Review Exercise

1. Determine whether each of the following statements is true or false. Provide a brief explanation for each answer.
   a. $|\vec{a} + \vec{b}| \geq |\vec{a}|$
   b. $|\vec{a} + \vec{b}| = |\vec{a} + \vec{c}|$ implies $|\vec{b}| = |\vec{c}|$
   c. $\vec{a} + \vec{b} = \vec{a} + \vec{c}$ implies $\vec{b} = \vec{c}$
   d. $\overrightarrow{RF} = \overrightarrow{SW}$ implies $\overrightarrow{RS} = \overrightarrow{FW}$
   e. $m\vec{a} + n\vec{a} = (m + n)\vec{a}$
   f. If $|\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$, then $|\vec{a} + \vec{b}| = |\vec{c} + \vec{d}|$.

2. If $\vec{x} = 2\vec{a} - 3\vec{b} - 4\vec{c}$, $\vec{y} = -2\vec{a} + 3\vec{b} + 3\vec{c}$, and $\vec{z} = 2\vec{a} - 3\vec{b} + 5\vec{c}$, determine simplified expressions for each of the following:
   a. $2\vec{x} - 3\vec{y} + 5\vec{z}$
   b. $3(-2\vec{x} - 4\vec{y} + \vec{z}) - (2\vec{x} - \vec{y} + \vec{z}) - 2(-4\vec{x} - 5\vec{y} + \vec{z})$

3. If $X(-2, 1, 2)$ and $Y(-4, 4, 8)$ are two points in R^3, determine the following:
   a. $\overrightarrow{XY}$ and $|\overrightarrow{XY}|$
   b. The coordinates of a unit vector in the same direction as $\overrightarrow{XY}$.

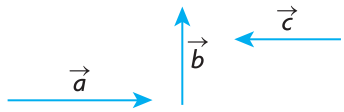
4. $X(-1, 2, 6)$ and $Y(5, 5, 12)$ are two points in R^3.
   a. Determine the components of a position vector equivalent to $\overrightarrow{YX}$.
   b. Determine the components of a *unit* vector that is in the same direction as $\overrightarrow{YX}$.

5. Find the components of the unit vector with the opposite direction to that of the vector from $M(2, 3, 5)$ to $N(8, 1, 2)$.

6. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
   a. Determine the components of the vectors representing the diagonals.
   b. Determine the angles between the sides of the parallelogram.

7. The points $A(-1, 1, 1)$, $B(2, 0, 3)$, and $C(3, 3, -4)$ are vertices of a triangle.
   a. Show that this triangle is a right triangle.
   b. Calculate the area of triangle ABC.
   c. Calculate the perimeter of triangle ABC.
   d. Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices.

8. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



a. Construct the vector $\vec{a} - \vec{b} + \vec{c}$.

b. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and if $|\vec{a}| = 4$ and $|\vec{b}| = 3$, determine $|\vec{a} + \vec{b}|$.

9. Given $\vec{p} = (-11, 7)$, $\vec{q} = (-3, 1)$, and $\vec{r} = (-1, 2)$, express each vector as a linear combination of the other two.

10. a. Find an equation to describe the set of points equidistant from $A(2, -1, 3)$ and $B(1, 2, -3)$.

b. Find the coordinates of two points that are equidistant from A and B.

11. Calculate the values of a, b, and c in each of the following:

a. $2(a, b, 4) + \frac{1}{2}(6, 8, c) - 3(7, c, -4) = (-24, 3, 25)$

b. $2\left(a, a, \frac{1}{2}a\right) + (3b, 0, -5c) + 2\left(c, \frac{3}{2}c, 0\right) = (3, -22, 54)$

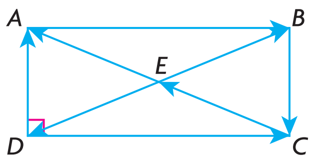
12. a. Determine whether the points $A(1, -1, 1)$, $B(2, 2, 2)$, and $C(4, -2, 1)$ represent the vertices of a right triangle.

b. Determine whether the points $P(1, 2, 3)$, $Q(2, 4, 6)$, and $R(-1, -2, -3)$ are collinear.

