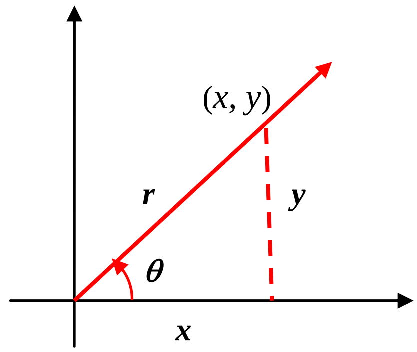
Section 2.2 – Trigonometric Functions

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$



Six Trigonometry Functions

$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{opp}{hyp} = \frac{y}{r} = \frac{a}{c} = \cos B$$

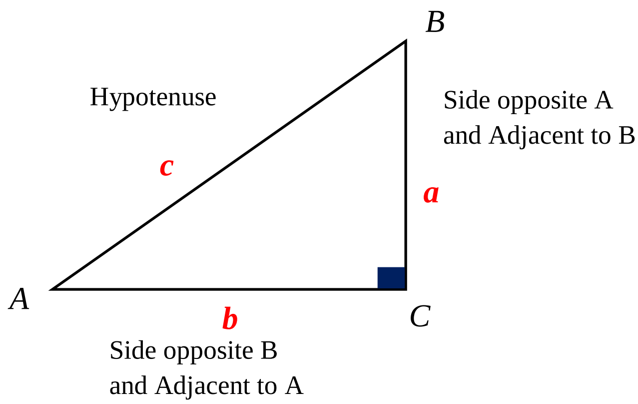
$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{adj}{hyp} = \frac{x}{r} = \frac{b}{c} = \sin B$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = \frac{a}{b} = \cot B$$

$$\cot\theta = \frac{adj}{opp} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y} = \frac{b}{a} = \tan B$$

$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta} = \frac{r}{x} = \frac{c}{b} = \csc B$$

$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{c}{a} = \sec B$$



Example

Find the six trigonometry functions of θ if θ is in the standard position and the point $(8, 15)$ is on the terminal side of θ.

<u>*Solution*</u>

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{8^2 + 15^2} = 17$$

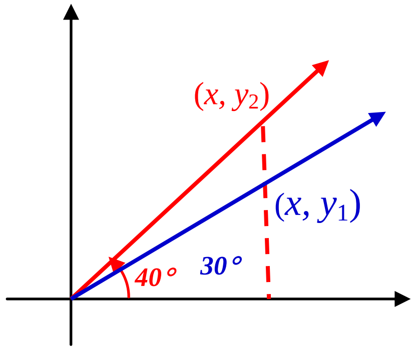
$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$

Example

Which will be greater, tan 30° or tan 40°? How large could $\tan\theta$ be?

<u>*Solution*</u>



$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$Ratio: \frac{y_2}{x} > \frac{y_1}{x}$$

$\to$ tan 40°> tan 30°

No limit as to how large $\tan\theta$ can be.

Function	**I**	**II**	**III**	**IV**
y = sin x	+	+	–	–
y = cosx	+	–	–	+
y = tan x	+	–	+	–
y = cot x	+	–	+	–
y = cscx	+	+	–	–
y = sec x	+	–	–	+

Example

If $\cos\theta = \frac{\sqrt{3}}{2}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

<u>*Solution*</u>

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \quad \to x = \sqrt{3}, \quad r = 2$$

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{2^2 - \left(\sqrt{3}\right)^2} = \sqrt{4-3} = 1 \qquad \text{Since } \theta \text{ is Q IV} \Rightarrow \boxed{y = -1}$$

$$\sin\theta = \frac{y}{r} = -\frac{1}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \Rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Solving for $\cos\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Solving for $\sin\theta$

$$\sin^2\theta = 1 - \cos^2\theta \quad \Rightarrow \quad \boxed{\sin\theta = \pm\sqrt{1 - \cos^2\theta}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$
$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Example

Write $\tan\theta$ in terms of $\sin\theta$.

<u>*Solution*</u>

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$
$$= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

Example

If $\cos\theta = \frac{1}{2}$ and θ terminated in QIV, find the remaining trigonometric ratios for θ.

