Section 2.3 – Linear and Angular Velocities

The most intuitive measure of the rate at which the rider is traveling around the wheel is what we call ***linear velocity***.

Another way to specify how fast the rider is traveling around the wheel is with what we call ***angular velocity***.

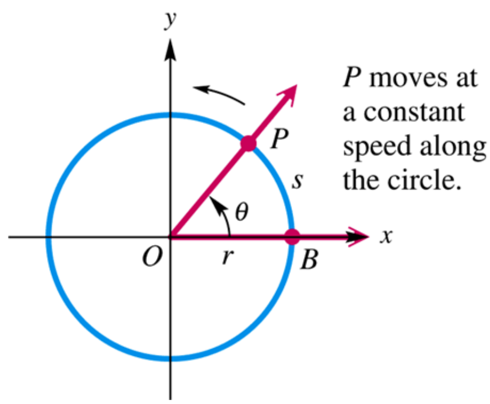
Linear Speed

Definition

If P is a point on a circle of radius r, and P moves a distance s on the circumference of the circle in an amount of time t, then the ***linear velocity***, v, of P is given by the formula

$$speed = \frac{distance}{time}$$

$$v = \frac{s}{t}$$



Example

A point on a circle travels 5 cm in 2 sec. Find the linear velocity of the point.

<u>Solution</u>

Given: $s = 5cm$

$t = 2\ sec$

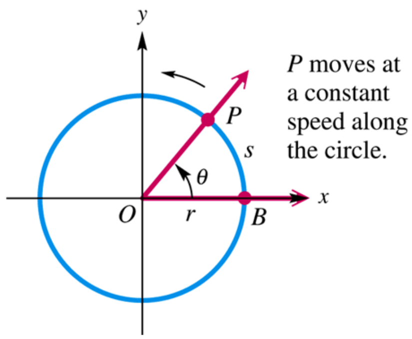
$$v = \frac{s}{t} = \frac{5\ cm}{2\ sec}$$
$$= 2.5\ cm / sec$$

Angular Speed

Definition

If P is a point moving with uniform circular motion on a circle of radius r, and the line from the center of the circle through P sweeps out a central angle θ in an amount of time t, then the *angular velocity*, ω (omega), of P is given by the formula

$$\omega = \frac{\theta}{t} \qquad \textit{where } \theta \textit{ is measured in radians}$$



Example

A point on a circle rotates through $\frac{3\pi}{4}$ radians in 3 sec. Give the angular velocity of the point.

<u>*Solution*</u>

Given: $\theta = \frac{3\pi}{4}\, rad$

$t = 3 sec$

$$\omega = \frac{\frac{3\pi}{4}\ rad}{3\ sec}$$

$$= \frac{\pi}{4}\ rad\,/\,sec$$

Example

A bicycle wheel with a radius of 13.0 *in*. turns with an angular velocity of 3 radians per seconds. Find the distance traveled by a point on the bicycle tire in 1 minute.

<u>*Solution*</u>

Given: $r = 13.0\ in.$

$\omega = 3\text{rad}/sec$

$t = 1\ min = 60\ sec.$

$$\omega = \frac{\theta}{t}$$

$$\omega\, t = \theta \qquad\qquad s = r\theta \quad \Rightarrow \theta = \frac{s}{r}$$

$$\omega\, t = \frac{s}{r}$$

$$s = \omega\, tr$$

$$= 3 \times 60 \times 13$$

$$= 2{,}340\ inches$$

$$or\ \frac{2{,}340}{12} = 195\, ft$$

Relationship between the Two Velocities

If $s = r\theta$

$$\frac{s}{t} = \frac{r\theta}{t}$$

$$\frac{s}{t} = r\frac{\theta}{t}$$

$$v = r\omega$$

$$v = r\frac{\theta}{t}$$

Linear and Angular Velocity

If a point is moving with uniform circular motion on a circle of radius r, then the linear velocity v and angular velocity ω of the point are related by the formula

$$v = r\omega$$

Example

A phonograph record is turning at 45 revolutions per minute (*rpm*). If the distance from the center of the record to a point on the edge of the record is 3 inches, find the angular velocity and the linear velocity of the point in feet per minute.

