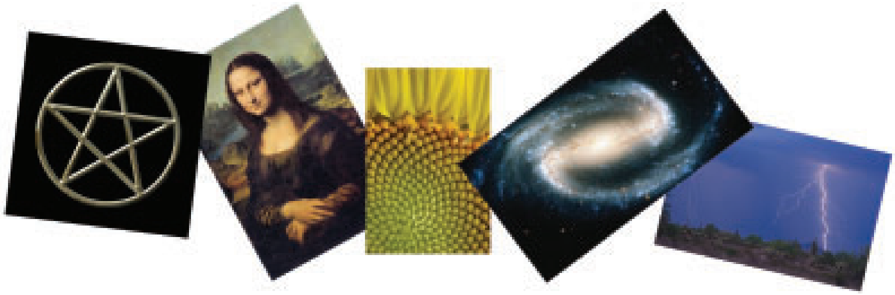
For help using a graphing calculator to generate sequences using recursive formulas, see Technical Appendix, B-16.

Curious | *Math*

The Golden Ratio

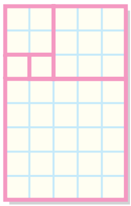
The golden ratio (symbolized by ϕ, Greek letter phi) was known to the ancient Greeks. Euclid defined the golden ratio by a point C on a line segment AB such that $\phi = \frac{AC}{CB} = \frac{AB}{CB}$.

The golden ratio, like the Fibonacci sequence, seems to pop up in unexpected places. The ancient Greeks thought that it defined the most pleasing ratio to the eye, so they used it in their architecture. Artists have been known to incorporate the golden ratio into their works. It has even received some exposure in an episode of the crime series NUMB3RS, as well as in the movie and book *The Da Vinci Code*.



Human works aren't the only places where the golden ratio occurs. The ratio of certain proportions in the human body are close to the golden ratio, and spirals in seed heads of flowers can be expressed using the golden ratio.

- On a piece of graph paper, trace a 1×1 square.
- Draw another 1×1 square touching the left side of the first square.
- On top of these two squares, draw a 2×2 square.
- On the right side of your picture, draw a 3×3 square touching one of the 1×1 squares and the 2×2 square.
- Below your picture, draw a 5×5 square touching both 1×1 squares and the 3×3 square.
- Repeat this process of adding squares around the picture, alternating directions left, up, right, down, and so on. The start of the spiral is shown at the right.



1. How is this spiral related to the Fibonacci sequence and the golden ratio?

7 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How do you know if a sequence is arithmetic?

A: A sequence is arithmetic if consecutive terms differ by a constant called the common difference, *d*.

$$t_2 - t_1 = d, \quad t_3 - t_2 = d, \quad t_4 - t_3 = d$$
$$\vdots$$

The recursive formula of an arithmetic sequence is based on adding the same value to the previous term. Its general term is defined by a discrete linear function since the graph of term versus position number gives a straight line.

Study *Aid*

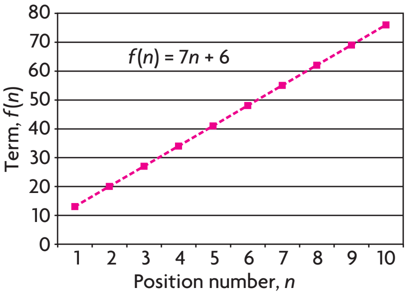
- See Lesson 7.1, Examples 1 to 4
- Try Mid-Chapter Review Questions 1 to 5.

EXAMPLE

If the sequence 13, 20, 27, 34, 41, ... is arithmetic, state the recursive formula and the general term.

Solution

Each term is 7 more than the previous term. So the recursive formula is $t_1 = 13$, $t_n = t_{n-1} + 7$, where $n \in \mathbf{N}$ and $n > 1$. The general term is $t_n = 13 + (n - 1)(7) = 7n + 6$ and its graph is a discrete linear function.



Q: How do you know if a sequence is geometric?

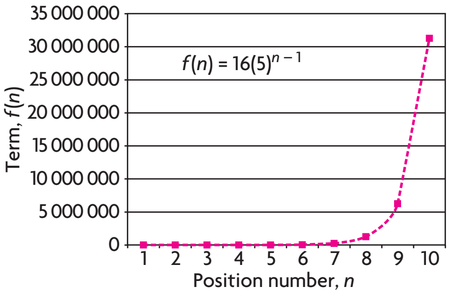
A: A sequence is geometric if the ratio of consecutive terms is a constant called the common ratio, *r*.

$$\frac{t_2}{t_1} = r, \quad \frac{t_3}{t_2} = r, \quad \frac{t_4}{t_3} = r$$
$$\vdots$$

The recursive formula of a geometric sequence is based on multiplying the previous term by the same value. Its general term is defined by a discrete exponential function since the graph of term versus position number gives an exponential curve.

Study *Aid*

- See Lesson 7.2, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 4 to 6.



EXAMPLE

If the sequence 16, 80, 400, 2000, 10 000, ... is geometric, state the recursive formula and the general term.

Solution

Each term is 5 times the previous term. So the recursive formula is $t_1 = 16$, $t_n = 5t_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$. The general term is $t_n = 16 \times 5^{n-1}$, and its graph is a discrete exponential function.

Q: How do you determine terms of a sequence that is neither arithmetic nor geometric?

A: Look for a pattern among the terms. It is also useful to look at the 1st, 2nd, 3rd, and possibly higher, differences. Once you find a pattern, you can use it to generate terms of the sequence.

Study *Aid*

- See Lesson 7.3, Examples 1 to 4.
- Try Mid-Chapter Review Questions 7, 8, and 9.

EXAMPLE

Determine the next three terms of the sequence 1, 6, 7, 6, 5, 6, 11, ...

Solution

Start by looking at the 1st, 2nd, and 3rd differences.

Term	1st Difference	2nd Difference	3rd Difference
1	5	−4	2
6	1	−2	2
7	−1	0	2
6	−1	2	2
5	1	4	**2**
6	5	**6**	**2**
11	**11**	**8**	**2**
22	**19**	**10**	
41	**29**		
70			

The 1st differences are not constant. Since the 2nd differences seem to be going up by a constant, the 3rd differences are the same. To determine the next three terms, work backward using the 3rd, 2nd, and then the 1st differences. Assuming that the pattern continues, the next three terms are 22, 41, and 70.

PRACTICE Questions

Lesson 7.1

1. For each arithmetic sequence, determine

i) the recursive formula

ii) the general term

iii) t_{10}

a) $29, 21, 13, \ldots$

b) $-8, -16, -24, \ldots$

c) $-17, -9, -1, \ldots$

d) $3.25, 9.5, 15.75, \ldots$

e) $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$

f) $x, 3x + 3y, 5x + 6y, \ldots$

2. Determine the recursive formula and the general term for the arithmetic sequence in which

a) the first term is 17 and the common difference is 11

b) $t_1 = 38$ and $d = -7$

c) the first term is 55 and the second term is 73

d) $t_3 = -34$ and $d = -38$

e) the fifth term is 91 and the seventh term is 57

3. The number of seats in the rows of a stadium form an arithmetic sequence. Two employees of the stadium determine that the 13th row has 189 seats and the 25th row has 225 seats. How many seats are in the 55th row?

Lesson 7.2

4. i) Determine whether each sequence is arithmetic or geometric.

ii) Determine the general term, the recursive formula, and t_6.

a) $15, 30, 45, \ldots$

b) $640, 320, 160, \ldots$

c) $23, -46, 92, \ldots$

d) $3000, 900, 270, \ldots$

e) $3.8, 5, 6.2, \ldots$

f) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$

5. i) Determine the type of each sequence (arithmetic, geometric, or neither), where $n \in \mathbf{N}$.

ii) State the first five terms.

a) $t_n = 5n$

b) $t_n = \dfrac{3}{4^n + 3}$

c) $t_1 = 5, t_n = t_{n-1} - 12$, where $n > 1$

d) $t_1 = -2, \dfrac{t_n}{t_{n-1}} = -2$, where $n > 1$

e) $t_1 = 8, t_2 = 11, t_n = 2t_{n-1} - t_{n-2}$, where $n > 2$

6. A work of art is priced at $10 000. After one week, if the art isn't sold, its price is reduced by 10%. Each week after that, if it hasn't sold, its price is reduced by another 10%. Your mother really likes the art and you would like to purchase it for her, but you have only $100. If the art is not sold, how many weeks will you have to wait before being able to afford it?

Lesson 7.3

7. An IQ test has the question "Determine the next three numbers in the sequence 1, 9, 29, 67, 129, 221, ___, ___, ___." What are the next three terms? Explain your reasoning.

8. Determine the general term of the sequence $x + y, x^2 + 2y, x^3 + 3y, \ldots$. Explain your reasoning.

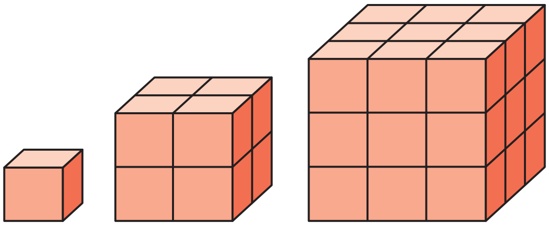
9. Sarah built a sequence of large cubes using unit cubes.

a) State the sequence of the number of unit cubes in each larger cube.

b) Determine the next three terms of the sequence.

c) State the general term of the sequence.

d) How many unit cubes does Sarah need to build the 15th cube?



Lesson 7.4

10. a) Determine the 15th term of the sequence 3, 2, 5, 7, 12, ... Explain your reasoning.

b) Write the recursive formula for the sequence in part (a).

7.5

Arithmetic Series

YOU WILL NEED

- linking cubes

GOAL

Calculate the sum of the terms of an arithmetic sequence.

INVESTIGATE the Math

Marian goes to a party where there are 23 people present, including her. Each person shakes hands with every other person once and only once.

? How can Marian determine the total number of handshakes that take place?

A. Suppose the people join the party one at a time. When they enter, they shake hands with the host and everyone who is already there. Create a sequence representing the number of handshakes each person will make. What type of sequence is this?

B. Write your sequence from part A, but include plus signs between terms. This expression is a **series** and represents the total number of handshakes.

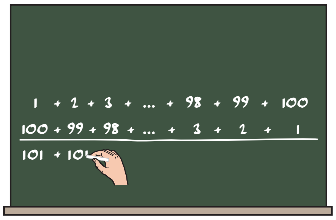
series

the sum of the terms of a sequence

arithmetic series

the sum of the terms of an arithmetic sequence

C. When German mathematician Karl Friedrich Gauss (1777−1855) was a child, his teacher asked him to calculate the sum of the numbers from 1 to 100. Gauss wrote the list of numbers twice, once forward and once backward. He then paired terms from the two lists to solve the problem. Use this method to determine the sum of your **arithmetic series.**



D. Solve the handshake problem without using Gauss's method.

Reflecting

E. Suppose the partial sums of an arithmetic series are the terms of an arithmetic sequence. What would you notice about the 1st and 2nd differences?

F. Why is Gauss's method for determining the sum of an arithmetic series efficient?

G. Consider the arithmetic series 1 + 6 + 11 + 16 + 21 + 26 + 31 + 36. Use Gauss's method to determine the sum of this series. Do you think this method will work for *any* arithmetic series? Justify your answer.

partial sum

the sum, S_n, of the first n terms of a sequence

APPLY the Math

EXAMPLE 1 Representing the sum of an arithmetic series

Determine the sum of the first n terms of the arithmetic series $a + (a + d) + (a + 2d) + (a + 3d) + \ldots$.

Barbara's Solution

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + t_n$$

The series is arithmetic. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of an arithmetic sequence, $t_n = a + (n - 1)d$.

$$\begin{array}{rlllll}
S_n = & a & + (a + d) & + \ldots + [a + (n - 2)d] & + [a + (n - 1)d] \\
+\ S_n = & [a + (n - 1)d] & + [a + (n - 2)d] & + \ldots + (a + d) & + a \\
\hline
2S_n = & [2a + (n - 1)d] & + [2a + (n - 1)d] & + \ldots + [2a + (n - 1)d] & + [2a + (n - 1)d]
\end{array}$$

$$2S_n = n \times [2a + (n - 1)d]$$

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

Using Gauss's method, I wrote the sum out twice, first forward and then backward. Next, I added each column. Since the terms in the top row go *up* by d and the terms in the bottom row go *down* by d, each pair of terms has the same sum.