<u>*Solution*</u>

$\sin\theta = -\sqrt{1-\cos^2\theta}$ $= -\sqrt{1-\left(\frac{1}{2}\right)^2}$ $= -\sqrt{1-\frac{1}{4}}$ $= -\sqrt{\frac{3}{4}}$ $= -\frac{\sqrt{3}}{2}$	$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{1/2} = 2$
	$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$
	$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$
	$\cot\theta = -\frac{1}{\sqrt{3}}$

Example

Simplify the expression $\sqrt{x^2+9}$ as much as possible after substituting $3\tan\theta$ for x

<u>*Solution*</u>

$$x = 3\tan\theta$$
$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$
$$= \sqrt{9\tan^2\theta+9}$$
$$= \sqrt{9\left(\tan^2\theta+1\right)}$$
$$= 3\sqrt{\sec^2\theta}$$
$$= 3\sec\theta$$

Example

Triangle ABC is a right triangle with $C = 90°$. If $a = 6$ and $c = 10$, find the six trigonometric functions of A.

Solution

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 6^2} = 8$$

$\sin A = \frac{a}{c} = \frac{6}{10} = \frac{3}{5}$	$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$	$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$
$\csc A = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$	$\sec A = \frac{c}{b} = \frac{10}{8} = \frac{5}{4}$	$\cot A = \frac{b}{a} = \frac{8}{6} = \frac{4}{3}$

$$\textit{if }\ A + B = 90° \Rightarrow \begin{cases} \sin A = \cos B \\ \sec A = \csc B \\ \tan A = \cot B \end{cases}$$

Cofunction Theorem

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

Example

Write each function in terms of its cofunction

a) $\cos 52°$

Solution

$$\cos 52° = \sin\left(90° - 52°\right) = \sin 38°$$

b) $\tan 71°$

Solution

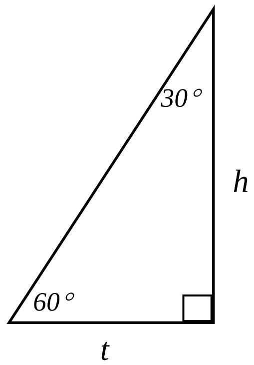
$$\tan 71° = \cot\left(90° - 71°\right) = \cot 19°$$

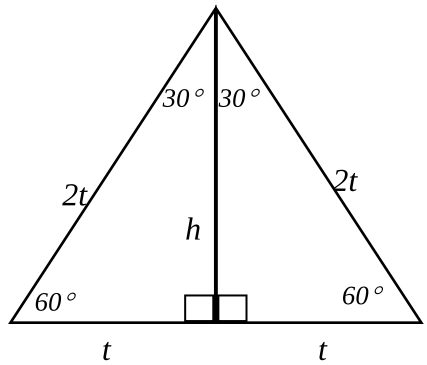
c) $\sec 24°$

Solution

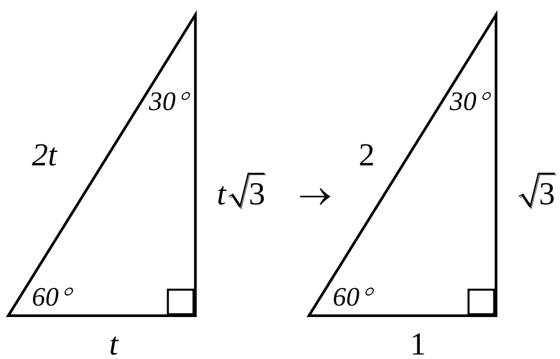
$$\sec 24° = \csc\left(90° - 24°\right) = \csc 66°$$

The 30° - 60° - 90° *Triangle*





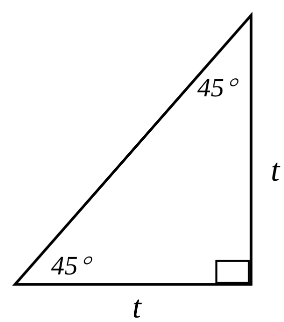
$$t^2 + h^2 = (2t)^2$$
$$t^2 + h^2 = 4t^2$$
$$h^2 = 4t^2 - t^2$$
$$h^2 = 3t^2$$
$$h = t\sqrt{3}$$