<u>*Solution*</u>

$$\begin{aligned} \omega &= 45\ rpm \\ &= 45\ \frac{rev}{min} \qquad\qquad 1\ revolution = 2\pi rad \\ &= 45\ \frac{rev}{min}\,\frac{2\pi\ rad}{1\ rev} \\ &= 90\pi\ rad\,/\,min \end{aligned}$$

$$\begin{aligned} v &= r\omega \\ &= (3\ in.)\left(90\pi\frac{rad}{min}\right) \\ &= 270\pi\frac{in}{min} \\ &= 848\ in\,/\,min \end{aligned}$$

$$v = 848\ \frac{in}{min}\,\frac{1\,ft}{12in}$$

$$v = 70.7\ ft\,/\,\text{min}$$

Example

Suppose that P is on a circle with radius 10 cm, and ray OP is rotating with angular speed $\frac{\pi}{18}\ rad/\sec$.

a) Find the angle generated by P in 6 seconds

b) Find the distance traveled by P along the circle in 6 seconds.

c) Find the linear speed of P in cm per sec.

<u>*Solution*</u>

a) $\theta = \omega t$

$$\boxed{\theta = \frac{\pi}{18}.6 = \frac{\pi}{3}\ rad}$$

b) $s = r\theta$

$$\boxed{s = 10\left(\frac{\pi}{3}\right) = \frac{10\pi}{3}\ cm}$$

c) $v = \frac{s}{t}$

$$v = \frac{\frac{10\pi}{3}}{6}$$

$$= \frac{10\pi}{18}$$

$$= \frac{5\pi}{9}\ cm/\sec$$

Example

A belt runs a pulley of radius 6 cm at $80\ rev/\min$.

a) Find the angular speed of the pulley in radians per sec.

b) Find the linear speed of the belt in cm per sec.

<u>*Solution*</u>

a) $\omega = 80\frac{rev}{\min}\cdot\frac{1\min}{60\sec}\cdot\frac{2\pi}{1rev}$

$$= \frac{8\pi}{3}\ rad/\sec$$

b) $v = r\omega$

$$= 6\left(\frac{8\pi}{3}\right)$$

$$\approx 50\ cm/\sec$$

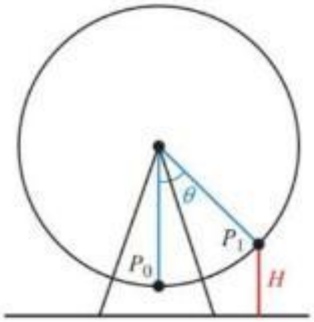
Example

The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft*, and one complete revolution takes 20 minutes, find

a. The linear velocity, in miles per hour, of a person riding on the wheel.

b. The height of the rider in terms of the time *t*, where *t* is measured in minutes.

<u>Solution</u>



Given: $\theta = 1\ rev = 2\pi\ rad$

$t = 20\ min.$

$r = \frac{D}{2} = \frac{250}{2} = 125\,ft$

a. $\omega = \frac{\theta}{t}$ *or* $\boxed{v = \frac{r\theta}{t}}$

$= \frac{2\pi}{20}$

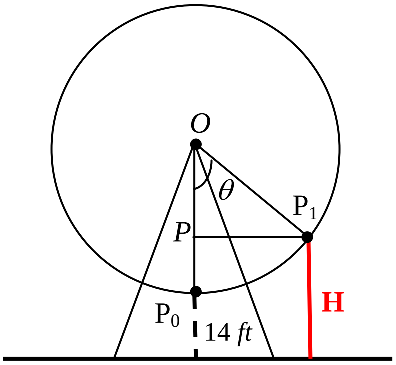
$= \frac{\pi}{10}\ rad / min$

$$\begin{aligned} v &= r\omega \\ &= (125\,ft)\left(\frac{\pi}{10} rad/\min\right) \\ &\approx 39.27\ ft/\min \end{aligned}$$

$$\begin{aligned} v &\approx 39.27\ \frac{ft}{\min}\ \frac{60\min}{1hr}\ \frac{1mile}{5{,}280\,ft} \\ &\approx 0.45\ mi/hr \end{aligned}$$

b. $\cos\theta = \frac{OP}{OP_1}$

$= \frac{OP}{125}$



$$\begin{aligned} OP &= 125\cos\theta \\ H &= PP_0 + 14 \\ &= OP_0 - OP + 14 \\ &= 125 - 125\cos\theta + 14 \\ &= 139 - 125\cos\theta \end{aligned}$$

$$\begin{aligned} \omega &= \frac{\theta}{t} \\ \theta &= \omega t \\ \theta &= \frac{\pi}{10}\ t \\ H &= 139 - 125\cos\left(\frac{\pi}{10}t\right) \end{aligned}$$