There are n pairs that add up to $2a + (n - 1)d$, but that represents $2S_n$, so I divided by 2.

The sum of the first n terms of an arithmetic series is

$$\begin{aligned}
S_n &= \frac{n[2a + (n - 1)d]}{2} \\
&= \frac{n[a + a + (n - 1)d]}{2} \\
&= \frac{n[a + (a + (n - 1)d)]}{2} \\
&= \frac{n(t_1 + t_n)}{2}
\end{aligned}$$

I knew that $2a = a + a$, so I wrote this formula another way. I regrouped the terms in the numerator. I noticed that a is the first term of the series and $a + (n - 1)d$ is the nth term.

If a problem involves adding the terms of an arithmetic sequence, you can use the formula for the sum of an arithmetic series.

EXAMPLE 2 Solving a problem by using an arithmetic series

In an amphitheatre, seats are arranged in 50 semicircular rows facing a domed stage. The first row contains 23 seats, and each row contains 4 more seats than the previous row. How many seats are in the amphitheatre?

Kew's Solution

$a = 23, d = 4$

Since each row has 4 more seats than the previous row, the number of seats in each row forms an arithmetic sequence.

$23 + 27 + 31 + \ldots + t_{50}$

I wrote an arithmetic series to represent the total number of seats in the amphitheatre.

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{50} = \frac{(50)[2(23) + (50 - 1)(4)]}{2}$$

$$= 6050$$

Since I knew the first term and the common difference, I used the formula for the sum of an arithmetic series in terms of *a* and *d*. I substituted $n = 50$ since there are 50 rows of seats.

There are 6050 seats in the amphitheatre.

In order to determine the sum of any arithmetic series, you need to know the number of terms in the series.

EXAMPLE 3 Selecting a strategy to calculate the sum of a series when the number of terms is unknown

Determine the sum of $-31 - 35 - 39 - \ldots -403$.

Jasmine's Solution

$$t_2 - t_1 = -35 - (-31) = -4$$
$$t_3 - t_2 = -39 - (-35) = -4$$

I checked to see if the series was arithmetic. So I calculated a few 1st differences. The differences were the same, so the series is arithmetic.

$$\begin{aligned} t_n &= a + (n - 1)d \\ -403 &= -31 + (n - 1)(-4) \\ -403 + 31 &= (n - 1)(-4) \\ -372 &= (n - 1)(-4) \\ \frac{-372}{-4} &= \frac{(n - 1)(-4)}{-4} \\ 93 &= n - 1 \\ 93 + 1 &= n \\ 94 &= n \end{aligned}$$

I needed to determine the value of n when $t_n = -403$. So I substituted $a = 31$, $d = 4$, and t_n into the formula for the general term of an arithmetic sequence and solved for n.

There are 94 terms in this sequence.

$$\begin{aligned} S_n &= \frac{n(t_1 + t_n)}{2} \\ S_{94} &= \frac{94[-31 + (-403)]}{2} \\ &= -20\,398 \end{aligned}$$

Since I knew the first and last terms of the series, I used the formula for the sum of an arithmetic series in terms of t_1 and t_n. I substituted $n = 94$, $t_1 = -31$, and $t_{94} = -403$.

The sum of the series $-31 - 35 - 39 - \dots -403$ is $-20\,398$.

In Summary

Key Idea

- An arithmetic series is created by adding the terms of an arithmetic sequence. For the sequence 2, 10, 18, 26, ... , the related arithmetic series is $2 + 10 + 18 + 26 + \dots$.
- The partial sum, S_n, of a series is the sum of a finite number of terms from the series, $S_n = t_1 + t_2 + t_3 + \dots + t_n$.
  For example, for the sequence 2, 10, 18, 26, ... ,

$$\begin{aligned} S_4 &= t_1 + t_2 + t_3 + t_4 \\ &= 2 + 10 + 18 + 26 \\ &= 56 \end{aligned}$$

Need to Know

- The sum of the first n terms of an arithmetic sequence can be calculated using
  - $S_n = \dfrac{n[2a + (n - 1)d]}{2}$ or
  - $S_n = \dfrac{n(t_1 + t_n)}{2}$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference.
- You can use either formula, but you need to know the number of terms in the series and the first term. If you know the last term, use the formula in terms of t_1 and t_n. If you can calculate the common difference, use the formula in terms of a and d.

CHECK Your Understanding

1. Calculate the sum of the first 10 terms of each arithmetic series.

a) $59 + 64 + 69 + \dots$

b) $31 + 23 + 15 + \dots$

c) $-103 - 110 - 117 - \dots$

d) $-78 - 56 - 34 - \dots$

2. Calculate the sum of the first 20 terms of an arithmetic sequence with first term 18 and common difference 11.

3. Bricks are stacked in 20 rows such that each row has a fixed number of bricks more than the row above it. The top row has 5 bricks and the bottom row 62 bricks. How many bricks are in the stack?

PRACTISING

4. i) Determine whether each series is arithmetic.

ii) If the series is arithmetic, calculate the sum of the first 25 terms.

a) $-5 + 1 + 7 + 13 + \dots$

b) $2 + 10 + 50 + 250 + \dots$

c) $1 + 1 + 2 + 3 + \dots$

d) $18 + 22 + 26 + 30 + \dots$

e) $31 + 22 + 13 + 4 + \dots$

f) $1 - 3 + 5 - 7 + \dots$

5. For each series, calculate t_{12} and S_{12}.

K **a)** $37 + 41 + 45 + 49 + \dots$

b) $-13 - 24 - 35 - 46 - \dots$

c) $-18 - 12 - 6 + 0 + \dots$

d) $\frac{1}{5} + \frac{7}{10} + \frac{6}{5} + \frac{17}{10} + \dots$

e) $3.19 + 4.31 + 5.43 + 6.55 + \dots$

f) $p + (2p + 2q) + (3p + 4q) + (4p + 6q) + \dots$

6. Determine the sum of the first 20 terms of the arithmetic series in which

a) the first term is 8 and the common difference is 5

b) $t_1 = 31$ and $t_{20} = 109$

c) $t_1 = 53$ and $t_2 = 37$

d) the 4th term is 18 and the terms increase by 17

e) the 15th term is 107 and the terms decrease by 3

f) the 7th term is 43 and the 13th term is 109

7. Calculate the sums of these arithmetic series.

a) $1 + 6 + 11 + \dots + 96$

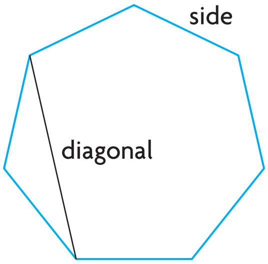
b) $24 + 37 + 50 + \dots + 349$

c) $85 + 77 + 69 + \dots - 99$

d) $5 + 8 + 11 + \dots + 2135$

e) $-31 - 38 - 45 - \dots - 136$

f) $-63 - 57 - 51 - \dots + 63$



8. A diagonal in a regular polygon is a line segment joining two nonadjacent vertices.

A **a)** Develop a formula for the number of diagonals for a regular polygon with n sides.

b) Show that your formula works for a regular heptagon (a seven-sided polygon).

9. Joe invests \$1000 at the start of each year for five years and earns 6.3% simple interest on his investments. How much will all his investments be worth at the start of the fifth year?

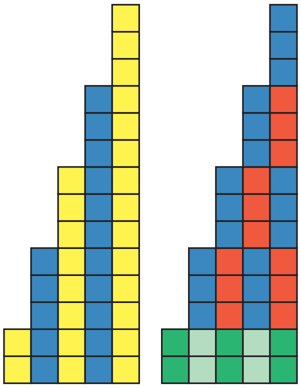
10. During a skydiving lesson, Chandra jumps out of a plane and falls 4.9 m during the first second. For each second afterward, she continues to fall 9.8 m more than the previous second. After 15 s, she opens her parachute. How far did Chandra fall before she opened her parachute?

11. Jamal got a job working on an assembly line in a toy factory. On the 20th day of work, he assembled 137 toys. He noticed that since he started, every day he assembled 3 more toys than the day before. How many toys did Jamal assemble altogether during his first 20 days?

12. In the video game "Geometric Constructors," a number of shapes have to be arranged into a predefined form. In level 1, you are given 3 min 20 s to complete the task. At each level afterward, a fixed number of seconds are removed from the time until, at level 20, 1 min 45 s are given. What would be the total amount of time given if you were to complete the first 20 levels?

13. **T** Sara is training to run a marathon. The first week she runs 5 km each day. The next week, she runs 7 km each day. During each successive week, each day she runs 2 km farther than she ran the days of the previous week. If she runs for five days each week, what total distance will Sara run in a 10 week training session?

14. **C** Joan is helping a friend understand the formulas for an arithmetic series. She uses linking cubes to represent the sum of the series $2 + 5 + 8 + 11 + 14$ two ways. These representations are shown at the right. Explain how the linking-cube representations can be used to explain the formulas for an arithmetic series.



Extending

15. The 10th term of an arithmetic series is 34, and the sum of the first 20 terms is 710. Determine the 25th term.

16. The arithmetic series $1 + 4 + 7 + \dots + t_n$ has a sum of 1001. How many terms does the series have?

7.6 Geometric Series

YOU WILL NEED

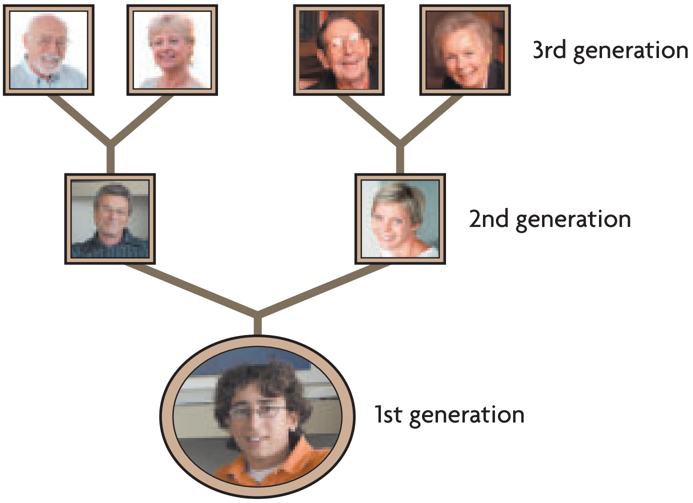
- spreadsheet software

GOAL

Calculate the sum of the terms of a geometric sequence.

INVESTIGATE the Math

An ancestor tree is a family tree that shows only the parents in each generation. John started to draw his ancestor tree, starting with his own parents. His complete ancestor tree includes 13 generations.



? How many people are in John's ancestor tree?

A. Create a sequence to represent the number of people in each generation for the first six generations. How do you know that this sequence is geometric?

B. Based on your sequence, create a geometric series to represent the total number of people in John's ancestor tree.

geometric series
the sum of the terms of a geometric sequence

C. Write the series again, but this time multiply each term by the common ratio. Write both series, rS_n and S_n, so that equal terms are aligned one above the other. Subtract S_n from rS_n.

D. Based on your calculation in part C, determine how many people are in John's ancestor tree.

Reflecting

E. How is the sum of a geometric series related to an exponential function?

F. Why did lining up equal terms make the subtraction easier?

APPLY the Math

EXAMPLE 1 Representing the sum of a geometric series

Determine the sum of the first n terms of the geometric series.