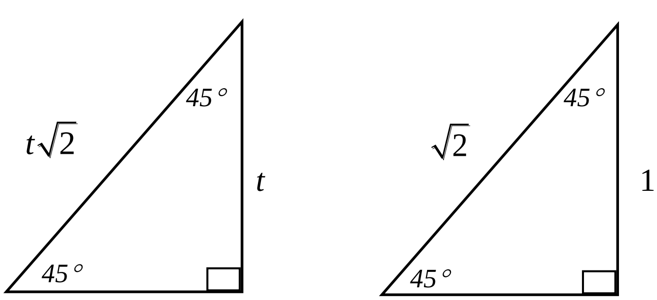


$$\Rightarrow \boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

The 45° - 45° - 90° *Triangle*

$$hypotenuse^2 = t^2 + t^2$$
$$hypotenuse = \sqrt{2t^2}$$
$$hypotenuse = t\sqrt{2}$$





$$\Rightarrow \boxed{\cos 45° = \frac{1}{\sqrt{2}}}$$

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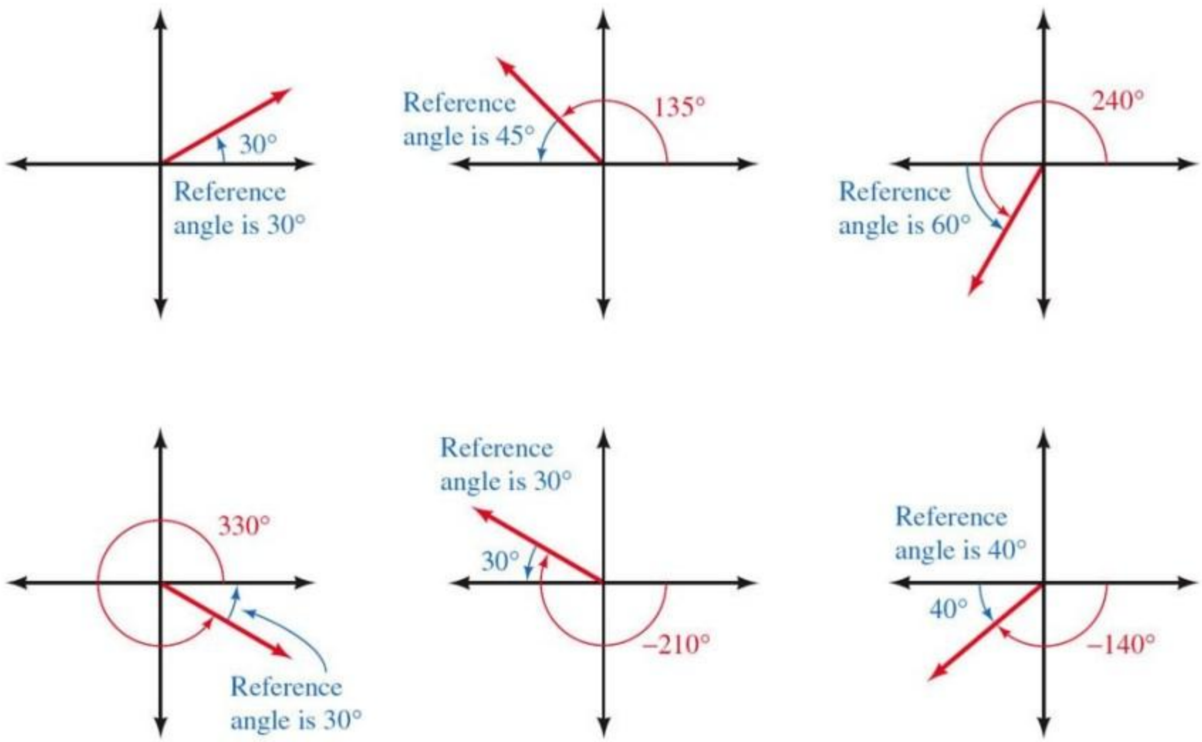
Reference Angle

Definition

The reference angle or related angle for any angle θ in standard position ifs the positive acute angle between the terminal side of θ and the *x*-axis, and it is denoted $\hat{\theta}$



$f\ \ \theta \in QI$ then $\hat{\theta} = \theta$ $\leftrightarrow$ $\theta = \hat{\theta}$

$f\ \ \theta \in QII$ then $\hat{\theta} = 180° - \theta$ $\leftrightarrow$ $\theta = 180° - \hat{\theta}$

$f\ \ \theta \in QIII$ then $\hat{\theta} = \theta - 180°$ $\leftrightarrow$ $\theta = \hat{\theta} + 180°$

$f\ \ \theta \in QIV$ then $\hat{\theta} = 360° - \theta$ $\leftrightarrow$ $\theta = 360° - \hat{\theta}$

