A Mathematician's Lament

by Paul Lockhart

A musician wakes from a terrible nightmare. In his dream he finds himself in a society where music education has been made mandatory. "We are helping our students become more competitive in an increasingly sound-filled world." Educators, school systems, and the state are put in charge of this vital project. Studies are commissioned, committees are formed, and decisions are made— all without the advice or participation of a single working musician or composer.

Since musicians are known to set down their ideas in the form of sheet music, these curious black dots and lines must constitute the "language of music." It is imperative that students become fluent in this language if they are to attain any degree of musical competence; indeed, it would be ludicrous to expect a child to sing a song or play an instrument without having a thorough grounding in music notation and theory. Playing and listening to music, let alone composing an original piece, are considered very advanced topics and are generally put off until college, and more often graduate school.

As for the primary and secondary schools, their mission is to train students to use this language— to jiggle symbols around according to a fixed set of rules: "Music class is where we take out our staff paper, our teacher puts some notes on the board, and we copy them or transpose them into a different key. We have to make sure to get the clefs and key signatures right, and our teacher is very picky about making sure we fill in our quarter-notes completely. One time we had a chromatic scale problem and I did it right, but the teacher gave me no credit because I had the stems pointing the wrong way."

In their wisdom, educators soon realize that even very young children can be given this kind of musical instruction. In fact it is considered quite shameful if one's third-grader hasn't completely memorized his circle of fifths. "I'll have to get my son a music tutor. He simply won't apply himself to his music homework. He says it's boring. He just sits there staring out the window, humming tunes to himself and making up silly songs."

In the higher grades the pressure is really on. After all, the students must be prepared for the standardized tests and college admissions exams. Students must take courses in Scales and Modes, Meter, Harmony, and Counterpoint. "It's a lot for them to learn, but later in college when they finally get to hear all this stuff, they'll really appreciate all the work they did in high school." Of course, not many students actually go on to concentrate in music, so only a few will ever get to hear the sounds that the black dots represent. Nevertheless, it is important that every member of society be able to recognize a modulation or a fugal passage, regardless of the fact that they will never hear one. "To tell you the truth, most students just aren't very good at music. They are bored in class, their skills are terrible, and their homework is barely legible. Most of them couldn't care less about how important music is in today's world; they just want to take the minimum number of music courses and be done with it. I guess there are just music people and non-music people. I had this one kid, though, man was she sensational! Her sheets were impeccable— every note in the right place, perfect calligraphy, sharps, flats, just beautiful. She's going to make one hell of a musician someday."

Waking up in a cold sweat, the musician realizes, gratefully, that it was all just a crazy dream. "Of course!" he reassures himself, "No society would ever reduce such a beautiful and meaningful art form to something so mindless and trivial; no culture could be so cruel to its children as to deprive them of such a natural, satisfying means of human expression. How absurd!"

Meanwhile, on the other side of town, a painter has just awakened from a similar nightmare...

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Sadly, our present system of mathematics education is precisely this kind of nightmare. In fact, if I had to design a mechanism for the express purpose of *destroying* a child's natural curiosity and love of pattern-making, I couldn't possibly do as good a job as is currently being done— I simply wouldn't have the imagination to come up with the kind of senseless, soul-crushing ideas that constitute contemporary mathematics education.

Everyone knows that something is wrong. The politicians say, "we need higher standards." The schools say, "we need more money and equipment." Educators say one thing, and teachers say another. They are all wrong. The only people who understand what is going on are the ones most often blamed and least often heard: the students. They say, "math class is stupid and boring," and they are right.

Mathematics and Culture

The first thing to understand is that mathematics is an art. The difference between math and the other arts, such as music and painting, is that our culture does not recognize it as such. Everyone understands that poets, painters, and musicians create works of art, and are expressing themselves in word, image, and sound. In fact, our society is rather generous when it comes to creative expression; architects, chefs, and even television directors are considered to be working artists. So why not mathematicians?

Part of the problem is that nobody has the faintest idea what it is that mathematicians do. The common perception seems to be that mathematicians are somehow connected with science— perhaps they help the scientists with their formulas, or feed big numbers into computers for some reason or other. There is no question that if the world had to be divided into the "poetic dreamers" and the "rational thinkers" most people would place mathematicians in the latter category.