Tara's Solution

$$t_n = ar^{n-1}$$

The series is geometric. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of a geometric sequence.

$$\begin{aligned}
rS_n &= \phantom{-(a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \\
-S_n &= -(a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}) \\
\hline
(r-1)S_n &= -a + 0 + 0 + 0 + \ldots + 0 + 0 + ar^n \\
(r-1)S_n &= -a + ar^n
\end{aligned}$$

I wrote the sum out. If I multiplied every term by the common ratio, most of the terms would be repeated. I wrote this new series above the original series and lined up equal terms, so I would get zero for most of the terms when I subtracted. Only the first term of S_n and the last term of rS_n would remain.

$$\begin{aligned}
(r-1)S_n &= a(r^n - 1) \\
\left(\frac{\overset{1}{\cancel{r-1}}}{\underset{1}{\cancel{r-1}}}\right)S_n &= \frac{a(r^n-1)}{r-1} \\
S_n &= \frac{a(r^n-1)}{r-1}
\end{aligned}$$

I solved for S_n by dividing both sides by $r - 1$.

The sum of the first n terms of a geometric series is

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ where } r \neq 1.$$

$$S_n = \frac{ar^n - a}{r-1}$$

$$S_n = \frac{t_{n+1} - t_1}{r-1}, \text{ where } r \neq 1.$$

I wrote this formula another way by expanding the numerator. I noticed that a is the first term in the series and ar^n is the $(n + 1)$th term.

If a problem involves adding together the terms of a geometric sequence, you can use the formula for the sum of geometric series.

EXAMPLE 2 Solving a problem by using a geometric series

At a fish hatchery, fish hatch at different times even though the eggs were all fertilized at the same time. The number of fish that hatched on each of the first four days after fertilization was 2, 10, 50, and 250, respectively. If the pattern continues, calculate the total number of fish hatched during the first 10 days.

Joel's Solution

$$\frac{t_2}{t_1} = \frac{10}{2} \qquad \frac{t_3}{t_2} = \frac{50}{10} \qquad \frac{t_4}{t_3} = \frac{250}{50}$$
$$= 5 \qquad = 5 \qquad = 5$$

I checked to see if the sequence 2, 10, 50, 250, ... is geometric. So I calculated the ratio of consecutive terms. Since all the ratios are the same, the sequence is geometric.

$$\therefore r = 5$$
$$a = 2$$
$$n = 10$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{2(5^{10} - 1)}{5 - 1}$$
$$= 4\ 882\ 812$$

The first term is 2 and there are 10 terms. Since I knew the first term, the common ratio, and the number of terms, I used the formula for the sum of a geometric series in terms of *a*, *r*, and *n*. I substituted $a = 2$, $r = 5$, and $n = 10$.

A total of 4 882 812 fish hatched during the first 10 days.

EXAMPLE 3 Selecting a strategy to calculate the sum of a geometric series when the number of terms is unknown

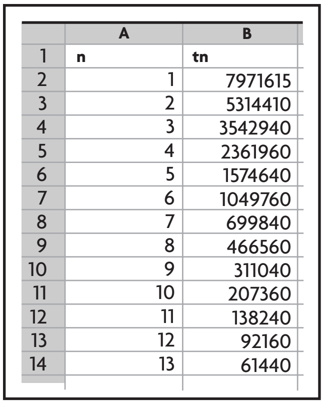
Calculate the sum of the geometric series
$7\,971\,615 + 5\,314\,410 + 3\,542\,940 + \ldots + 92\,160$.

Jasmine's Solution: Using a Spreadsheet

$$\frac{t_2}{t_1} = \frac{5\,314\,410}{7\,971\,615} = \frac{2}{3}$$

$$\therefore r = \frac{2}{3}$$

I knew that the series is geometric. So I calculated the common ratio.



	A	B
1	n	tn
2	1	7971615
3	2	5314410
4	3	3542940
5	4	2361960
6	5	1574640
7	6	1049760
8	7	699840
9	8	466560
10	9	311040
11	10	207360
12	11	138240
13	12	92160
14	13	61440

I needed to determine the number of terms, n, to get to $t_n = 92\,160$. So I set up a spreadsheet to generate the terms of the series. I saw that the 12th term is 92 160.

$$S_n = \frac{t_{n+1} - t_1}{r - 1}$$

$$S_{12} = \frac{61\,440 - 7\,971\,615}{\frac{2}{3} - 1}$$

$$= 23\,730\,525$$

From the spreadsheet, 61 440 corresponds to the $(n + 1)$th term. Since I knew the first term and the $(n + 1)$th term, I used the formula for the sum of a geometric series in terms of t_1 and t_{n+1}.

The sum of the series $7\,971\,615 + 5\,314\,410 + 3\,542\,940 + \ldots + 92\,160$ is 23 730 525.

Mario's Solution: Using a Graphing Calculator

$$\frac{t_2}{t_1} = \frac{5\ 314\ 410}{7\ 971\ 615}$$
$$= \frac{2}{3}$$

I knew that the series is geometric. So I calculated the common ratio.

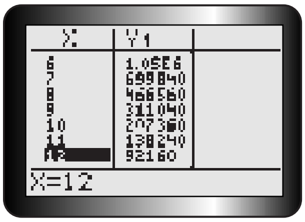
$$\therefore\ r = \frac{2}{3}$$
$$t_n = ar^{n-1}$$
$$= 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I wrote the formula for the general term of a geometric sequence. I substituted $a = 7\ 971\ 615$ and $r = \frac{2}{3}$ into the formula.

$$92\ 160 = 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I needed to determine the number of terms, n, to get to $t_n = 92\ 160$.

```
Plot1  Plot2  Plot3
\Y1■7971615(2/3)
^(X-1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
  X      Y1
  6      1.05E6
  7      699840
  8      466560
  9      311040
  10     207360
  11     138240
  12     92160
X=12
```

I entered the function Y1 = 7 971 615(2/3)^(X−1) into my graphing calculator. Then I used the table function to determine the term number whose value was 92 160. It was $n = 12$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

```
(7971615((2/3)^1
2-1))/((2/3)-1)
                23730525
```

Since I knew that $a = 7\ 971\ 615$, $r = \frac{2}{3}$, and $n = 12$, I substituted these values into the formula for the sum of a geometric series in terms of a, r, and n. I used my calculator to evaluate.

The sum of the series 7 971 615 + 5 314 410 + 3 542 940 + ... + 92 160 is 23 730 525.

Tech | ***Support***

For help using the table on a graphing calculator, see Technical Appendix, B-6.

In Summary

Key Idea

- A geometric series is created by adding the terms of a geometric sequence. For the sequence 3, 6, 12, 24, ... , the related geometric series is $3 + 6 + 12 + 24 + ...$

Need to Know

- The sum of the first n terms of a geometric sequence can be calculated using
  - $S_n = \dfrac{a(r^n - 1)}{r - 1}$, where $r \neq 1$ or
  - $S_n = \dfrac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.
- You can use either formula, but you need to know the common ratio and the first term. If you know the $(n + 1)$th term, use the formula in terms of t_1 and t_{n+1}. If you can calculate the number of terms, use the formula in terms of a, r, and n.

CHECK *Your Understanding*

1. Calculate the sum of the first seven terms of each geometric series.

a) $6 + 18 + 54 + ...$

b) $100 + 50 + 25 + ...$

c) $8 - 24 + 72 - ...$

d) $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{12} + ...$

2. Calculate the sum of the first six terms of a geometric sequence with first term 11 and common ratio 4.

PRACTISING

3. For each geometric series, calculate t_6 and S_6.

a) $6 + 30 + 150 + ...$

b) $-11 - 33 - 99 - ...$

c) $21\,000\,000 + 4\,200\,000 + 840\,000 + ...$

d) $\dfrac{4}{5} + \dfrac{8}{15} + \dfrac{16}{45} + ...$

e) $3.4 - 7.14 + 14.994 - ...$

f) $1 + 3x^2 + 9x^4 + ...$

4. **i)** Determine whether each series is arithmetic, geometric, or neither.

K **ii)** If the series is geometric, calculate the sum of the first eight terms.

a) $5 + 10 + 15 + 20 + ...$

b) $7 + 21 + 63 + 189 + ...$

c) $2048 - 512 + 128 - 32 + ...$

d) $10 - 20 + 30 - 40 + ...$

e) $1.1 + 1.21 + 1.331 + 1.4641 + ...$

f) $81 + 63 + 45 + 27 + ...$

5. Determine the sum of the first seven terms of the geometric series in which

a) $t_1 = 13$ and $r = 5$
b) the first term is 11 and the seventh term is 704
c) $t_1 = 120$ and $t_2 = 30$
d) the third term is 18 and the terms increase by a factor of 3
e) $t_8 = 1024$ and the terms decrease by a factor of $\frac{2}{3}$
f) $t_5 = 5$ and $t_8 = -40$

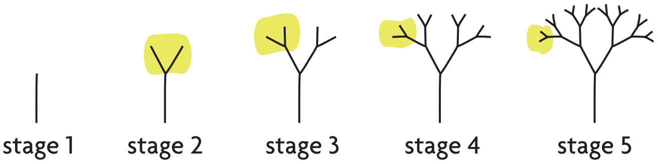
6. Calculate the sum of each geometric series.

a) $1 + 6 + 36 + \ldots + 279\,936$
b) $960 + 480 + 240 + \ldots + 15$
c) $17 - 51 + 153 - \ldots - 334\,611$
d) $24\,000 + 3600 + 540 + \ldots + 1.8225$
e) $-6 + 24 - 96 + \ldots + 98\,304$
f) $4 + 2 + 1 + \ldots + \frac{1}{1024}$

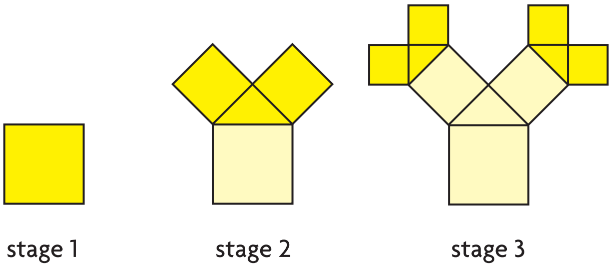
7. A ball is dropped from a height of 3 m and bounces on the ground. At the top of each bounce, the ball reaches 60% of its previous height. Calculate the total distance travelled by the ball when it hits the ground for the fifth time.

8. The formula for the sum of a geometric series is $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_{n+1} - t_1}{r - 1}$, each of which is valid only if $r \neq 1$. Explain how you would determine the sum of a geometric series if $r = 1$.

9. A simple fractal tree grows in stages. At each new stage, two new line segments branch out from each segment at the top of the tree. The first five stages are shown. How many line segments need to be drawn to create stage 20?



10. A Pythagorean fractal tree starts at stage 1 with a square of side length 1 m. At every consecutive stage, an isosceles right triangle and two squares are attached to the last square(s) drawn. The first three stages are shown. Calculate the area of the tree at the 10th stage.



11. A large company has a phone tree to contact its employees in case of an emergency factory shutdown. Each of the five senior managers calls three employees, who each call three other employees, and so on. If the tree consists of seven levels, how many employees does the company have?

12. John wants to calculate the sum of a geometric series with 10 terms, where the 10th term is 5 859 375 and the common ratio is $\frac{5}{3}$. John solved the problem by considering another geometric series with common ratio $\frac{3}{5}$. Use John's method to calculate the sum. Justify your reasoning.

13. **T** A cereal company attempts to promote its product by placing certificates for a cash prize in selected boxes. The company wants to come up with a number of prizes that satisfy all of these conditions:

a) The total of the prizes is at most \$2 000 000.

b) Each prize is in whole dollars (no cents).

c) When the prizes are arranged from least to greatest, each prize is a constant integral multiple of the next smaller prize and is
- more than double the next smaller prize
- less than 10 times the next smaller prize

Determine a set of prizes that satisfies these conditions.

14. **C** Describe several methods for calculating the partial sums of an arithmetic and a geometric series. How are the methods similar? different?

Extending

15. In a geometric series, $t_1 = 12$ and $S_3 = 372$. What is the greatest possible value for t_5? Justify your answer.

16. In a geometric series, $t_1 = 23$, $t_3 = 92$, and the sum of all of the terms of the series is 62 813. How many terms are in the series?