Example

Find the exact value of $\sin 240°$

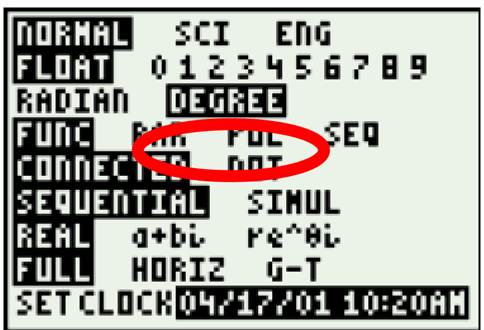
<u>*Solution*</u>

$\hat{\theta} = 240° - 180° = 60°$ $\quad \rightarrow 240° \in QIII$

$\sin 240° = -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$

Approximation- Simply using calculator



$\sin 250° \approx -0.9397$

$\cos 250° \approx -0.3420$

$\tan 250° \approx 2.7475$

$\csc 250° = \frac{1}{\sin 250°} \approx -1.0642$

To find the angle by using the inverse trigonometry functions, always enter a ***positive*** *value.*

Example

Find θ if $\sin\theta = -0.5592$ and θ terminates in QIII with $0° \le \theta < 360°$.

Solution

$\hat{\theta} = \sin^{-1} 0.5592 \approx 34°$

$\theta \in$ QIII

$\Rightarrow \theta = 180° + 34° = 214°$

Example

Find θ to the nearest degree if $\cot\theta = -1.6003$ and θ terminates in QII with $0° \le \theta < 360°$.

Solution

$\cot\theta = -1.6003 = \frac{1}{\tan\theta}$

$\tan\theta = \frac{1}{-1.6003}$

$\hat{\theta} = \tan^{-1} \frac{1}{1.6003}$

$= 32°$

$\theta \in$ QII $\quad \Rightarrow \theta = 180° - 32° = 148°$

Exercise ***Section* 2.2 – Trigonometric Functions**

1. Find the six trigonometry functions of θ if θ is in the standard position and the point $(-2, 3)$ is on the terminal side of θ.

2. Find the six trigonometry functions of θ if θ is in the standard position and the point $(-3, -4)$ is on the terminal side of θ.

3. Find the six trigonometry functions of θ in standard position with terminal side through the point $(-3,\ 0)$.

4. Find the six trigonometry functions of θ if θ is in the standard position and the point $(12, -5)$ is on the terminal side of θ.

5. Find the values of the six trigonometric functions for an angle of $90°$.

6. Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

7. Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

8. Find the remaining trigonometric function of θ if $\sin\theta = \frac{12}{13}$ and θ terminates in QI.

9. Find the remaining trigonometric function of θ if $\cot\theta = -2$ and θ terminates in QII.

10. Find the remaining trigonometric function of θ if $\tan\theta = \frac{3}{4}$ and θ terminates in QIII.

11. Find the remaining trigonometric function of θ if $\cos\theta = \frac{24}{25}$ and θ terminates in QIV.

12. Find the remaining trigonometric functions of θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV.

13. Find the remaining trigonometric functions of θ if $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$.

14. If $\sin\theta = -\frac{5}{13}$, and θ is QIII, find $\cos\theta$ and $\tan\theta$.

15. If $\cos\theta = \frac{3}{5}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

16. Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

17. Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

18. Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

19. Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

20. If $\cos\theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin\theta$

21. If $\sin\theta = \frac{3}{5}$ and θ terminated in QII, find $\cos\theta$ and $\tan\theta$.

22. Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

23. Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

24. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$

25. Write $\frac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

26. Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

27. Write $\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

28. Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

29. Multiply $(1-\cos\theta)(1+\cos\theta)$

30. Multiply $(\sin\theta+2)(\sin\theta-5)$

31. Simplify the expression $\sqrt{25-x^2}$ as much as possible after substituting $5\sin\theta$ for x.

32. Simplify the expression $\sqrt{4x^2+16}$ as much as possible after substituting $2\tan\theta$ for x

33. Simplify by using the table. $5\sin^2 30°$

34. Simplify by using the table $\sin^2 60° + \cos^2 60°$

35. Simplify by using the table $(\tan 45° + \tan 60°)^2$

36. Find the exact value of $\csc 300°$

37. Find θ if $\sin\theta = -\frac{1}{2}$ and θ terminates in QIII with $0° \le \theta \le 360°$.

38. Find θ to the nearest degree if $\sec\theta = 3.8637$ and θ terminates in QIV with $0° \le \theta < 360°$.

39. Find the exact value of $\cos 225°$

40. Find the exact value of $\tan 315°$

41. Find the exact value of $\cos 420°$

42. Find the exact value of $\cot 480°$

43. Use the calculator to find the value of $\csc 166.7°$

44. Use the calculator to find the value of $\sec 590.9°$

45. Use the calculator to find the value of $\tan 195° \, 10'$

46. Use the calculator to find θ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in$ QIV with $0° \le \theta < 360°$

47. Use the calculator to find θ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in$ QIII with $0° \le \theta < 360°$