13. a. Show that the points $A(3, 0, 4)$, $B(1, 2, 5)$, and $C(2, 1, 3)$ represent the vertices of a right triangle.

b. Determine $\cos \angle ABC$.

14. In the following rectangle, vectors are indicated by the direction of the arrows.

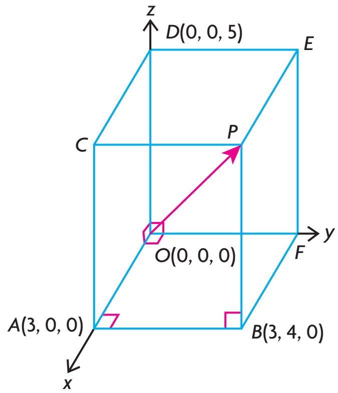


a. Name two pairs of vectors that are opposites.

b. Name two pairs of identical vectors.

c. Explain why $|\vec{AD}|^2 + |\vec{DC}|^2 = |\vec{DB}|^2$.

15. A rectangular prism measuring 3 by 4 by 5 is drawn on a coordinate axis as shown in the diagram.



a. Determine the coordinates of points C, P, E, and F.

b. Determine position vectors for $\overrightarrow{DB}$ and $\overrightarrow{CF}$.

c. By drawing the rectangle containing $\overrightarrow{DB}$ and $\overrightarrow{OP}$, determine the acute angle between these vectors.

d. Determine the angle between $\overrightarrow{OP}$ and $\overrightarrow{AE}$.

16. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, and the angle between them is 30°. Determine each of the following:

a. $|\vec{d} + \vec{e}|$ b. $|\vec{d} - \vec{e}|$ c. $|\vec{e} - \vec{d}|$

17. An airplane is headed south at speed 400 km/h. The airplane encounters a wind from the east blowing at 100 km/h.

a. How far will the airplane travel in 3 h?

b. What is the direction of the airplane?

18. a. Explain why the set of vectors: $\{(2, 3), (3, 5)\}$ spans R^2.

b. Find m and n in the following: $m(2, 3) + n(3, 5) = (323, 795)$.

19. a. Show that the vector $\vec{a} = (5, 9, 14)$ can be written as a linear combination of the vectors $\vec{b}$ and $\vec{c}$, where $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$. Explain why $\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$.

b. Is the vector $\vec{a} = (-13, 36, 23)$ in the span of $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$? Explain your answer.

20. A cube is placed so that it has three of its edges located along the positive *x*-, *y*-, and *z*-axes (one edge along each axis) and one of its vertices at the origin.

    a. If the cube has a side length of 4, draw a sketch of this cube and write the coordinates of its vertices on your sketch.

    b. Write the coordinates of the vector with its head at the origin and its tail at the opposite vertex.

    c. Write the coordinates of a vector that starts at $(4, 4, 4)$ and is a diagonal in the plane parallel to the *xz*-plane.

    d. What vector starts at the origin and is a diagonal in the *xy*-plane?

21. If $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$, and $\vec{c} = 2\vec{i} + 13\vec{k}$, determine $\left|2\left(\vec{a} + \vec{b} - \vec{c}\right) - \left(\vec{a} + 2\vec{b}\right) + 3\left(\vec{a} - \vec{b} + \vec{c}\right)\right|$.

22. The three points $A(-3, 4)$, $B(3, -4)$, and $C(5, 0)$ are on a circle with radius 5 and centre at the origin. Points A and B are the endpoints of a diameter, and point C is on the circle.

    a. Calculate $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{BC}|$.

    b. Show that A, B, and C are the vertices of a right triangle.