17. Factor $x^{15} - 1$.

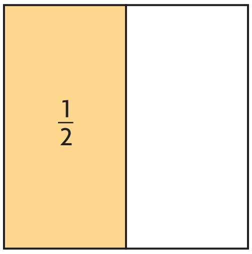
18. Suppose you want to calculate the sum of the *infinite* geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.$$

a) The diagram shown illustrates the first term of this series. Represent the next three terms on the diagram.

b) How can the formula for the sum of a geometric series be used in this case?

c) Does it make sense to talk about adding together an infinite number of terms? Justify your reasoning.



7.7

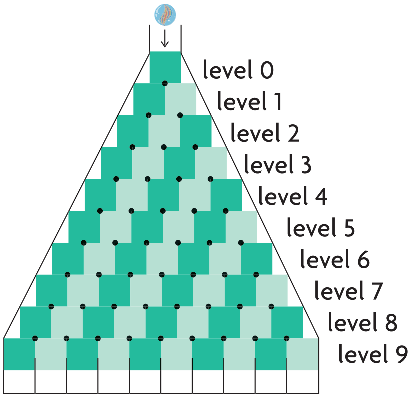
Pascal's Triangle and Binomial Expansions

GOAL

Investigate patterns in Pascal's triangle, and use one of these patterns to expand binomials efficiently.

INVESTIGATE the Math

A child's toy called "Rockin' Rollers" involves dropping a marble into its top. When the marble hits a pin, it has the same chance of going either left or right. A version of the toy with nine levels is shown at the right.



? How many paths are there to each of the bins at the bottom of this version of "Rockin' Rollers"?

A. Consider a "Rockin' Rollers" toy that has only one level. Calculate the number of paths to each bin at the bottom. Repeat the calculation with a toy having two and three levels.

B. How is the number of paths for a toy with three levels related to the number of paths for a toy with two levels? Why is this so?

C. Use the pattern to predict how many paths lead to each bin in a toy with four levels. Check your prediction by counting the number of paths.

D. Use your pattern to calculate the number of paths to each bin in a toy with nine levels.

Blaise Pascal

Reflecting

E. How is the number of paths for each bin in a given level related to the number of paths in the level above it?

F. The triangular pattern of numbers in the "Rockin' Rollers" toy is known as Pascal's triangle, named after French mathematician Blaise Pascal (1623–62), who explored many of its properties. What other pattern(s) can you find in Pascal's triangle?

APPLY the Math

EXAMPLE 1 Connecting Pascal's triangle to the expansion of a binomial power

Expand $(x + y)^6$.

Pedro's Solution

$$(x + y)^1 = 1x + 1y$$

The binomial to the 1st power is the same as the binomial itself.

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

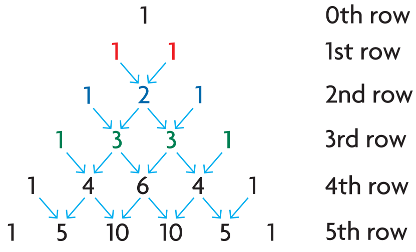
I expanded the square of a binomial.

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)^2\\ &= (x + y)(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= 1x^3 + 3x^2y + 3xy^2 + 1y^3\end{aligned}$$

I expanded the binomial to the 3rd power. I noticed that the coefficients in each of the expansions so far are numbers in each row of Pascal's triangle.

1
1 1
1 2 1
1 3 3 1

Each term in the expansion is in terms of a product of an exponent of x and an exponent of y. The exponents of x start from 3 (the exponent of the binomial), and go down to zero, while the exponents of y start at zero and go up to 3. In each term, the sum of the x and y exponents is always 3.



I wrote out 6 rows of Pascal's triangle. The one at the top must correspond to $(x + y)^0 = 1$.

$$\begin{aligned}(x + y)^4 &= (x + y)(x + y)^3\\ &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3)\\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + 3x^2y^2 + 3xy^3 + y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

I tried one more expansion to check that these patterns continue.

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^3 + 6xy^5 + y^6$$

To expand $(x + y)^6$, I used my patterns and the 6th row of the triangle.

Any binomial can be expanded by using Pascal's triangle to help determine the coefficients of each term.

EXAMPLE 2 Selecting a strategy to expand a binomial power involving a variable in one term

Expand and simplify $(x-2)^5$.

Tanya's Solution

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Since the exponent of the binomial is 5, I used the 5th row of Pascal's triangle to determine the coefficients.

$$(x-2)^5 = 1(x)^5 + 5(x)^4(-2)^1 + 10(x)^3(-2)^2 + 10(x)^2(-2)^3 + 5(x)^1(-2)^4 + 1(-2)^5$$

I used the terms *x* and -2 and applied the pattern for expanding a binomial. The exponents in each term always add up to 5. As the *x* exponents decrease by 1 each time, the exponents of -2 increase by 1.

$$= x^5 + 5(x^4)(-2) + 10(x^3)(4) + 10(x^2)(-8) + 5(x)(16) + (-32)$$

I simplified each term.

$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

EXAMPLE 3 Selecting a strategy to expand a binomial power involving a variable in each term

Expand and simplify $(5x+2y)^3$.

Jason's Solution

1
1 1
1 2 1
1 3 3 1

Since the exponent of the binomial is 3, I wrote out the 3rd row of Pascal's triangle.

$$(5x + 2y)^3$$
$$= 1(5x)^3 + 3(5x)^2(2y)^1 + 3(5x)^1(2y)^2 + 1(2y)^3$$

I used the terms 5*x* and 2*y* and applied the pattern for expanding a binomial.

$$= 1(125x^3) + 3(25x^2)(2y) + 3(5x)(4y^2) + 1(8y^3)$$

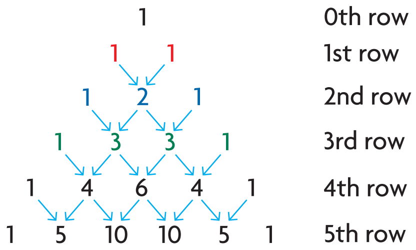
I simplified each term.

$$= 125x^3 + 150x^2y + 60xy^2 + 8y^3$$

In Summary

Key Ideas

- The arrangement of numbers shown is called Pascal's triangle. Each row is generated by calculating the sum of pairs of consecutive terms in the previous row.



- The numbers in Pascal's triangle correspond to the coefficients in the expansion of binomials raised to whole-number exponents.

Need to Know

- Pascal's triangle has many interesting relationships among its numbers. Some of these relationships are recursive.
  - For example, down the sides are constant sequences: 1, 1, 1, ...
  - The diagonal beside that is the counting numbers, 1, 2, 3, ... , which form an arithmetic sequence.
  - The next diagonal is the triangular numbers, 1, 3, 6, 10, ... , which can be defined by the recursive formula

$$t_1 = 1, t_n = t_{n-1} + n, \text{ where } n \in \mathbf{N} \text{ and } n > 1.$$

- There are patterns in the expansions of a binomial $(a + b)^n$:
  - Each term in the expansion is the product of a number from Pascal's triangle, a power of *a*, and a power of *b*.
  - The coefficients in the expansion correspond to the numbers in the *n*th row in Pascal's triangle.
  - In the expansion, the exponents of *a* start at *n* and decrease by 1 down to zero, while the exponents of *b* start at zero and increase by 1 up to *n*.
  - In each term, the sum of the exponents of *a* and *b* is always *n*.

CHECK Your Understanding

1. The first four entries of the 12th row of Pascal's triangle are 1, 12, 66, and 220. Determine the first four entries of the 13th row of the triangle.

2. Expand and simplify each binomial power.

a) $(x + 2)^5$ **b)** $(x - 1)^6$ **c)** $(2x - 3)^3$

3. Expand and simplify the first three terms of each binomial power.

a) $(x + 5)^{10}$ **b)** $(x - 2)^8$ **c)** $(2x - 7)^9$

PRACTISING

4. Expand and simplify each binomial power.

K **a)** $(k + 3)^4$ **c)** $(3q - 4)^4$ **e)** $\left(\sqrt{2}x + \sqrt{3}\right)^6$

b) $(y - 5)^6$ **d)** $(2x + 7y)^3$ **f)** $(2z^3 - 3y^2)^5$

5. Expand and simplify the first three terms of each binomial power.

a) $(x - 2)^{13}$ **c)** $(z^5 - z^3)^{11}$ **e)** $\left(3b^2 - \frac{2}{b}\right)^{14}$

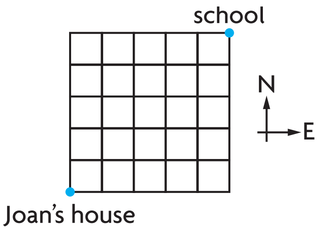
b) $(3y + 5)^9$ **d)** $\left(\sqrt{a} + \sqrt{5}\right)^{10}$ **f)** $(5x^3 + 3y^2)^8$

6. Using the pattern for expanding a binomial, expand each binomial power to describe a pattern in Pascal's triangle.

a) $2^n = (1 + 1)^n$

b) $0 = (1 - 1)^n$, where $n \geq 1$

7. Using the pattern for expanding a binomial, expand and simplify the expression $\frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]$, where $n = 1, 2, 3$, and 4. How are the terms related?



8. Using the diagram at the left, determine the number of different ways that Joan
A could walk to school from her house if she always travels either north or east.

9. Explain, without calculating, how you can use the pattern for expanding a
T binomial to expand $(x + y + z)^{10}$.

10. Expand and simplify $(3x - 5y)^6$.

11. Summarize the methods of expanding a binomial power and determining a
C term in an expansion.

Extending

12. If a relation is linear, the 1st differences are constant. If the 2nd differences are also constant, the relation is quadratic. Use the pattern for expanding a binomial to demonstrate that if a relation is cubic, the third differences are constant. (*Hint:* You may want to look at x^3 and $(x + 1)^3$.)

13. When a fair coin is tossed, the probability of getting heads or tails is $\frac{1}{2}$. Expand and simplify the first three terms in the expression $\left(\frac{1}{2} + \frac{1}{2}\right)^{10}$. How are the terms related to tossing the coin 10 times?

7 Chapter Review

FREQUENTLY ASKED Questions

Q: What strategies can you use to determine the sum of an arithmetic sequence?

A1: Write the series out twice, one above the other, once forward and once backward. When the terms of the two series are paired together, they have the same sum. This method works for calculating the sum of any arithmetic series.

A2: You can use either of the formulas $S_n = \frac{n[2a + (n - 1)d]}{2}$ or $S_n = \frac{n(t_1 + t_n)}{2}$. In both cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference. For either formula, you need to know the number of terms in the series and the first term. If you know the last term, use the formula in terms of t_1 and t_n. If you can calculate the common difference, use the formula in terms of a and d.

Study *Aid*

- See Lesson 7.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 14 to 17.

Q: How do you determine the sum of a geometric series?

A: You can use either of the formulas $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$. In both cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio. For either formula, you need to know the common ratio and the first term. If you know the $(n + 1)$th term, use the formula in terms of t_1 and t_{n+1}. If you can calculate the number of terms, use the formula in terms of a, r, and n.

Study *Aid*

- See Lesson 7.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 18 to 22.

Q: How do you expand a binomial power?

A: Use the pattern for expanding a binomial. Suppose you have the binomial $(a + b)^n$, where n is a whole number. Choose the nth row of Pascal's triangle for the coefficients. Each term in the expansion is a product of a number from Pascal's triangle, a power of a, and a power of b. The exponents of a start at n and decrease by 1 down to zero, while the exponents of b start at zero and increase by 1 up to n. In each term of the expansion, the sum of the exponents of a and b is always n.

Study *Aid*

- See Lesson 7.7, Examples 1, 2, and 3.
- Try Chapter Review Question 23.

PRACTICE Questions

Lesson 7.1

1. Represent the sequence 2, 8, 14, 20, ...
- **a)** in words
- **b)** algebraically
- **c)** graphically

2. How can you determine whether a sequence is arithmetic?

3. For each arithmetic sequence, state
- **i)** the general term
- **ii)** the recursive formula

- **a)** 58, 73, 88, ...
- **b)** $-49, -40, -31, \ldots$
- **c)** 81, 75, 69, ...

4. Determine the 100th term of the arithmetic sequence with $t_7 = 465$ and $t_{13} = 219$.

5. A student plants a seed. After the seed sprouts, the student monitors the growth of the plant by measuring its height every week. The height after each of the first three weeks was 7 mm, 20 mm, and 33 mm, respectively. If this pattern of growth continues, in what week will the plant be more than 100 mm tall?