48. Use the calculator to find θ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in$ QII with $0° \le \theta < 360°$

49. Find θ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and θ terminates in QIV with $0° \le \theta < 360°$

50. Use the calculator to find θ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in$ QII with $0° \le \theta < 360°$

Solution Section 2.2 – Trigonometric Functions

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-2, 3)$ is on the terminal side of θ.

Solution

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} \qquad \tan\theta = \frac{y}{x} = -\frac{3}{2} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} = -\frac{\sqrt{13}}{2}$$

$$\cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{13}} \qquad \cot\theta = \frac{x}{y} = -\frac{2}{3} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{13}}{3}$$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-3, -4)$ is on the terminal side of θ.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-4)^2} = 5$$

$$\sin\theta = -\frac{4}{5} \qquad \tan\theta = \frac{-4}{-3} = \frac{4}{3} \qquad \csc\theta = -\frac{5}{4}$$

$$\cos\theta = -\frac{3}{5} \qquad \cot\theta = \frac{3}{4} \qquad \sec\theta = -\frac{5}{3}$$

Exercise

Find the six trigonometry functions of θ in standard position with terminal side through the point $(-3, 0)$.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 0^2} = 3$$

$$\sin\theta = \frac{0}{3} = 0 \qquad \tan\theta = \frac{0}{-3} = 0 \qquad \csc\theta = \frac{1}{0} \to \infty$$

$$\cos\theta = \frac{-3}{3} = -1 \qquad \cot\theta = \frac{1}{0} = \infty \qquad \sec\theta = \frac{1}{-1} = -1$$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point (12, −5) is on the terminal side of θ.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{12^2 + (-5)^2} = \underline{13|}$$

$\sin\theta = -\frac{5}{13}$ $\tan\theta = -\frac{5}{12}$ $\csc\theta = -\frac{13}{5}$

$\cos\theta = \frac{12}{13}$ $\cot\theta = -\frac{12}{5}$ $\sec\theta = \frac{13}{12}$

Exercise

Find the values of the six trigonometric functions for an angle of 90°.

Solution

$\sin 90° = 1$ $\tan 90° = \infty$ $\csc 90° = 1$

$\cos 90° = 0$ $\cot 90° = 0$ $\sec 90° = \infty$

Exercise

Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

Solution

$\cos\theta = \frac{1}{2}$ $\rightarrow$ QI & QIV

Exercise

Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

Solution

$\csc\theta = -2.45 = \frac{1}{\sin\theta}$ $\rightarrow$ QIII & QIV

Exercise

Find the remaining trigonometric function of θ if $\sin\theta = \frac{12}{13}$ and θ terminates in QI

Solution

$$x = \sqrt{13^2 - 12^2} = 5$$

$$\sin\theta = \frac{12}{13} = \frac{y}{r} \qquad \tan\theta = \frac{y}{x} = \frac{12}{5} \qquad \csc\theta = \frac{13}{12}$$

$$\cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \cot\theta = \frac{x}{y} = \frac{5}{12} \qquad \sec\theta = \frac{13}{5}$$

Exercise

Find the remaining trigonometric function of θ if $\cot\theta = -2$ and θ terminates in QII.

Solution

$$\cot\theta = -2 = \frac{x}{y} \quad (\theta \in QII) \quad \Rightarrow \boxed{x = -2, \ \ y = 1}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (1)^2} = \underline{\sqrt{5}|}$$

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{5}}$$

$$\tan\theta = \frac{y}{x} = -\frac{1}{2}, \quad \sec\theta = \frac{r}{x} = -\frac{\sqrt{5}}{2}, \quad \csc\theta = \frac{r}{y} = \sqrt{5}$$

Exercise

Find the remaining trigonometric function of θ if $\tan\theta = \frac{3}{4}$ and θ terminates in QIII.