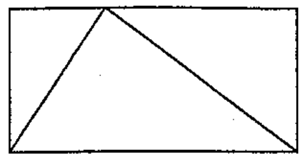
Nevertheless, the fact is that there is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics. It is every bit as mind blowing as cosmology or physics (mathematicians *conceived* of black holes long before astronomers actually found any), and allows more freedom of expression than poetry, art, or music (which depend heavily on properties of the physical universe). Mathematics is the purest of the arts, as well as the most misunderstood.

So let me try to explain what mathematics is, and what mathematicians do. I can hardly do better than to begin with G.H. Hardy's excellent description:

> A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*.

So mathematicians sit around making patterns of ideas. What sort of patterns? What sort of ideas? Ideas about the rhinoceros? No, those we leave to the biologists. Ideas about language and culture? No, not usually. These things are all far too complicated for most mathematicians' taste. If there is anything like a unifying aesthetic principle in mathematics, it is this: *simple is beautiful*. Mathematicians enjoy thinking about the simplest possible things, and the simplest possible things are *imaginary*.

For example, if I'm in the mood to think about shapes— and I often am— I might imagine a triangle inside a rectangular box:

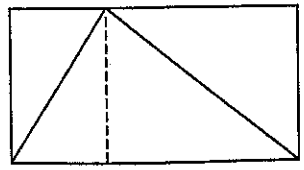


I wonder how much of the box the triangle takes up? Two-thirds maybe? The important thing to understand is that I'm not talking about this *drawing* of a triangle in a box. Nor am I talking about some metal triangle forming part of a girder system for a bridge. There's no ulterior practical purpose here. I'm just *playing*. That's what math is— wondering, playing, amusing yourself with your imagination. For one thing, the question of how much of the box the triangle takes up doesn't even make any *sense* for real, physical objects. Even the most carefully made physical triangle is still a hopelessly complicated collection of jiggling atoms; it changes its size from one minute to the next. That is, unless you want to talk about some sort of *approximate* measurements. Well, that's where the aesthetic comes in. That's just not simple, and consequently it is an ugly question which depends on all sorts of real-world details. Let's leave that to the scientists. The *mathematical* question is about an imaginary triangle inside an imaginary box. The edges are perfect because I want them to be— that is the sort of object I prefer to think about. This is a major theme in mathematics: things are what you want them to be. You have endless choices; there is no reality to get in your way.

On the other hand, once you have made your choices (for example I might choose to make my triangle symmetrical, or not) then your new creations do what they do, whether you like it or not. This is the amazing thing about making imaginary patterns: they talk back! The triangle takes up a certain amount of its box, and I don't have any control over what that amount is. There is a number out there, maybe it's two-thirds, maybe it isn't, but I don't get to say what it is. I have to *find out* what it is.

So we get to play and imagine whatever we want and make patterns and ask questions about them. But how do we answer these questions? It's not at all like science. There's no experiment I can do with test tubes and equipment and whatnot that will tell me the truth about a figment of my imagination. The only way to get at the truth about our imaginations is to use our imaginations, and that is hard work.

In the case of the triangle in its box, I do see something simple and pretty:



If I chop the rectangle into two pieces like this, I can see that each piece is cut diagonally in half by the sides of the triangle. So there is just as much space inside the triangle as outside. That means that the triangle must take up exactly half the box!

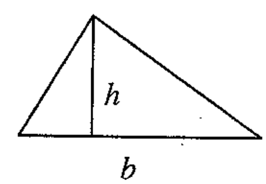
This is what a piece of mathematics looks and feels like. That little narrative is an example of the mathematician's art: asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations. There is really nothing else quite like this realm of pure idea; it's fascinating, it's fun, and it's free!

Now where did this idea of mine come from? How did I know to draw that line? How does a painter know where to put his brush? Inspiration, experience, trial and error, dumb luck. That's the art of it, creating these beautiful little poems of thought, these sonnets of pure reason. There is something so wonderfully transformational about this art form. The relationship between the triangle and the rectangle was a mystery, and then that one little line made it obvious. I couldn't see, and then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing, and change myself in the process. Isn't that what art is all about?