Lesson 7.2

6. How can you determine whether a sequence is geometric?

7. **i)** Determine whether each sequence is arithmetic, geometric, or neither.
ii) If a sequence is arithmetic or geometric, determine t_6.

- **a)** 5, 15, 45, ...
- **b)** 0, 3, 8, ...
- **c)** 288, 14.4, 0.72, ...
- **d)** 10, 50, 90, ...
- **e)** 19, 10, 1, ...
- **f)** 512, 384, 288, ...

8. For each geometric sequence, determine
- **i)** the recursive formula
- **ii)** the general term
- **iii)** the first five terms

- **a)** the first term is 7 and the common ratio is -3
- **b)** $a = 12$ and $r = \frac{1}{2}$
- **c)** the second term is 36 and the third term is 144

9. **i)** Determine the type of each sequence (arithmetic, geometric, or neither), where $n \in \mathbf{N}$.
ii) State the first five terms.

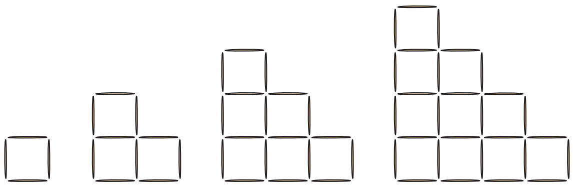
- **a)** $t_n = 4n + 5$
- **b)** $t_n = \dfrac{1}{7n - 3}$
- **c)** $t_n = n^2 - 1$
- **d)** $t_1 = -17, t_n = t_{n-1} + n - 1$, where $n > 1$

10. In a laboratory experiment, the count of a certain bacteria doubles every hour.
- **a)** At 1 p.m., there were 23 000 bacteria present. How many bacteria will be present at midnight?
- **b)** Can this model be used to determine the bacterial population at any time? Explain.

11. Guy purchased a rare stamp for $820 in 2001. If the value of the stamp increases by 10% per year, how much will the stamp be worth in 2010?

Lesson 7.3

12. Toothpicks are used to make a sequence of stacked squares as shown. Determine a rule for calculating t_n, the number of toothpicks needed for a stack of squares n high. Explain your reasoning.



13. Determine the 100th term of the sequence

$\frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \ldots$. Explain your reasoning.

Lesson 7.5

14. For each arithmetic series, calculate the sum of the first 50 terms.
- **a)** $1 + 9 + 17 + \ldots$
- **b)** $21 + 17 + 13 + \ldots$
- **c)** $31 + 52 + 73 + \ldots$
- **d)** $-9 - 14 - 19 - \ldots$
- **e)** $17.5 + 18.9 + 20.3 + \ldots$
- **f)** $-39 - 31 - 23 - \ldots$

15. Determine the sum of the first 25 terms of an arithmetic series in which

a) the first term is 24 and the common difference is 11

b) $t_1 = 91$ and $t_{25} = 374$

c) $t_1 = 84$ and $t_2 = 57$

d) the third term is 42 and the terms decrease by 11

e) the 12th term is 19 and the terms decrease by 4

f) $t_5 = 142$ and $t_{15} = 12$

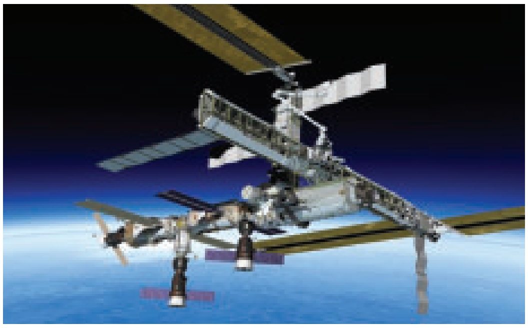
16. Calculate the sum of each series.

a) $1 + 13 + 25 + ... + 145$

b) $9 + 42 + 75 + ... + 4068$

c) $123 + 118 + 113 + ... - 122$

17. A spacecraft leaves an orbiting space station to descend to the planet below. The spacecraft descends 64 m during the first second and then engages its reverse thrusters to slow down its descent. It travels 7 m less during each second afterward. If the spacecraft lands after 10 s, how far did it descend?



Lesson 7.6

18. For each geometric series, calculate t_6 and S_6.

a) $11 + 33 + 99 + ...$

b) $0.111\,111 + 1.111\,11 + 11.1111 + ...$

c) $6 - 12 + 24 - ...$

d) $32\,805 + 21\,870 + 14\,580 + ...$

e) $17 - 25.5 + 38.25 - ...$

f) $\frac{1}{2} + \frac{3}{10} + \frac{9}{50} + ...$

19. Determine the sum of the first eight terms of the geometric series in which

a) the first term is -6 and the common ratio is 4

b) $t_1 = 42$ and $t_9 = 2112$

c) the first term is 320 and the second term is 80

d) the third term is 35 and the terms increase by a factor of 5

20. A catering company has 15 customer orders during its first month. For each month afterward, the company has double the number of orders than the previous month. How many orders in total did the company fill at the end of its first year?

21. The 1st, 5th, and 13th terms of an arithmetic sequence are the first three terms of a geometric sequence with common ratio 2. If the 21st term of the arithmetic sequence is 72, calculate the sum of the first 10 terms of the geometric sequence.

22. Calculate the sum of each series.

a) $7 + 14 + 28 + ... + 3584$

b) $-3 - 6 - 12 - 24 - ... - 768$

c) $1 + \frac{5}{2} + \frac{25}{4} + ... + \frac{15\,625}{64}$

d) $96\,000 - 48\,000 + 24\,000 - ... + 375$

e) $1000 + 1000(1.06) + 1000(1.06)^2 + ... + 1000(1.06)^{12}$

Lesson 7.7

23. Expand and simplify.

a) $(a + 6)^4$

b) $(b - 3)^5$

c) $(2c + 5)^3$

d) $(4 - 3d)^6$

e) $(5e - 2f)^4$

f) $\left(3f^2 - \frac{2}{f}\right)^4$

7 Chapter Self-Test

1. **i)** Determine the first five terms of each sequence, where $n \in \mathbf{N}$.
ii) Determine whether each sequence is arithmetic, geometric, or neither.

a) $t_n = 5 \times 3^{n+1}$

b) $t_n = \dfrac{3n + 2}{2n + 1}$

c) $t_n = 5n$

d) $t_1 = 5, t_n = 7t_{n-1}$, where $n > 1$

e) $t_1 = 19, t_n = 1 - t_{n-1}$, where $n > 1$

f) $t_1 = 7, t_2 = 13, t_n = 2t_{n-1} - t_{n-2}$, where $n > 2$

2. For each sequence, determine
i) the general term
ii) the recursive formula

a) a geometric sequence with $a = -9$ and $r = -11$

b) an arithmetic sequence with second term 123 and third term -456

3. Determine the number of terms in each sequence.
a) 18, 25, 32, ... , 193
b) 2, -10, 50, ... , $-156\,250$

4. Expand and simplify each binomial power.
a) $(x - 5)^4$
b) $(2x + 3y)^3$

5. Calculate the sum of each series.
a) $19 + 33 + 47 + ... + 439$
b) the first 10 terms of the series $10\,000 + 12\,000 + 14\,400 + ...$

6. A sequence is defined by the recursive formula $t_1 = 4$, $t_2 = 5$, $t_n = \dfrac{t_{n-1} + 1}{t_{n-2}}$, where $n \in \mathbf{N}$ and $n > 2$. Determine t_{123}. Explain your reasoning.

7. Your grandparents put aside \$100 for you on your first birthday. Every following year, they put away \$75 more than they did the previous year. How much money will have been put aside by the time you are 21?

8. Determine the next three terms of each sequence.
a) 1, 7, 8, 15, 23, 38, ...
b) $p^2 + 2q, p^3 - 3q, p^4 + 4q, p^5 - 5q, ...$
c) $\dfrac{25}{3}, \dfrac{19}{6}, \dfrac{13}{9}, \dfrac{7}{12}, \dfrac{1}{15}, ...$

7 Chapter Task

Allergy Medicine

It is estimated that 1 in 7 Canadians suffers from seasonal allergies such as hay fever. A typical treatment for hay fever is over-the-counter antihistamines. Tom decides to try a certain brand of antihistamine. The label says:

- The half-life of the antihistamine in the body is 16 h.
- For his size, maximum relief is felt when there are 150 mg to 180 mg in the body. Side effects (sleepiness, headaches, nausea) can occur when more than 180 mg are in the body.
- Each pill contains 30 mg of the active ingredient. It is unhealthy to ingest more than 180 mg within a 24 h period.

? How many hay fever pills should Tom take, and how often should he take them?

A. What are some conditions that would be reasonable when taking medication? For example, think about dosages, as well as times of the day when you would take the medication.

B. Determine three different schedules for taking the pills considering the appropriate amounts of the medication to ingest and your conditions in part A.

C. For each of your schedules, determine the amount of the medication present in Tom's body for the first few days.

D. Based on your calculations in part C, which schedule is best for Tom? Is another schedule more appropriate?

Task *Checklist*

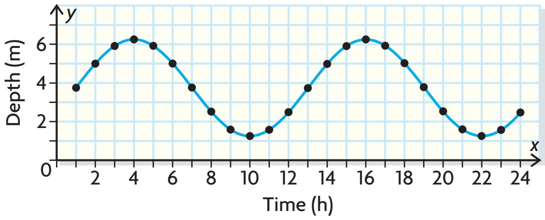
- ✓ Did you justify your "reasonable" conditions?
- ✓ Did you show your work?
- ✓ Did you support your choice of medication schedule?
- ✓ Did you explain your thinking clearly?

29. 23 folds

30.

Angle (degrees)	Length of BC (cm)
105	80.7
125	62.6
145	43.5
165	26.1
175	20.8

31. **a)**



$f(h) = 2.5\cos(30(h - 4))^\circ + 3.75$ or $f(h) = 2.5\sin(30(h - 1))^\circ + 3.75$

b) 6.25 m

c) The minimum depth of the water at this location is 1.25 m. Therefore, since the hull of the boat must have a clearance of at least 1 m at all times, if the bottom of the hull is more than 0.25 m below the surface of the water, then this location is not suitable for the dock. However, if the bottom of the hull is less than or equal to 0.25 m below the surface of the water, then this location is suitable for the dock.

Chapter 7

Getting Started, p. 414

1. **a)** $y = -\frac{2}{5}x + 8$
b) $y = -9x + 49$
c) $y = \frac{7}{5}x - 7$
d) $y = -2x - 7$

2. **a)** 6 **b)** $\frac{13}{10}$ or 1.3 **c)** 0 **d)** 4

3. **a)** linear **b)** neither **c)** quadratic

4. **a)** $x = 5$ **b)** $x = -5$ **c)** $x = \frac{33}{16}$ **d)** $x = 1.53$

5. about 2.2 g

6. 51.2%

7.

Definition:

A function of the form $f(x) = a \times b^x$, where a and b are constants.

Constant changes in the independent variable result in the dependent variable being multiplied by a constant.

Rules/Method:

The graph has a horizontal asymptote, and the graph looks like one of these shapes:

In a table of values, look at the 1st ratios. If they are constant, the function is an exponential.