Solution

$$\tan\theta = \frac{3}{4} = \frac{y}{x} \quad (\theta \in QIII) \quad \Rightarrow \boxed{x = -4, \ \ y = -3}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + (-3)^2} = \underline{5|}$$

$$\sin\theta = \frac{y}{r} = -\frac{3}{5}, \quad \cos\theta = \frac{x}{r} = -\frac{4}{5}$$

$$\cot\theta = \frac{x}{y} = \frac{4}{3}, \quad \sec\theta = \frac{r}{x} = -\frac{5}{4}, \quad \csc\theta = \frac{r}{y} = -\frac{5}{3}$$

Exercise

Find the remaining trigonometric function of θ if $\cos\theta = \frac{24}{25}$ and θ terminates in QIV.

Solution

$$\cos\theta = \frac{24}{25} = \frac{x}{r} \quad (\theta \in QIV) \quad \Rightarrow \boxed{x = 24}$$

$$y = -\sqrt{r^2 - x^2} = -\sqrt{(25)^2 - (24)^2} = \underline{-7}$$

$$\sin\theta = \frac{y}{r} = -\frac{7}{25}$$

$$\tan\theta = \frac{y}{x} = -\frac{7}{24}, \quad \cot\theta = \frac{x}{y} = -\frac{24}{7}$$

$$\sec\theta = \frac{r}{x} = \frac{25}{24}, \quad \csc\theta = \frac{r}{y} = -\frac{25}{7}$$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV.

Solution

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \quad \Rightarrow x = \sqrt{3}, \ r = 2$$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

Since θ is QIV $\Rightarrow y = -\sqrt{2^2 - \sqrt{3}^2}$

$$= -\sqrt{4-3}$$

$$\underline{= -1}$$

$$\sin\theta = \frac{y}{r} = -\frac{1}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \qquad \cot\theta = -\sqrt{3}$$

$$\sec\theta = \frac{r}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \csc\theta = \frac{r}{y} = \frac{2}{-1} = -2$$

Exercise

Find the remaining trigonometric functions of θ if $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$.

Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta} < 0 \;\; \& \;\; \cos\theta > 0 \;\; \Rightarrow \sin\theta < 0 \Rightarrow \theta \text{ in QIV}$$

$$\tan\theta = -\frac{1}{2} = \frac{y}{x}$$

$$\Rightarrow y = 1, x = 2 \rightarrow r = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\sin\theta = \frac{y}{r} = -\frac{1}{\sqrt{5}} \qquad \cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}}$$

$$\cot\theta = -2$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{1} \qquad \csc\theta = \frac{r}{y} = \frac{2}{-1} = -2$$

Exercise

If $\sin\theta = -\frac{5}{13}$, and θ is QIII, find $\cos\theta$ and $\tan\theta$.

Solution

$$\sin\theta = -\frac{5}{13} = \frac{y}{r} \quad \rightarrow y = -5, \;\; r = 13$$

$$r^2 = x^2 + y^2$$

$$\Rightarrow x^2 = r^2 - y^2$$

$$\Rightarrow x = \sqrt{r^2 - y^2}$$

$$\Rightarrow x = \sqrt{13^2 - 5^2} = \pm 12 \quad \text{Since } \theta \text{ is Q III} \Rightarrow x = -12$$

$$\cos\theta = \frac{x}{r} = -\frac{12}{13}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-12} = \frac{5}{12}$$

Exercise

If $\cos\theta = \frac{3}{5}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

Solution

$$\cos\theta = \frac{3}{5} = \frac{x}{r} \quad (\theta \in QIV) \quad \Rightarrow \boxed{x = 3} \qquad y = -4$$

$$\sin\theta = -\frac{4}{5}, \quad \tan = -\frac{4}{3}$$

Exercise

Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

Solution

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

Exercise

Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

Solution

$$\cos\theta = \frac{1}{\sec\theta}$$

$$= \frac{1}{\frac{5}{3}}$$

$$= \frac{3}{5}$$

Exercise

Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

Solution

$$\sin\theta = \frac{1}{\csc\theta}$$

$$= -\frac{2}{\sqrt{12}}\frac{\sqrt{12}}{\sqrt{12}}$$

$$= -\frac{2\sqrt{12}}{12}$$

$$= -\frac{\sqrt{12}}{6}$$

Exercise

Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{3}{5}}{-\frac{4}{5}}$$

$$= -\frac{3}{4}$$

Exercise

If $\cos\theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin\theta$

Solution

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{1}{4}}$$

$$= \sqrt{\frac{3}{4}}$$

$$= \frac{\sqrt{3}}{2}$$

Exercise

If $\sin\theta = \frac{3}{5}$ and θ terminated in QII, find $\cos\theta$ and $\tan\theta$.