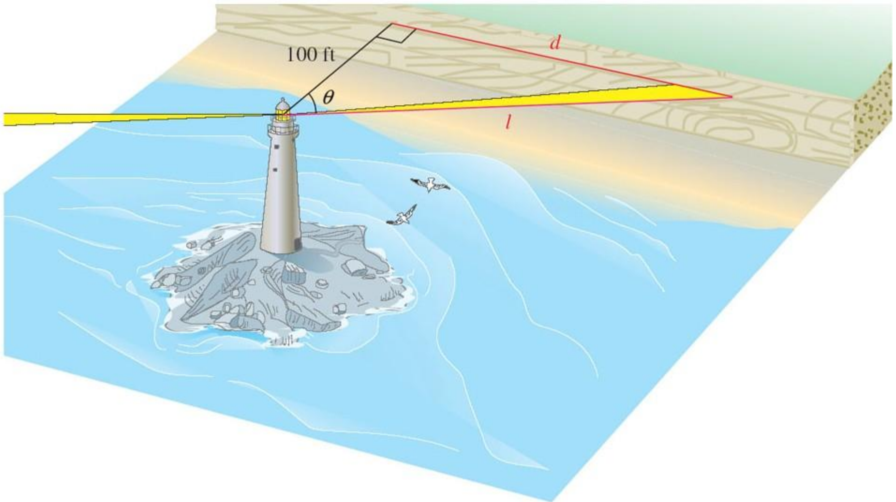
Exercises *Section* 2.3 – Linear and Angular Velocities

1. Find the linear velocity of a point moving with uniform circular motion, if $s = 12$ cm and $t = 2$ sec.

2. Find the distance s covered by a point moving with linear velocity $v = 55$ mi/hr and $t = 0.5$ hr.

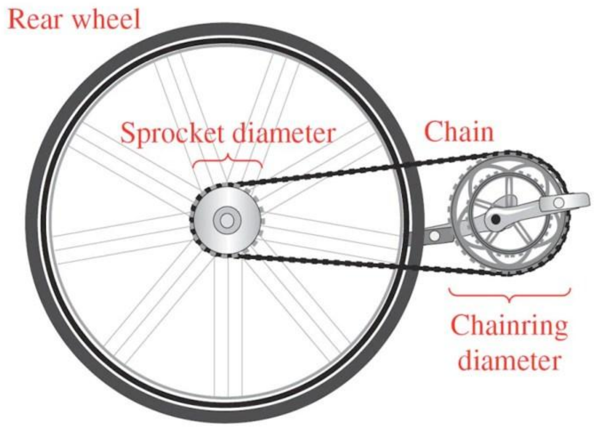
3. Point ***P*** sweeps out central angle $\theta = 12\pi$ as it rotates on a circle of radius r with $t = 5\pi$ sec. Find the angular velocity of point ***P***.

4. Find an equation that expresses ***l*** in terms of time *t*. Find ***l*** when *t* is 0.5 sec, 1.0 sec, and 1.5 sec. (assume the light goes through one rotation every 4 seconds.)



5. Find the angular velocity, in radians per minute, associated with given 7.2 *rpm*.

6. When Lance Armstrong blazed up Mount Ventoux in the 2002 tour, he was equipped with a 150-millimeter-diameter chainring and a 95-millimeter-diameter sprocket. Lance is known for maintaining a very high cadence, or pedal rate. The sprocket and rear wheel rotate at the same rate, and the diameter of the rear wheel is 700 mm. If he was pedaling at a rate of 90 revolutions per minute, find his speed in kilometers per hour. (1 km = 1,000,000 mm or 10^6 mm)