Exponential Functions

Examples:

$f(x) = 9 \times 5^x$
$f(x) = \frac{2}{3} \times \left(\frac{5}{11}\right)^x$

Non-examples:

$y = \frac{2}{3}x - 7$ (linear function)
$y = x^3$ (cubic function)
7×2^x (exponential expression)

Lesson 7.1, pp. 424–425

1. **a)** arithmetic, $d = 4$ **b)** not arithmetic **c)** not arithmetic **d)** arithmetic, $d = -11$

2. **a)** General term: $t_n = 14n + 14$
Recursive formula: $t_1 = 28$, $t_n = t_{n-1} + 14$, where $n > 1$
b) General term: $t_n = 57 - 4n$
Recursive formula: $t_1 = 53$, $t_n = t_{n-1} - 4$, where $n > 1$
c) General term: $t_n = 109 - 110n$
Recursive formula: $t_1 = -1$, $t_n = t_{n-1} - 110$, where $n > 1$

3. $t_{12} = 53$

4. $t_{15} = 323$

5.

	i)	**ii)**
a)	arithmetic	General term: $t_n = 3n + 5$ Recursive formula: $t_1 = 8$, $t_n = t_{n-1} + 3$, where $n > 1$
b)	not arithmetic	—
c)	not arithmetic	—
d)	not arithmetic	—
e)	arithmetic	General term: $t_n = 11n + 12$ Recursive formula: $t_1 = 23$, $t_n = t_{n-1} + 11$, where $n > 1$
f)	arithmetic	General term: $t_n = \left(\frac{1}{6}\right)n$ Recursive formula: $t_1 = \frac{1}{6}$, $t_n = t_{n-1} + \frac{1}{6}$, where $n > 1$

6. **a)** Recursive formula: $t_1 = 19$, $t_n = t_{n-1} + 8$, where $n > 1$
General term: $t_n = 8n + 11$
b) Recursive formula: $t_1 = 4$, $t_n = t_{n-1} - 5$, where $n > 1$
General term: $t_n = 9 - 5n$
c) Recursive formula: $t_1 = 21$, $t_n = t_{n-1} + 5$, where $n > 1$
General term: $t_n = 5n + 16$
d) Recursive formula: $t_1 = 71$, $t_n = t_{n-1} - 12$, where $n > 1$
General term: $t_n = 83 - 12n$

7.

	i)	**ii)**
a)	arithmetic	13, 27, 41, 55, 69; $d = 14$
b)	not arithmetic	—
c)	not arithmetic	—
d)	arithmetic	1, 2, 3, 4, 5; $d = 1$

8.

	i)	**ii)**	**iii)**
a)	$t_n = 5n + 30$	$t_1 = 35$, $t_n = t_{n-1} + 5$, where $n > 1$	$t_{11} = 85$
b)	$t_n = 42 - 11n$	$t_1 = 31$, $t_n = t_{n-1} - 11$, where $n > 1$	$t_{11} = -79$
c)	$t_n = -17 - 12n$	$t_1 = -29$, $t_n = t_{n-1} - 12$, where $n > 1$	$t_{11} = -149$
d)	$t_n = 11$	$t_1 = 11$, $t_n = t_{n-1}$, where $n > 1$	$t_{11} = 11$
e)	$t_n = \left(\frac{1}{5}\right)n + \frac{4}{5}$	$t_1 = 1$, $t_n = t_{n-1} + \frac{1}{5}$, where $n > 1$	$t_{11} = 3$
f)	$t_n = 0.17n + 0.23$	$t_1 = 0.4$, $t_n = t_{n-1} + 0.17$, where $n > 1$	$t_{11} = 2.1$

9.

	i)	**ii)**
a)	arithmetic	6, 4, 2, 0, −2; $d = -2$
b)	not arithmetic	—
c)	arithmetic	$\frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$; $d = \frac{1}{4}$
d)	not arithmetic	—

10. **a)** 90 seats **b)** 23 rows

11. 63rd month

12. 16 years

13. **a)** 29 **b)** 38 **c)** 15 **d)** 14 **e)** 18 **f)** 9

14. Answers may vary. For example, $t_{100} = (t_8 - t_4) \times 23 + t_8$
The 4th and 8th terms differ by $4d$. The 8th and 100th term differ by $92d = 23 \times 4d$.

15. $t_n = 7n - 112$, where $n \in \mathbf{N}$

16. **a)** Answers will vary. For example,
20, 50, 80, … with $a = 20$ and $d = 30$
50, 20, −10, … with $a = 50$ and $d = -30$
5, 20, 35, 50, … with $a = 5$ and $d = 15$

b) The common difference must divide 30 (50 − 20) or −30 (20 − 50) evenly. The first term must be an integer multiple of the common difference away from 20 and 50.

17. $t_{100} = 112, 211, 310, 409, 607,$ or 1201

18. yes

Lesson 7.2, pp. 430–432

1. **a)** not geometric **b)** geometric, $r = 3$ **c)** not geometric **d)** geometric, $r = \frac{1}{2}$

2. **a)** General term: $t_n = 9 \times 4^{n-1}$
Recursive formula: $t_1 = 9$, $t_n = 4t_{n-1}$, where $n > 1$

b) General term: $t_n = 625 \times 2^{n-1}$
Recursive formula: $t_1 = 625$, $t_n = 2t_{n-1}$, where $n > 1$

c) General term: $t_n = 10\,125 \times \left(\frac{2}{3}\right)^{n-1}$

Recursive formula: $t_1 = 10\,125$, $t_n = \left(\frac{2}{3}\right) t_{n-1}$, where $n > 1$

3. $t_{33} = 9963$

4. $t_{10} = 180$

5.

	i)	**ii)**
a)	geometric	General term: $t_n = 12 \times 2^{n-1}$ or $t_n = 3 \times 2^{n+1}$ Recursive formula: $t_1 = 12$, $t_n = 2t_{n-1}$, where $n > 1$
b)	not geometric	—
c)	not geometric	—
d)	geometric	General term: $t_n = 5 \times (-3)^{n-1}$ Recursive formula: $t_1 = 5$, $t_n = -3t_{n-1}$, where $n > 1$
e)	not geometric	—
f)	geometric	General term: $t_n = 125 \times \left(\frac{2}{5}\right)^{n-1}$ Recursive formula: $t_1 = 125$, $t_n = \left(\frac{2}{5}\right) t_{n-1}$, where $n > 1$

6.

	i)	**ii)**	**iii)**
a)	$t_n = 4 \times 5^{n-1}$	$t_1 = 4$, $t_n = 5t_{n-1}$, where $n > 1$	$t_6 = 12\,500$
b)	$t_n = -11 \times 2^{n-1}$	$t_1 = -11$, $t_n = 2t_{n-1}$, where $n > 1$	$t_6 = -352$
c)	$t_n = 15 \times (-4)^{n-1}$	$t_1 = 15$, $t_n = -4t_{n-1}$, where $n > 1$	$t_6 = -15\,360$
d)	$t_n = 896 \times \left(\frac{1}{2}\right)^{n-1}$ or $t_n = 7 \times 2^{8-n}$	$t_1 = 896$, $t_n = \left(\frac{1}{2}\right) t_{n-1}$, where $n > 1$	$t_6 = 28$
e)	$t_n = 6 \times \left(\frac{1}{3}\right)^{n-1}$ or $t_n = 2 \times 3^{2-n}$	$t_1 = 6$, $t_n = \left(\frac{1}{3}\right) t_{n-1}$, where $n > 1$	$t_6 = \frac{2}{81}$
f)	$t_n = 0.2^{n-1}$	$t_1 = 1$, $t_n = 0.2t_{n-1}$, where $n > 1$	$t_6 = 0.000\,32$

7.

	i)	**ii)**
a)	arithmetic	$t_n = 4n + 5$
b)	geometric	$t_n = 7 \times (-3)^{n-1}$
c)	geometric	$t_n = 18 \times (-1)^{n-1}$
d)	neither	—
e)	arithmetic	$t_n = 39 - 10n$
f)	geometric	$t_n = 128 \times \left(\frac{3}{4}\right)^{n-1}$

8. **a)** Recursive formula: $t_1 = 19$, $t_n = 5t_{n-1}$, where $n > 1$
General term: $t_n = 19 \times 5^{n-1}$

b) Recursive formula: $t_1 = -9$, $t_n = -4t_{n-1}$, where $n > 1$
General term: $t_n = -9 \times (-4)^{n-1}$

c) Recursive formula: $t_1 = 144$, $t_n = \left(\frac{1}{4}\right)t_{n-1}$, where $n > 1$

General term: $t_n = 144 \times \left(\frac{1}{4}\right)^{n-1}$ or $t_n = 9 \times 4^{3-n}$

d) Recursive formula: $t_1 = 900$, $t_n = \left(\frac{1}{6}\right)t_{n-1}$, where $n > 1$

General term: $t_n = 900 \times \left(\frac{1}{6}\right)^{n-1}$ or $t_n = 25 \times 6^{3-n}$

9.

	i)	**ii)**
a)	not geometric	—
b)	geometric	$-8, 24, -72, 216, -648; r = -3$
c)	geometric	$123, 41, \frac{41}{3}, \frac{41}{9}, \frac{41}{27}; r = \frac{1}{3}$
d)	geometric	$10, 20, 40, 80, 160; r = 2$

10.

	i)	**ii)**
a)	geometric	$4, 16, 64, 256, 1024; r = 4$
b)	not geometric	—
c)	not geometric	—
d)	geometric	$-\frac{7}{125}, \frac{7}{25}, -\frac{7}{5}, 7, -35; r = -5$
e)	not geometric	—
f)	geometric	$\frac{11}{13}, \frac{11}{169}, \frac{11}{2197}, \frac{11}{28\,561}, \frac{11}{371\,293}; r = \frac{1}{13}$

11. 1 474 560

12. 131 220 bacteria

13. \$10 794.62

14. **a)** 65.61% **b)** 29 dosages

15. Answers may vary. For example,
Yes, $t_{29} = t_7 \times \left(\frac{t_7}{t_5}\right)^{11}$. Use the formula for the general term to write t_{29} in terms of a and r. Then write the equation for t_{29} using the laws of exponents. Evaluate $\frac{t_7}{t_5}$ and then rewrite t_{29} in terms of t_5 and t_7.

16. **a)** 243 shaded triangles **b)** about 0.338 cm²

17. Both sequences are recursive, so the recursive formulas look similar, except that you add with an arithmetic sequence but multiply with a geometric sequence. The general terms also look similar, except that with an arithmetic sequence, you add a multiple of the common difference to the first term but with a geometric sequence, you multiply a power of the common ratio with the first term.

18. $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}$; The sum gets closer to 2.

19. $t_{10} = 10\,752$, general term: $t_n = 2^{n-1}(2n + 1)$

20. Yes, only for the sequence $a, a, a, \ldots$, where $d = 0$ and $r = 1$.

21. Answers will vary. For example, 1, 2, 3, 4, 5, 6, 7, 8, … (arithmetic sequence). To form the geometric sequence (shown in red), the previous term determines which term you select.

22. about 35.44 cm²

Lesson 7.3, pp. 439–440

1. Yes for $n > 2$, each term depends only on the two previous terms (difference), so the sequence will repeat. Check Sam's formula with other terms to see that it works.

2. $t_n = \dfrac{n}{n + 1}$

3. **a)** $t_n = 2n + 1$ **c)** $t_n = 2n^2 + 2n$
b) $t_n = 3n + 1$ **d)** (triangles) $t_4 = 2(4) + 1 = 9$ toothpicks
(squares) $t_4 = 2(4)^2 + 2(4) = 40$ toothpicks

4. **a)** $t_n = \begin{cases} \cancel{-\left(\dfrac{n-1}{2}\right)} & \text{if } n \text{ is odd} \quad 1-n \\ \cancel{\dfrac{n}{2}}\; n & \text{if } n \text{ is even} \end{cases}$

b) Another rule is $t_1 = 0$, $t_n = \dfrac{n}{2}$ for even n, and $t_n = -t_{n-1}$ for odd n. The other rule is better because each term can be calculated directly, instead of having to calculate terms for even n before calculating terms for odd n.

c) $t_{12345} = \cancel{-6172}$ -12344

5. $nx + \dfrac{1}{y^n}$

6. $t_n = \dfrac{N_n}{D_n}$, where $N_n = 3 \times 7^{n-1}$ and D_n has n fives or

$$t_n = \frac{3 \times 7^{n-1}}{\frac{5}{9}(10^n - 1)}$$

7. **a)** 159, 319, 639 **c)** 34, 55, 89 **e)** −216, 343, −512
b) 85, 79, 72 **d)** 54, 108, 110 **f)** 111, 223, 447

8. ~~9900~~ comparisons 4950

9. 169 271, 846 354, 4 231 771

10. $t_{1000} = 20$

11. Answers will vary. For example, 2, 3, 6, 11, 18, 27, ... The 1st differences form a sequence of odd numbers. So to generate new terms, work backward from the 1st differences.