Solution

$$\cos\theta = -\sqrt{1-\sin^2\theta}$$
$$= -\sqrt{1-\left(\frac{3}{5}\right)^2}$$
$$= -\sqrt{1-\frac{9}{25}}$$
$$= -\sqrt{\frac{16}{25}}$$
$$= -\frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{3/5}{-4/5}$$
$$= -\frac{3}{4}$$

Exercise

Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

Solution

$$\cos\theta = \sqrt{1-\sin^2\theta}$$
$$= \sqrt{1-\frac{1}{9}}$$
$$= \sqrt{\frac{8}{9}}$$
$$= \frac{2\sqrt{2}}{3}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}}$$
$$= \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{2}}{4}$$

Exercise

Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

Solution

$$\sec\theta = \frac{1}{\cos\theta} = -3 \qquad \Rightarrow \cos\theta = -\frac{1}{3}$$

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

$$\tan\theta = \frac{-\frac{2\sqrt{2}}{3}}{-\frac{1}{3}} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

$$\cot\theta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$$

Exercise

Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$

Solution

$$\sin\theta = \frac{1}{\csc} = \frac{1}{-2.45} = -.41$$

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-.41^2} = -.91$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-.41}{-.91} = .45$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{.45} = 2.22$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-.91} = -1.1$$

Exercise

Write $\frac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\frac{\sec\theta}{\csc\theta} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}$$

$$= \frac{1}{\cos\theta}\frac{\sin\theta}{1}$$

$$= \frac{\sin\theta}{\cos\theta}$$

Exercise

Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\cot\theta - \csc\theta = \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\cos\theta - 1}{\sin\theta}$$

Exercise

Write $\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

Solution

$$\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta} = \frac{\sin^2\theta + \cos\theta}{\cos\theta\sin\theta}$$

Exercise

Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\sin\theta\cot\theta + \cos\theta = \sin\theta\frac{\cos\theta}{\sin\theta} + \cos\theta$$

$$= \cos\theta + \cos\theta$$

$$= 2\cos\theta$$

Exercise

Multiply $(1-\cos\theta)(1+\cos\theta)$

Solution

$$(1-\cos\theta)(1+\cos\theta)=1-\cos^2\theta$$
$$=\sin^2\theta$$

Exercise

Multiply $(\sin\theta+2)(\sin\theta-5)$

Solution

$$(\sin\theta+2)(\sin\theta-5)=\sin^2\theta-3\sin\theta-10$$

Exercise

Simplify the expression $\sqrt{25-x^2}$ as much as possible after substituting $5\sin\theta$ for x.

Solution

$$\sqrt{25-x^2}=\sqrt{25-(5\sin\theta)^2}$$
$$=\sqrt{25-25\sin^2\theta}$$
$$=\sqrt{25(1-\sin^2\theta)}$$
$$=\sqrt{25}\sqrt{\cos^2\theta}$$
$$=5\cos\theta$$

Exercise

Simplify the expression $\sqrt{4x^2+16}$ as much as possible after substituting $2\tan\theta$ for x

Solution

$$\sqrt{4x^2+16}=\sqrt{4(2\tan\theta)^2+16}$$
$$=\sqrt{16\tan^2\theta+16}$$
$$=\sqrt{16(\tan^2\theta+1)}$$
$$=4\sqrt{\tan^2\theta+1}$$
$$=4\sqrt{\sec^2\theta}$$
$$=4\sec\theta$$

Exercise

Simplify by using the table. $5\sin^2 30°$

Solution

$$5\sin^2 30° = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}$$

Exercise

Simplify by using the table. $\sin^2 60° + \cos^2 60°$

Solution

$$\begin{aligned}\sin^2 60° + \cos^2 60° &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1\end{aligned}$$

Exercise

Simplify by using the table. $(\tan 45° + \tan 60°)^2$

Solution

$$\begin{aligned}(\tan 45° + \tan 60°)^2 &= \left(1 + \sqrt{3}\right)^2 \\ &= 1 + 3 + 2\sqrt{3} \\ &= 4 + 2\sqrt{3}\end{aligned}$$

Exercise

Find the exact value of $\csc 300°$

Solution

$$\hat{\theta} = 360° - 300° = 60° \quad \rightarrow 300° \in QIV$$

$$\csc 300° = -\frac{1}{\sin 60°} = -\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$$

Exercise

Find θ if $\sin\theta = -\frac{1}{2}$ and θ terminates in QIII with $0° \le \theta \le 360°$.