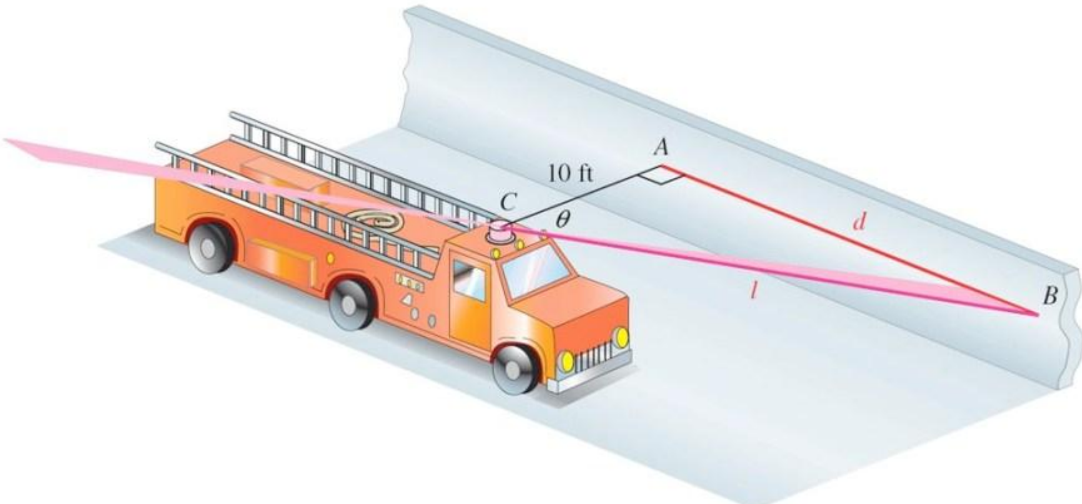


7. A Ferris wheel has a radius 50.0 ft. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ *rad* .

a) How far is the person above the ground?

b) If it takes 30 sec for the wheel to turn $\frac{2\pi}{3}$ *rad* , what is the angular speed of the wheel?

8. A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 feet from the wall and rotates through a complete revolution every 2 seconds. Find the equations that give the lengths d and ℓ in terms of time.



9. Suppose that point P is on a circle with radius 60 cm, and ray OP is rotating with angular speed $\frac{\pi}{12}$ radian per sec.

a) Find the angle generated by P in 8 sec.

b) Find the distance traveled by P along the circle in 8 sec.

c) Find the linear speed of P in 8 sec.

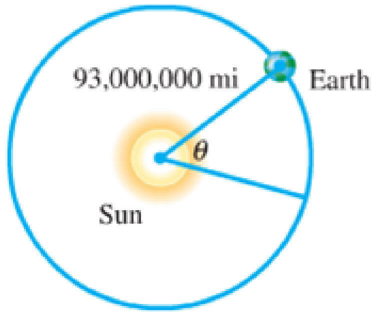
10. Tires of a bicycle have radius 13 in. and are turning at the rate of 215 revolutions per min. How fast is the bicycle traveling in miles per hour? (*Hint*: 1 mi = 5280 ft.)

11. Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.

a) Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.

b) Give the angular speed in radians per hour.

c) Find the linear speed of Earth in miles per hour.



12. Earth revolves on its axis once every 24 hr. Assuming that earth's radius is 6400 km, find the following.

a) Angular speed of Earth in radians per day and radians per hour.

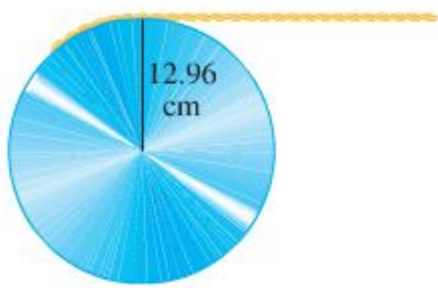
b) Linear speed at the North Pole or South Pole

c) Linear speed ar a city on the equator

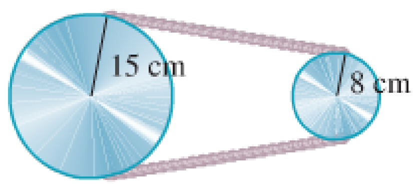
13. The pulley has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.

a) Find the linear speed of the belt in cm per sec.

b) Find the angular speed of the pulley in rad per sec.



14. The two pulleys have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in rad per sec.



15. A thread is being pulled off a spool at the rate of 59.4 cm per sec. Find the radius of the spool if it makes 152 revolutions per min.

16. A railroad track is laid along the arc of a circle of radius 1800 ft. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30 mph to go around this portion of the track?

17. A 90-horsepower outboard motor at full throttle will rotate it propeller at exactly 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

18. The shoulder joint can rotate at 25 rad/min. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 ft, find the linear speed of the club head from the shoulder rotation

Solution   Section 2.3 – Linear and Angular Velocities

Exercise

Find the linear velocity of a point moving with uniform circular motion, if s = 12 cm and t = 2 sec.