Lesson 7.4, p. 443

1. The ratio of consecutive terms tends toward the same value as the Fibonacci and Lucas sequences, and the sequences have similar identities.

2. $r = \dfrac{1 \pm \sqrt{5}}{2}$; the positive root $r = \dfrac{1 + \sqrt{5}}{2}$ approximates the ratio of $\dfrac{F_n}{F_{n-1}}$ and $\dfrac{L_n}{L_{n-1}}$ as n increases.

3. **a)** 1, 5, 7, 17, 31, 65, 127, 257, 511, and 1025
b) about 5, 1.4, 2.43, 1.82, 2.10, 1.95, 2.02, 1.99, and 2.00. Ratios get close to 2.
c) $t_n = 2^n + (-1)^n$

Mid-Chapter Review, p. 447

1.

	i)	ii)	iii)
a)	$t_1 = 29, t_n = t_{n-1} - 8$, where $n > 1$	$t_n = 37 - 8n$	$t_{10} = -43$
b)	$t_1 = -8, t_n = t_{n-1} - 8$, where $n > 1$	$t_n = -8n$	$t_{10} = -80$
c)	$t_1 = -17, t_n = t_{n-1} + 8$, where $n > 1$	$t_n = 8n - 25$	$t_{10} = 55$
d)	$t_1 = 3.25, t_n = t_{n-1} + 6.25$, where $n > 1$	$t_n = 6.25n - 3$	$t_{10} = 59.5$
e)	$t_1 = \frac{1}{2}, t_n = t_{n-1} + \frac{1}{6}$, where $n > 1$	$t_n = \frac{1}{6}n + \frac{1}{3}$	$t_{10} = 2$
f)	$t_1 = x, t_n = t_{n-1} + 2x + 3y$, where $n > 1$	$t_n = (2n - 1) \times x + 3(n - 1)y$	$t_{10} = 19x + 27y$

2. **a)** Recursive formula: $t_1 = 17, t_n = t_{n-1} + 11$, where $n > 1$
General term: $t_n = 11n + 6$
b) Recursive formula: $t_1 = 38, t_n = t_{n-1} - 7$, where $n > 1$
General term: $t_n = 45 - 7n$
c) Recursive formula: $t_1 = 55, t_n = t_{n-1} + 18$, where $n > 1$
General term: $t_n = 18n + 37$
d) Recursive formula: $t_1 = 42, t_n = t_{n-1} - 38$, where $n > 1$
General term: $t_n = 80 - 38n$
e) Recursive formula: $t_1 = 159, t_n = t_{n-1} - 17$, where $n > 1$
General term: $t_n = 176 - 17n$

3. 315 seats

4.

	i)	ii)
a)	arithmetic	General term: $t_n = 15n$ Recursive formula: $t_1 = 15$, $t_n = t_{n-1} + 15$, where $n > 1$ $t_6 = 90$
b)	geometric	General term: $t_n = 640 \times \left(\frac{1}{2}\right)^{n-1}$ Recursive formula: $t_1 = 640$, $t_n = \left(\frac{1}{2}\right)t_{n-1}$, where $n > 1$ $t_6 = 20$
c)	geometric	General term: $t_n = 23 \times (-2)^{n-1}$ Recursive formula: $t_1 = 23$, $t_n = -2t_{n-1}$, where $n > 1$ $t_6 = -736$
d)	geometric	General term: $t_n = 3000 \times (0.3)^{n-1}$ Recursive formula: $t_1 = 3000$, $t_n = 0.3t_{n-1}$, where $n > 1$ $t_6 = 7.29$
e)	arithmetic	General term: $t_n = 1.2n + 2.6$ Recursive formula: $t_1 = 3.8$, $t_n = t_{n-1} + 1.2$, where $n > 1$ $t_6 = 9.8$
f)	geometric	General term: $t_n = \left(\frac{1}{2}\right) \times \left(\frac{2}{3}\right)^{n-1}$ Recursive formula: $t_1 = \frac{1}{2}$, $t_n = \left(\frac{2}{3}\right)t_{n-1}$, where $n > 1$ $t_6 = \frac{16}{243}$

5.

	i)	ii)
a)	geometric	5, 25, 125, 625, 3125
b)	neither	$\frac{3}{7}, \frac{3}{19}, \frac{3}{67}, \frac{3}{259}, \frac{3}{1027}$
c)	arithmetic	5, −7, −19, −31, −43
d)	geometric	−2, 4, −8, 16, −32
e)	arithmetic	8, 11, 14, 17, 20

6. 45 weeks

7. 349, 519, 737
The 3rd differences are constant, so use them to determine terms.

8. $t_n = x^n + ny$

9. **a)** 1, 8, 27 **c)** $t_n = n^3$
b) 64, 125, 216 **d)** 3375

10. **a)** $t_{15} = 1453$
b) $t_1 = 3, t_2 = 2, t_n = t_{n-2} + t_{n-1}$, where $n > 2$

Lesson 7.5, pp. 452–453

1. **a)** $S_{10} = 815$ **c)** $S_{10} = -1345$
b) $S_{10} = -50$ **d)** $S_{10} = 210$

2. $S_{20} = 2450$

3. 670 bricks

4.

	i)	ii)
a)	arithmetic	$S_{25} = 1675$
b)	not arithmetic	—
c)	not arithmetic	—
d)	arithmetic	$S_{25} = 1650$
e)	arithmetic	$S_{25} = -1925$
f)	not arithmetic	—

5. **a)** $t_{12} = 81, S_{12} = 708$ **d)** $t_{12} = \frac{57}{10}$ or 5.7, $S_{12} = \frac{177}{5}$ or 35.4
b) $t_{12} = -134, S_{12} = -882$ **e)** $t_{12} = 15.51, S_{12} = 112.2$
c) $t_{12} = 48, S_{12} = 180$ **f)** $t_{12} = 12p + 22q, S_{12} = 78p + 132q$

6. **a)** $S_{20} = 1110$ **c)** $S_{20} = -1980$ **e)** $S_{20} = 2410$
b) $S_{20} = 1400$ **d)** $S_{20} = 2570$ **f)** $S_{20} = 1630$

7. **a)** $S_{20} = 970$ **c)** $S_{24} = -168$ **e)** $S_{16} = -1336$
b) $S_{26} = 4849$ **d)** $S_{711} = 760\ 770$ **f)** $S_{22} = 0$

8. **a)** $D_n = \frac{n(n-3)}{2}$, where $n > 2$ **b)** 14 diagonals

9. $~~5630~~ 6945

10. 1102.5 m

11. 2170 toys

12. 3050 s or 50 min 50 s

13. 700 km

14. Two copies of the first representation fit together to form a rectangle $t_1 + t_n$ by n, yielding the formula $S_n = \frac{n(t_1 + t_n)}{2}$.

Two copies of the second representation fit together to form a rectangle. You can see a and d and get the formula $S_n = \frac{n[2a + (n-1)d]}{2}$.

15. $t_{25} = 79$

16. 26 terms

Lesson 7.6, p. 459–461

1. **a)** $S_7 = 6558$ **c)** $S_7 = 4376$
b) $S_7 = \frac{3175}{16}$ or 198.4375 **d)** $S_7 = \frac{127}{192}$

2. $S_6 = 15\ 015$

3. **a)** $t_6 = 18\ 750, S_6 = 23\ 436$ **d)** $t_6 = \frac{128}{1215}, S_6 = \frac{532}{243}$
b) $t_6 = -2673, S_6 = -4004$ **e)** $t_6 \doteq -138.859, S_6 \doteq -92.969$
c) $t_6 = 6720, S_6 = 26\ 248\ 320$ **f)** $t_6 = 243x^{10}, S_6 = \frac{729x^{12} - 1}{3x^2 - 1}$

4.

	i)	ii)
a)	arithmetic	—
b)	geometric	$S_8 = 22\ 960$
c)	geometric	$S_8 = \frac{13\ 107}{8}$ or 1638.375
d)	neither	—
e)	geometric	$S_8 \doteq 12.579$
f)	arithmetic	—

5. **a)** $S_7 = 253\ 903$ **d)** $S_7 = 2186$
b) $S_7 = 1397$ **e)** $S_7 = 49\ 416$
c) $S_7 = \frac{163\ 830}{1024}$ or about 159.990 **f)** $S_7 = \frac{645}{48}$ or 13.4375

6. **a)** $S_8 = 335\ 923$ **c)** $S_{10} = -250\ 954$ **e)** $S_8 = 78\ 642$
b) $S_7 = 1905$ **d)** $S_6 = 28\ 234.9725$ **f)** $S_{13} = \frac{8191}{1024}$

7. about 10.8 m

8. If $r = 1$, all the terms are the same, $a + a + a + \ldots$. So the sum of n terms would be $S_n = na$.

9. 1 048 575 line segments

10. 12.25 m^2

11. 5465 employees

12. 14 559 864

13. Answers will vary. For example, the first prize is $1829, each prize is 3 times the previous one, and there are 7 prizes altogether. The total value of the prizes is $1 999 097.

14.

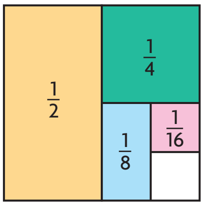
Arithmetic	Geometric	Similarities	Differences
$S_n = \frac{n(t_1 + t_n)}{2}$ Write the terms of S_n twice, once forward, then once backward above each other. Then add pairs of terms. The sum of the pairs is constant.	$S_n = \frac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$ Write the terms of S_n and rS_n above each other. Then subtract pairs of terms. The difference of all middle pairs is zero.	Both general formulas involve two "end" terms of the series.	You add with formula for arithmetic series, but subtract for geometric series. You divide by 2 for arithmetic series, but by $r - 1$ for geometric series.

15. $t_5 = 15\ 552$

16. 13 terms

17. $x^{15} - 1 = (x - 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1)$

18. **a)**



b) The formula for the sum of geometric series gives the sum of the first n terms. Using $a = \frac{1}{2}$, and $r = \frac{1}{2}$, $S_n = \frac{\frac{1}{2}\left[\left(\frac{1}{2}\right)^n - 1\right]}{-\frac{1}{2}}$.

As n approaches infinity, $S_n \doteq \frac{\frac{1}{2}(0 - 1)}{-\frac{1}{2}} \doteq 1$.

c) Yes. Consecutive terms of this series are getting smaller and smaller, so the sum is getting closer and closer to 1.

Lesson 7.7, p. 466

1. 1, 13, 78, and 286

2. **a)** $(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
b) $(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$
c) $(2x - 3)^3 = 8x^3 - 36x^2 + 54x - 27$

3. **a)** $(x + 5)^{10} = x^{10} + 50x^9 + 1125x^8 + \ldots$
b) $(x - 2)^8 = x^8 - 16x^7 + 112x^6 - \ldots$
c) $(2x - 7)^9 = 512x^9 - 16\ 128x^8 + 225\ 792x^7 - \ldots$

4. **a)** $(k + 3)^4 = k^4 + 12k^3 + 54k^2 + 108k + 81$
b) $(y - 5)^6 = y^6 - 30y^5 + 375y^4 - 2500y^3 + 9375y^2 - 18\ 750y + 15\ 625$
c) $(3q - 4)^4 = 81q^4 - 432q^3 + 864q^2 - 768q + 256$
d) $(2x + 7y)^3 = 8x^3 + 84x^2y + 294xy^2 + 343y^3$
e) $(\sqrt{2}x + \sqrt{3})^6 = 8x^6 + 24\sqrt{6}x^5 + 180x^4 + 120\sqrt{6}x^3 + 270x^2 + 54\sqrt{6}x + 27$

f) $(2z^3 - 3y^2)^5 = 32z^{15} - 240z^{12}y^2 + 720z^9y^4 - 1080z^6y^6 + 810z^3y^8 - 243y^{10}$

5. **a)** $(x - 2)^{13} = x^{13} - 26x^{12} + 312x^{11} - \ldots$
b) $(3y + 5)^9 = 19\,683y^9 + 295\,245y^8 + 1\,968\,300y^7 + \ldots$
c) $(z^5 - z^3)^{11} = z^{55} - 11z^{53} + 55z^{51} - \ldots$
d) $(\sqrt{a} + \sqrt{5})^{10} = a^5 + 10\sqrt{5a}\,a^4 + 225a^4 + \ldots$
e) $\left(3b^2 - \frac{2}{b}\right)^{14} = 4\,782\,969b^{28} - 44\,641\,044b^{25} + 193\,444\,524b^{22} - \ldots$
f) $(5x^3 + 3y^2)^8 = 390\,625x^{24} + 1\,875\,000x^{21}y^2 + 3\,937\,500x^{18}y^4 + \ldots$

6. **a)** The sum of all the numbers in a row of Pascal's triangle is equal to a power of 2.
b) If you alternately subtract and add the numbers in a row of Pascal's triangle, the result is always zero.