Solution

$$\hat{\theta} = \sin^{-1}\frac{1}{2}$$
$$= 30°$$

$$\theta \in \text{QIII}$$

$$\Rightarrow \theta = 180° + 30°$$
$$= 210°$$

Exercise

Find θ to the nearest degree if $\sec\theta = 3.8637$ and θ terminates in QIV with $0° \le \theta \le 360°$.

Solution

$$\sec\theta = 3.8637 = \frac{1}{\cos\theta}$$

$$\cos\theta = \frac{1}{3.8637}$$

$$\hat{\theta} = \cos^{-1}\frac{1}{3.8637}$$
$$= 75°$$

Calculator: $\cos^{-1}(1/3.8637)$

$$\theta \in \text{QIV}$$

$$\Rightarrow \theta = 360° - 75°$$
$$= 285°$$

Exercise

Find the exact value of $\cos 225°$

Solution

$$\hat{\theta} = 225° - 180° = 45°$$

$$\rightarrow 225° \in QIII$$

$$\cos 225° = -\cos 45°$$

$$= -\frac{\sqrt{2}}{2}$$

Exercise

Find the exact value of $\tan 315°$

Solution

$$\hat{\theta} = 360° - 315° = 45° \qquad \rightarrow 315° \in QIV$$

$$\tan 315° = -\tan 45° = -1$$

Exercise

Find the exact value of $\cos 420°$

Solution

$$\hat{\theta} = 420° - 360° = 60° \qquad \rightarrow 420° \in QI$$

$$\cos 420° = \cos 60° = \frac{1}{2}$$

Exercise

Find the exact value of $\cot 480°$

Solution

$$\hat{\theta} = 480° - 360° = 120°$$

$$\hat{\theta} = 180° - 120° = 60° \qquad \rightarrow 480° \in QII$$

$$\cot 480° = -\frac{\cos 60°}{\sin 60°}$$

$$= -\frac{1/2}{\sqrt{3}/2}$$

$$= -\frac{1}{\sqrt{3}}$$

Exercise

Use the calculator to find the value of $\csc 166.7°$

Solution

$$\csc 166.7° = \frac{1}{\sin 166.7°}$$

$$\approx 4.3469$$

Exercise

Use the calculator to find the value of $\sec 590.9°$

Solution

$$\sec 590.9° = \frac{1}{\cos 590.9°}$$

$$\approx -1.5856$$

Exercise

Use the calculator to find the value of $\tan 195°\ 10'$

Solution

$$\tan\left(195°\ 10'\right) = \tan\left(195° + \frac{10}{60}\right)$$

$$= \tan 195.1667°$$

$$\approx .271$$

Exercise

Use the calculator to find θ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in$ QIV with $0° \le \theta \le 360°$

Solution

$$\hat{\theta} = \sin^{-1}(0.3090) \approx 18.0°$$

Since $\theta \in$ QIV

$$\theta = 180° + 40.0°$$

$$= 220.0°$$

Exercise

Use the calculator to find θ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in$ QIII with $0° \le \theta \le 360°$

Solution

$$\hat{\theta} = \cos^{-1}(0.7660) \approx 40.0° \qquad \text{Since } \theta \in \text{ QIII}$$

$$\theta = 180° + 40.0°$$

$$= 220.0°$$

Exercise

Use the calculator to find θ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in$ QII with $0° \le \theta \le 360°$

Solution

$$\sec\theta = -3.4159$$

$$\cos\theta = -\frac{1}{3.4159}$$

$$\hat{\theta} = \cos^{-1}\left(\frac{1}{3.4159}\right) \approx 73.0° \qquad \text{Since } \theta \in \text{QII}$$

$$\theta = 180° - 73.0°$$

$$= 107.0°$$

Exercise

Find θ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and θ terminates in QIV with $0° \le \theta \le 360°$.

Solution

$$\hat{\theta} = \tan^{-1} 0.8541 \approx 40.5°$$

$$\theta \in \text{QIV}$$

$$\Rightarrow \theta = 360° - 40.5°$$

$$= 319.5°$$

Exercise

Use the calculator to find θ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in$ QII with $0° \le \theta < 360°$

Solution

$$\hat{\theta} = \sin^{-1} 0.49368329 = 29.6°$$

$$\theta \in \text{QII}$$

$$\Rightarrow \theta = 180° - 29.6°$$

$$= 150.4°$$