23. In terms of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{0}$, find a vector expression for each of the following:

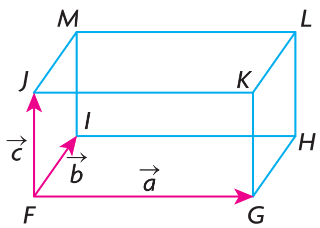
    a. $\overrightarrow{FL}$

    b. $\overrightarrow{MK}$

    c. $\overrightarrow{HJ}$

    d. $\overrightarrow{IH} + \overrightarrow{KJ}$

    e. $\overrightarrow{IK} - \overrightarrow{IH}$



24. Draw a diagram showing the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 2|\vec{b}|$ and $|\vec{b}| = |\vec{a} + \vec{b}|$ are both true. (Make sure to indicate the direction of the vectors.)

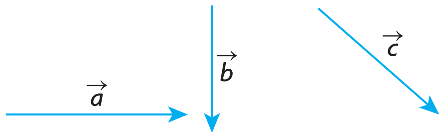
25. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, express each of the following in terms of $|\vec{a}|$ and $|\vec{b}|$:

    a. $|\vec{a} + \vec{b}|$
    b. $|\vec{a} - \vec{b}|$
    c. $|2\vec{a} + 3\vec{b}|$

26. Show that if $\vec{a}$ is perpendicular to each of the vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test

1. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are shown.



Using these three vectors, demonstrate that $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$. Name this property and explain how your answer shows this to be true.

2. $A(-2, 3, -5)$ and $B(6, 7, 3)$ are two points in R^3. Determine each of the following:
   a. $\overrightarrow{AB}$
   b. $|\overrightarrow{AB}|$
   c. a unit vector in the direction of $\overrightarrow{BA}$

3. The vectors $\vec{x}$ and $\vec{y}$ are each of length 3 units, i.e., $|\vec{x}| = |\vec{y}| = 3$. If $|\vec{x} + \vec{y}| = \sqrt{17}$, determine $|\vec{x} - \vec{y}|$.

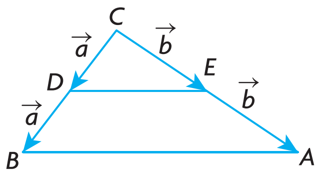
4. a. If $3\vec{x} - 2\vec{y} = \vec{a}$ and $5\vec{x} - 3\vec{y} = \vec{b}$, express the vectors $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.
   b. Solve for a, b, and c: $(2, -1, c) + (a, b, 1) - 3(2, a, 4) = (-3, 1, 2c)$.

5. a. Explain why the vectors $\vec{a} = (-2, 3)$ and $\vec{b} = (3, -1)$ span R^2.
   b. Determine the values of p and q in $p(-2, 3) + q(3, -1) = (13, -9)$.

6. a. Show that the vector $\vec{a} = (1, 12, -29)$ can be written as a linear combination of $\vec{b} = (3, 1, 4)$ and $\vec{c} = (1, 2, -3)$.
   b. Determine whether $\vec{r} = (16, 11, -24)$ can be written as a linear combination of $\vec{p} = (-2, 3, 4)$ and $\vec{q} = (4, 1, -6)$. Explain the significance of your result geometrically.

7. $\vec{x}$ and $\vec{y}$ are vectors of magnitude 1 and 2, respectively, with an angle of 120° between them. Determine $|3\vec{x} + 2\vec{y}|$ and the direction of $3\vec{x} + 2\vec{y}$.

8. In triangle ABC, point D is the midpoint of $\overrightarrow{BC}$ and point E is the midpoint of $\overrightarrow{AC}$. Vectors are marked as shown. Use vectors to prove that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$.



Answers

Review Exercise, pp. 344–347

1. a. false; Let $\vec{b} = -\vec{a} \neq 0$, then:
$|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})|$
$= |0|$
$= 0 < |\vec{a}|$

b. true; $|\vec{a} + \vec{b}|$ and $|\vec{a} + \vec{c}|$ both represent the lengths of the diagonal of a parallelogram, the first with sides $\vec{a}$ and $\vec{b}$ and the second with sides $\vec{a}$ and $\vec{c}$; since both parallelograms have $\vec{a}$ as a side and diagonals of equal length $|\vec{b}| = |\vec{c}|$.