Solution

$$v = \frac{s}{t}$$

$$= \frac{12}{2}\frac{cm}{\sec}$$

$$\underline{= 6\ cm/\sec}$$

Exercise

Find the distance ***s*** covered by a point moving with linear velocity v = 55 mi/hr and t = 0.5 hr.

Solution

$$s = vt$$

$$= 55\frac{mi}{hr} \times 0.5\ hr$$

$$\underline{= 27.5\ miles}$$

Exercise

Point P sweeps out central angle $\theta = 12\pi$ as it rotates on a circle of radius r with $t = 5\pi$ sec. Find the angular velocity of point P.

Solution

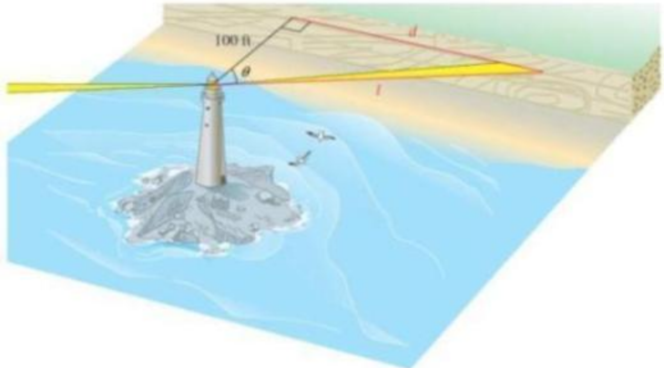
$$\omega = \frac{\theta}{t}$$

$$= \frac{12\pi}{5\pi}\frac{rad}{\sec}$$

$$\underline{= 2.4\ rad/\sec}$$

Exercise

Find an equation that expresses l in terms of time t. Find l when t is 0.5 sec, 1.0 sec, and 1.5 sec. (assume the light goes through one rotation every 4 seconds.)



Solution

$$\omega = \frac{\theta}{t} = \frac{2\pi}{4}\frac{rad}{\sec} = \frac{\pi}{2}\, rad/\sec$$

$$\Rightarrow \frac{\theta}{t} = \frac{\pi}{2}\, rad/\sec$$

$$\Rightarrow \theta = \frac{\pi}{2}t$$

$$\cos\left(\frac{\pi}{2}t\right) = \frac{100}{l}$$

$$\Rightarrow l\cos\left(\frac{\pi}{2}t\right) = 100$$

$$\Rightarrow l = \frac{100}{\cos\left(\frac{\pi}{2}t\right)} = 100\sec\left(\frac{\pi}{2}t\right)$$

For t = 0.5 sec $\Rightarrow l = \dfrac{100}{\cos\left(\frac{\pi}{2}\frac{1}{2}\right)} = \dfrac{100}{\cos\left(\frac{\pi}{4}\right)} = \dfrac{100}{\frac{1}{\sqrt{2}}} = 100\sqrt{2} \approx 141\, ft$

For t = 1.0 sec $\Rightarrow l = \dfrac{100}{\cos\left(\frac{\pi}{2}\right)} = \dfrac{100}{0} = Undefined$

For t = 1.5 sec $\Rightarrow l = \dfrac{100}{\cos\left(\frac{\pi}{2}\frac{3}{2}\right)} = \dfrac{100}{\cos\left(\frac{3\pi}{4}\right)} = \dfrac{100}{-\frac{1}{\sqrt{2}}} = -100\sqrt{2} \approx -141\, ft$

Exercise

Find the angular velocity, in radians per minute, associated with given 7.2 rpm.

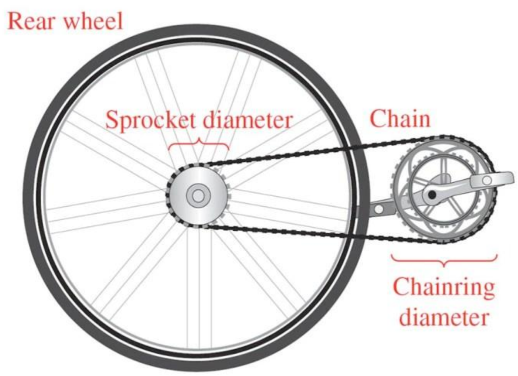
Solution

$$\omega = 7.2\frac{rev}{\min} \times 2\pi\frac{radians}{rev} = 14.4\pi \approx 45.2\frac{rad}{\min}$$