7. 1, 1, 2, 3; These are terms in the Fibonacci sequence.

8. 252 ways

9. Write $(x + y + z)^{10} = [x + (y + z)]^{10}$ and use the pattern for expanding a binomial twice.

10. $(3x - 5y)^6 = 729x^6 - 7290x^5y + 30\,375x^4y^2 - 67\,500x^3y^3 + 84\,375x^2y^4 - 56\,250xy^5 + 15\,625y^6$

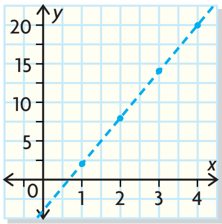
11. To expand $(a + b)^n$, where $n \in \mathbf{N}$, write the numbers from the nth row of Pascal's triangle. Each term in the expansion is the product of a number from Pascal's triangle, a power of a, and a power of b. The exponents of a start at n and go down, term by term, to zero. The exponents of b start at zero and go up, term by term, to n.

12. The 1st differences of a cubic correspond to the differences between the $(x + 1)$th value and the xth value. The 1st differences of a cubic are quadratic. The 2nd differences of a cubic correspond to the 1st differences of a quadratic, and the 3rd differences of a cubic correspond to the 2nd differences of a quadratic. Since the 2nd differences of a quadratic are constant, the 3rd differences of a cubic are constant.

13. $\left(\frac{1}{2} + \frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + 45\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \ldots$
The first three terms represent the probablility of getting heads 10, 9, and 8 times, respectively.

Chapter Review, pp. 468–469

1. **a)** Arithmetic sequence with first term 2, and each term afterward increases by 6.
b) General term: $t_n = 6n - 4$
Recursive formula: $t_1 = 2$, $t_n = t_{n-1} + 6$, where $n > 1$
c)



2. Check if the difference between consecutive terms is constant.

3.

	i)	ii)
a)	$t_n = 15n + 43$	$t_1 = 58$, $t_n = t_{n-1} + 15$, where $n > 1$
b)	$t_n = 9n - 58$	$t_1 = -49$, $t_n = t_{n-1} + 9$, where $n > 1$
c)	$t_n = 87 - 6n$	$t_1 = 81$, $t_n = t_{n-1} - 6$, where $n > 1$

4. $t_{100} = -3348$

5. 9 weeks

6. Check if the ratio of consecutive terms is constant.

7.

	i)	ii)
a)	geometric	$t_6 = 1215$
b)	neither	—
c)	geometric	$t_6 = 0.000\,09$
d)	arithmetic	$t_6 = 210$
e)	arithmetic	$t_6 = -26$
f)	geometric	$t_6 = 121.5$

8.

	i)	ii)	iii)
a)	$t_1 = 7$, $t_n = -3t_{n-1}$, where $n > 1$	$t_n = 7 \times (-3)^{n-1}$	7, −21, 63, −189, 567
b)	$t_1 = 12$, $t_n = \left(\frac{1}{2}\right)t_{n-1}$, where $n > 1$	$t_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$	12, 6, 3, $\frac{3}{2}$, $\frac{3}{4}$
c)	$t_1 = 9$, $t_n = 4t_{n-1}$, 576, where $n > 1$	$t_n = 9 \times 4^{n-1}$	9, 36, 144, 2304

9.

	i)	ii)
a)	arithmetic	9, 13, 17, 21, 25
b)	neither	$\frac{1}{4}, \frac{1}{11}, \frac{1}{18}, \frac{1}{25}, \frac{1}{32}$
c)	neither	0, 3, 8, 15, 24
d)	neither	−17, −16, −14, −11, −7

10. **a)** 47 104 000 bacteria
b) No. The bacteria would eventually run out of food and space in the culture.

11. \$1933.52

12. $t_n = n^2 + 3n$

13. $t_{100} = \frac{100}{299}$

14. **a)** $S_{50} = 9850$ **d)** $S_{50} = -6575$
b) $S_{50} = -3850$ **e)** $S_{50} = 2590$
c) $S_{50} = 27\,275$ **f)** $S_{50} = 11\,750$

15. **a)** $S_{25} = 3900$ **c)** $S_{25} = -6000$ **e)** $S_{25} = 375$
b) $S_{25} = 5812.5$ **d)** $S_{25} = -1700$ **f)** $S_{25} = 950$

16. **a)** $S_{13} = 949$ **b)** $S_{124} = 252\,774$ **c)** $S_{50} = 25$

17. 325 m

18. **a)** $t_6 = 2673$, $S_6 = 4004$
b) $t_6 = 11\,111.1$, $S_6 \doteq 12\,345.654$
c) $t_6 = -192$, $S_6 = -126$
d) $t_6 = 4320$, $S_6 = 89\,775$
e) $t_6 = -\frac{4131}{32}$ or $-129.093\,75$, $S_6 = -\frac{2261}{32}$
f) $t_6 = \frac{243}{6250}$, $S_6 = \frac{3724}{3125}$

19. **a)** $S_8 = -131\,070$ **c)** $S_8 \doteq 426.660$
b) $S_8 \doteq 3276.087\,34$ **d)** $S_8 = 136\,718.4$

20. 61 425 orders

21. $S_{10} = 12\,276$

22. **a)** 7161 **c)** $\frac{25\,999}{64}$
b) 1533 **d)** 64 125
e) about 18 882.14

23. **a)** $(a + 6)^4 = a^4 + 24a^3 + 216a^2 + 864a + 1296$
b) $(b - 3)^5 = b^5 - 15b^4 + 90b^3 - 270b^2 + 405b - 243$
c) $(2c + 5)^3 = 8c^3 + 60c^2 + 150c + 125$
d) $(4 - 3d)^6 = 4096 - 18\,432d + 34\,560d^2 - 34\,560d^3 + 19\,440d^4 - 5832d^5 + 729d^6$
e) $(5e - 2f)^4 = 625e^4 - 1000e^3f + 600e^2f^2 - 160ef^3 + 16f^4$
f) $\left(3f^2 - \frac{2}{f}\right)^4 = 81f^8 - 216f^5 + 216f^2 - \frac{96}{f} + \frac{16}{f^4}$

Chapter Self-Test, p. 470

1.

	i)	ii)
a)	45, 135, 405, 1215, 3645	geometric
b)	$\frac{5}{3}, \frac{8}{5}, \frac{11}{7}, \frac{14}{9}, \frac{17}{11}$	neither
c)	5, 10 , 15, 20, 25	arithmetic
d)	5, 35, 245, 1715, 12 005	geometric
e)	19, −18, 19, −18, 19	neither
f)	7, 13, 19, 25, 31	arithmetic

2.

	i)	ii)
a)	$t_n = (-9) \times (-11)^{n-1}$	$t_1 = -9, t_n = -11t_{n-1}$, where $n > 1$
b)	$t_n = 1281 - 579n$	$t_1 = 702, t_n = t_{n-1} - 579$, where $n > 1$

3. **a)** 26 terms
b) 8 terms

4. **a)** $(x - 5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$
b) $(2x + 3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$

5. **a)** $S_{31} = 7099$
b) $S_{10} = 259\ 586.8211$

6. $t_{123} = \frac{3}{2}$

7. $17 850

8. **a)** 61, 99, 160
b) $p^6 + 6q, p^7 - 7q, p^8 + 8q$
c) $-\frac{5}{18}, -\frac{11}{21}, -\frac{17}{24}$

Chapter 8

Getting Started, p. 474

1.

	i)	ii)	iii)
a)	23, 27	$t_n = 3 + 4n$	$t_1 = 7, t_n = t_{n-1} + 4$, where $n > 1$
b)	−50, −77	$t_n = 85 - 27n$	$t_1 = 58, t_n = t_{n-1} - 27$, where $n > 1$
c)	1280, 5120	$t_n = 5 \times 4^{n-1}$	$t_1 = 5, t_n = 4t_{n-1}$, where $n > 1$
d)	−125, 62.5	$t_n = 1000 \times \left(\frac{-1}{2}\right)^{n-1}$	$t_1 = 1000$, $t_n = \left(\frac{-1}{2}\right)t_{n-1}$, where $n > 1$

2. **a)** $t_5 = 147$ **b)** $d = 101$ **c)** $a = -257$ **d)** $t_{100} = 9742$

3. **a)** geometric—There is a constant rate between the terms.
b) $t_1 = 8000, t_n = (1.05)t_{n-1}$, where $n > 1$
c) $t_n = 8000 \times (1.05)^{n-1}$
d) $t_{10} \doteq 12\ 410.6257$

4. **a)** $S_{10} = 120$ **c)** $S_{10} = -285$
b) $S_{10} = 0$ **d)** $S_{10} = 5\ 456\ 000$

5. **a)** (1st year) 210 000, (2nd year) 220 500, (3rd year) 231 525
b) about 325 779

6. $x = 12$

7. **a)** $x \doteq 19.93$ **c)** $x \doteq 11.26$
b) $x \doteq 3.48$ **d)** $x \doteq 8.72$

8. Answers may vary. For example,

Example:	**Visual representation:**
$f(x) = 5 \times 2^x$	The graph looks like one of these shapes: 
Definition in your own words:	**Personal association:**
A function of the form $f(x) = a \times b^x$, where a and b are constants. When x increases or decreases by a constant, y is multiplied by a constant.	half-life geometric sequences

Lesson 8.1, pp. 481–482

1. **a)** **i)** (1st year) $532, (2nd year) $564, (3rd year) $596
ii) $980
b) **i)** (1st year) $1301.25, (2nd year) $1352.50, (3rd year) $1403.75
ii) $2018.75
c) **i)** (1st year) $26 250, (2nd year) $27 500, (3rd year) $28 750
ii) $43 750
d) **i)** (1st year) $1739.10, (2nd year) $1778.20, (3rd year) $1817.30
ii) $2286.50

2. **a)** $P = \$2000$ **c)** $r = 6\%/\text{a}$
b) $I = \$600$ **d)** $A(t) = 2000 + 120t$

3. 3 years and 132 days

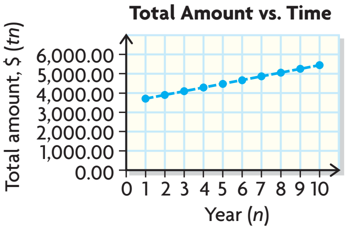
4. about 28%/a

5. **a)** $I = \$192, A = \$692$ **d)** $I = \$9.60, A = \$137.60$
b) $I = \$3763.20, A = \$6963.20$ **e)** $I = \$3923.08, A = \$53\ 923.08$
c) $I = \$260, A = \$5260$ **f)** $I = \$147.95, A = \$4647.95$

6. about 7.84%/a

7. $47 619.05

8. **a)** $192.50
b) (1st year) $3692.50, (2nd year) $3885.00, (3rd year) $4077.50, (4th year) $4270.00, (5th year) $4462.50
c) $t_n = 3500 + 192.50n$
d)



9. **a)** $3740 **b)** 27.2%/a

10. **a)** $1850
b) $t_n = 1850 + 231.25n$
c) 24 years and 158 days

11. 66 years and 8 months

12. $P = \$750$, $r = 3.7\%/\text{a}$ $A(t) = P + Prt; P = 750; Prt = 27.75t;$ $750rt = 27.75t; 750r = 27.75; r = 0.037$

13. $D = \frac{1}{r}$

14. $23 400