c. true; Subtracting $\vec{a}$ from both sides shows that $\vec{b} = \vec{c}$.

d. true; Draw the parallelogram formed by $\overrightarrow{RF}$ and $\overrightarrow{SW}$. $\overrightarrow{FW}$ and $\overrightarrow{RS}$ are the opposite sides of a parallelogram and must be equal.

e. true; the distributive law for scalars

f. false; Let $\vec{b} = -\vec{a}$ and let $\vec{c} = \vec{d} \neq 0$. Then,
$|\vec{a}| = |-\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$
but $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})| = 0$
$|\vec{c} + \vec{b}| = |\vec{c} + \vec{c}| = |2\vec{c}|$
so $|\vec{a} + \vec{b}| \neq |\vec{c} + \vec{d}|$

2. a. $20\vec{a} - 30\vec{b} + 8\vec{c}$
b. $\vec{a} - 3\vec{b} - 3\vec{c}$

3. a. $\overrightarrow{XY} = (-2, 3, 6)$,
$|\overrightarrow{XY}| = 7$
b. $\left(-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

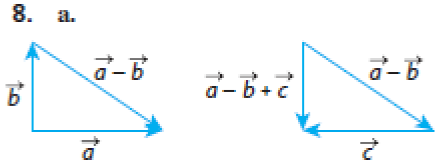
4. a. $(-6, -3, -6)$
b. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

5. $\left(-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

6. a. $\overrightarrow{OA} + \overrightarrow{OB} = (-3, 8, -8)$,
$\overrightarrow{OA} - \overrightarrow{OB} = (9, -4, -4)$
b. $\theta \doteq 84.4°$

7. a. $|\overrightarrow{AB}| = \sqrt{14}$,
$|\overrightarrow{BC}| = \sqrt{59}$,
$|\overrightarrow{CA}| = \sqrt{45}$
b. 12.5
c. 18.13
d. $(6, 2, -2)$

8. a.



b. 5

9. $\frac{1}{2}(-11, 7) + \left(-\frac{3}{2}\right)(-3, 1) = (-1, 2)$,
$\frac{1}{3}(-11, 7) + \left(-\frac{2}{3}\right)(-1, 2) = (-3, 1)$,
$3(-3, 1) + 2(-1, 2) = (-11, 7)$

10. a. $x - 3y + 6z = 0$ where $P(x, y, z)$ is the point.
b. $(0, 0, 0)$ and $\left(1, \frac{1}{3}, 0\right)$

11. a. $a = -3, b = 26.5, c = 10$
b. $a = 8, b = \frac{7}{3}, c = -10$

12. a. yes
b. yes

13. a. $|\overrightarrow{AB}|^2 = 9, |\overrightarrow{AC}|^2 = 3, |\overrightarrow{BC}|^2 = 6$
Since $|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2$ the triangle is right-angled
b. $\frac{\sqrt{6}}{3}$

14. a. $\overrightarrow{DA}, \overrightarrow{BC}$ and $\overrightarrow{EB}, \overrightarrow{ED}$
b. $\overrightarrow{DC}, \overrightarrow{AB}$ and $\overrightarrow{CE}, \overrightarrow{EA}$
c. $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{AC}|^2$
But $|\overrightarrow{AC}|^2 = |\overrightarrow{DB}|^2$
Therefore, $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{DB}|^2$

15. a. $C(3, 0, 5), P(3, 4, 5), E(0, 4, 5), F(0, 4, 0)$
b. $\overrightarrow{DB} = (3, 4, -5)$,
$\overrightarrow{CF} = (-3, 4, -5)$
c. 90°
d. 50.2°