This is why it is so heartbreaking to see what is being done to mathematics in school. This rich and fascinating adventure of the imagination has been reduced to a sterile set of "facts" to be memorized and procedures to be followed. In place of a simple and natural question about shapes, and a creative and rewarding process of invention and discovery, students are treated to this:

Triangle Area Formula:

$$A = 1/2\, b\, h$$



"The area of a triangle is equal to one-half its base times its height." Students are asked to memorize this formula and then "apply" it over and over in the "exercises." Gone is the thrill, the joy, even the pain and frustration of the creative act. There is not even a *problem* anymore. The question has been asked and answered at the same time— there is nothing left for the student to do.

Now let me be clear about what I'm objecting to. It's not about formulas, or memorizing interesting facts. That's fine in context, and has its place just as learning a vocabulary does— it helps you to create richer, more nuanced works of art. But it's not the *fact* that triangles take up half their box that matters. What matters is the beautiful *idea* of chopping it with the line, and how that might inspire other beautiful ideas and lead to creative breakthroughs in other problems— something a mere statement of fact can never give you.

By removing the creative process and leaving only the results of that process, you virtually guarantee that no one will have any real engagement with the subject. It is like *saying* that Michelangelo created a beautiful sculpture, without letting me *see* it. How am I supposed to be inspired by that? (And of course it's actually much worse than this— at least it's understood that there *is* an art of sculpture that I am being prevented from appreciating).

By concentrating on *what*, and leaving out *why*, mathematics is reduced to an empty shell. The art is not in the "truth" but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is *the art of explanation.* If you deny students the opportunity to engage in this activity— to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs— you deny them mathematics itself. So no, I'm not complaining about the presence of facts and formulas in our mathematics classes, I'm complaining about the lack of *mathematics* in our mathematics classes.

The main problem with school mathematics is that there are no *problems*. Oh, I know what *passes* for problems in math classes, these insipid "exercises." "Here is a type of problem. Here is how to solve it. Yes it will be on the test. Do exercises 1-35 odd for homework." What a sad way to learn mathematics: to be a trained chimpanzee.

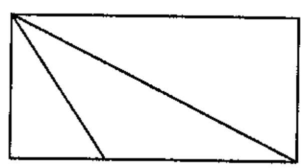
But a problem, a genuine honest-to-goodness natural human *question*— that's another thing. How long is the diagonal of a cube? Do prime numbers keep going on forever? Is infinity a number? How many ways can I symmetrically tile a surface? The history of mathematics is the history of mankind's engagement with questions like these, not the mindless regurgitation of formulas and algorithms (together with contrived exercises designed to make use of them).

A good problem is something you don't know *how* to solve. That's what makes it a good puzzle, and a good opportunity. A good problem does not just sit there in isolation, but serves as a springboard to *other* interesting questions. A triangle takes up half its box. What about a pyramid inside its three-dimensional box? Can we handle this problem in a similar way?

I can understand the idea of training students to master certain techniques— I do that too. But not as an end in itself. Technique in mathematics, as in any art, should be learned in context. The great problems, their history, the creative process— that is the proper setting. Give your students a good problem, let them struggle and get frustrated. See what they come up with. Wait until they are dying for an idea, *then* give them some technique. But not too much.

So put away your lesson plans and your overhead projectors, your full-color textbook abominations, your CD-ROMs and the whole rest of the traveling circus freak show of contemporary education, and simply do mathematics with your students! Art teachers don't waste their time with textbooks and rote training in specific techniques. They do what is natural to their subject— they get the kids painting. They go around from easel to easel, making suggestions and offering guidance:

"I was thinking about our triangle problem, and I noticed something. If the triangle is really slanted then it *doesn't* take up half it's box! See, look:



"Excellent observation! Our chopping argument assumes that the tip of the triangle lies directly over the base. Now we need a new idea."

"Should I try chopping it a different way?"

"Absolutely. Try all sorts of ideas. Let me know what you come up with!"