Exercise

When Lance Armstrong blazed up Mount Ventoux in the 2002 tour, he was equipped with a 150-millimeter-diameter chainring and a 95-millimeter-diameter sprocket. Lance is known for maintaining a very high cadence, or pedal rate. The sprocket and rear wheel rotate at the same rate, and the diameter of the rear wheel is 700 mm. If he was pedaling at a rate of 90 revolutions per minute, find his speed in kilometers per hour. (1 km = 1,000,000 mm or 10^6 mm)

Solution



Chainring:

$$\omega = \frac{v}{r}$$

$$= 90\,\frac{rev}{\min} \times 2\pi\,\frac{radians}{rev} \times \frac{60}{1}\,\frac{\min}{hr}$$

$$= 10800\pi\,\frac{rad}{hr}$$

$$v = r\omega$$

$$= \frac{150}{2}(mm) \times 10800\pi\,\frac{rad}{hr}$$

$$= 810000\pi\ \frac{mm}{hr}$$

Sprocket:

$$\omega = \frac{v}{r}$$

$$= \frac{810000\pi\frac{mm}{hr}}{\frac{95}{2}mm}$$

$$= 17052.63\pi\ \frac{rad}{hr}$$

$$v = r\omega$$

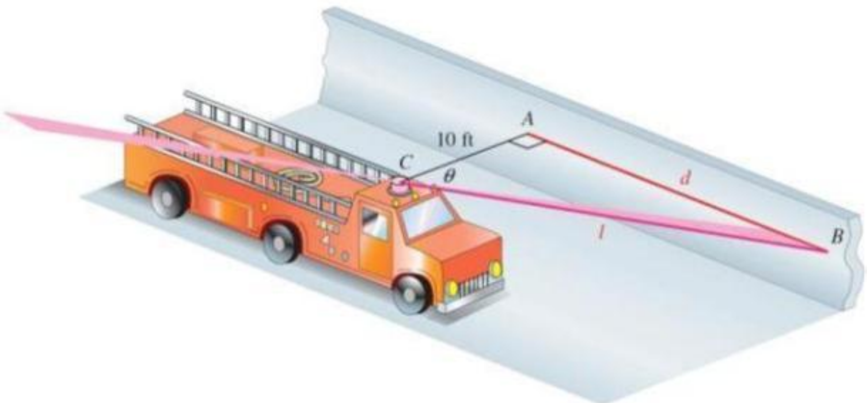
$$= 350(mm) \times \frac{1}{10^6}\,\frac{km}{mm} \times 17052.63\pi\,\frac{rad}{hr}$$

$$= 18.8\ \frac{km}{hr}$$

Exercise

A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 feet from the wall and rotates through a complete revolution every 2 seconds. Find the equations that give the lengths *d* and ℓ in terms of time.

Solution



$$\omega = \frac{\theta}{t}$$
$$= \frac{2\pi}{2}$$
$$= \pi \ rad / sec$$

$$\tan\theta = \frac{d}{10}$$
$$d = 10\tan\theta$$
$$= 10\tan \pi t$$

$$\sec\theta = \frac{l}{10}$$
$$l = 10\sec\theta$$
$$= 10\sec \pi t$$

Exercise

Suppose that point *P* is on a circle with radius 60 cm, and ray *OP* is rotating with angular speed $\frac{\pi}{12}$ radian per sec.

a) Find the angle generated by *P* in 8 sec.
b) Find the distance traveled by *P* along the circle in 8 sec.
c) Find the linear speed of *P* in 8 sec.

Solution

a) $\theta = \omega t$

$$\theta = \frac{\pi}{12} \cdot 8 = \boxed{\frac{2\pi}{3} \ rad}$$

b) $s = r\theta$

$$s = 60\left(\frac{2\pi}{3}\right) = \boxed{40\pi \ cm}$$

c) $v = \frac{s}{t}$

$$v = \frac{40\pi}{8} \ \boxed{= 5\pi \ cm/\sec}$$

Exercise

A Ferris wheel has a radius 50.0 ft. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ *rad* .

a) How far is the person above the ground?

b) If it takes 30 sec for the wheel to turn $\frac{2\pi}{3}$ *rad* , what is the angular speed of the wheel?