16. a. 7.74
b. 2.83
c. 2.83

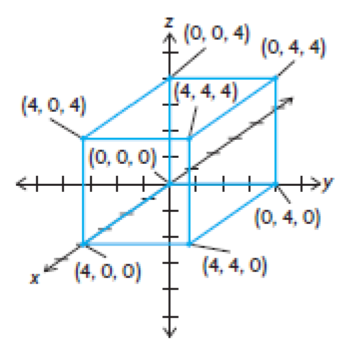
17. a. 1236.9 km
b. S14.0°W

18. a. Any pair of nonzero, noncollinear vectors will span R^2. To show that (2, 3) and (3, 5) are noncollinear, show that there does not exist any number k such that $k(2, 3) = (3, 5)$.
Solve the system of equations:
$2k = 3$
$3k = 5$
Solving both equations gives two different values for k, $\frac{3}{2}$ and $\frac{5}{3}$, so (2, 3) and (3, 5) are noncollinear and thus span R^2.
b. $m = -770, n = 621$

19. a. Find a and b such that
$(5, 9, 14) = a(-2, 3, 1) + b(3, 1, 4)$
$(5, 9, 14) = (-2a, 3a, a) + (3b, b, 4b)$
$(5, 9, 14) = (-2a + 3b, 3a + b, a + 4b)$
i. $5 = -2a + 3b$
ii. $9 = 3a + b$
iii. $14 = a + 4b$
Use the method of elimination with i. and iii.
$2(14) = 2(a + 4b)$
$28 = 2a + 8b$
$+ \;\; 5 = -2a + 3b$
$33 = 11b$
$3 = b$
By substitution, $a = 2$.
$\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$ because it can be written as a linear combination of $\vec{b}$ and $\vec{c}$.

b. If vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$, then $\vec{a}$ can be written as a linear combination of $\vec{b}$ and $\vec{c}$. Find m and n such that
$(-13, 36, 23) = m(-2, 3, 1) + n(3, 1, 4)$
$= (-2m, 3m, m) + (3n, n, 4n)$
$= (-2m + 3n, 3m + n, m + 4n)$
Solve the system of equations:
$-13 = -2m + 3n$
$36 = 3m + n$
$23 = m + 4n$
Use the method of elimination:
$2(23) = 2(m + 4n)$
$46 = 2m + 8n$
$+ -13 = -2m + 3n$
$33 = 11n$
$3 = n$
By substitution, $m = 11$.
So, vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$.

20. a.



b. $(-4, -4, -4)$
c. $(-4, 0, -4)$
d. $(4, 4, 0)$

21. 7

22. **a.** $|\overrightarrow{AB}| = 10,$
$|\overrightarrow{BC}| = 2\sqrt{5} \doteq 4.47,$
$|\overrightarrow{CA}| = \sqrt{80} \doteq 8.94$

b. If A, B, and C are vertices of a right triangle, then

$$|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2$$
$$|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = (2\sqrt{5})^2 + (\sqrt{80})^2 = 20 + 80 = 100$$
$$|\overrightarrow{AB}|^2 = 10^2 = 100$$

So, triangle ABC is a right triangle.

23. **a.** $\vec{a} + \vec{b} + \vec{c}$
b. $\vec{a} - \vec{b}$
c. $-\vec{b} - \vec{a} + \vec{c}$
d. $\vec{0}$
e. $\vec{b} + \vec{c}$

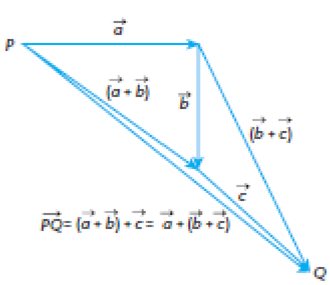
24.

25. **a.** $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$

26. **Case 1** If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.
Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition.
We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. **a.** (8, 4, 8)
b. 12
c. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

3. $\sqrt{19}$

4. **a.** $\vec{x} = 2\vec{b} - 3\vec{a}, \vec{y} = 3\vec{b} - 5\vec{a}$
b. $a = 1, b = 5, c = -11$

5. **a.** $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.
b. $p = -2, q = 3$

6. **a.** $(1, 12, -29) = -2(3, 1, 4) + 7(1, 2, -3)$
b. $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

7. $\sqrt{13}, \theta \doteq 3.61$; 73.9° relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$
$\overrightarrow{DE} = \vec{b} - \vec{a}$
Also,
$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$
$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$
Thus,
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$