Solution

a) $\alpha = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$$\cos\alpha = \frac{h_1}{r}$$

$$h_1 = r\cos\alpha$$

$$h_1 = 50\cos\frac{\pi}{6} = 43.3\ ft$$

Person is $50 + 43.3 = 93.3\ ft$ above the ground

b) $\omega = \frac{\theta}{t}$

$$\omega = \frac{\frac{2\pi}{3}\ rad}{30\ sec}$$

$$= \frac{\pi}{45}\ rad / sec$$

Exercise

Tires of a bicycle have radius 13 in. and are turning at the rate of 215 revolutions per min. How fast is the bicycle traveling in miles per hour? (*Hint*: 1 mi = 5280 ft.)

Solution

$$\omega = 215\ rev\ \frac{2\pi\ rad}{1\ rev} = 430\pi\ rad / \min$$

$$v = r\omega = 13(430\pi) = 5590\pi\ in / \min$$

$$v = 5590\pi \frac{in}{\min}\frac{60\min}{1hr}\frac{1ft}{12in}\frac{1mi}{5280\,ft}$$

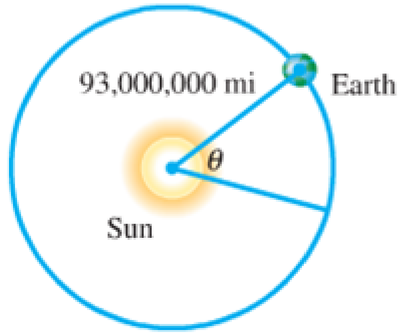
$$\approx 16.6\ mph$$

Exercise

Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.

a) Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.
b) Give the angular speed in radians per hour.
c) Find the linear speed of Earth in miles per hour.

Solution



a) $\theta = \frac{1}{365}(2\pi) = \frac{2\pi}{365}\ rad$

b) $\omega = \frac{2\pi\ rad}{365\ days}\frac{1\ day}{24\ hr} = \frac{\pi}{4380}\ rad\ /\ hr$

c) $v = r\omega = (93,000,000)\frac{\pi}{4380}\ \boxed{\approx 67,000\ mph}$

Exercise

Earth revolves on its axis once every 24 hr. Assuming that earth's radius is 6400 km, find the following.

a) Angular speed of Earth in radians per day and radians per hour.
b) Linear speed at the North Pole or South Pole
c) Linear speed ar a city on the equator

Solution

a) $\omega = \frac{\theta}{t}$

$= \frac{2\pi}{1}\frac{rad}{day}$

$= \frac{2\pi}{1}\frac{rad}{day}\frac{1\ day}{24\ hr}\ \boxed{= \frac{\pi}{12} rad\ /\ hr}$

b) At the poles, $r = 0$ so $\boldsymbol{v} = rw = 0$

c) At the equator, $r = 6400$ km

$v = rw$

$= 6400(2\pi)$

$= 12,800\pi\ km\ /\ day$

$= 12,800\pi\ \frac{km}{day}\ \frac{1\ day}{24\ hr}$

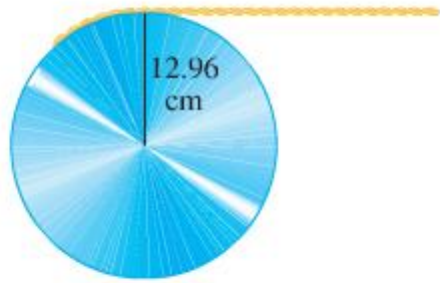
$\boxed{\approx 533\pi\ km\ /\ hr}$

Exercise

The pulley has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.

a) Find the linear speed of the belt in cm per sec.
b) Find the angular speed of the pulley in rad per sec.

Solution



Given: s = 56 cm in t = 18 sec

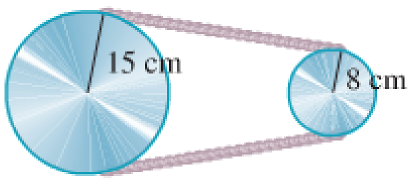
r = 12.96 cm

a) $v = \frac{s}{t} = \frac{56}{18} \approx 3.1\ cm/\sec$

b) $\omega = \frac{v}{r} = \frac{3.1}{12.96} \approx .24\ rad/\sec$

Exercise

The two pulleys have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in rad per sec.



Solution

Given: $\omega = \frac{25}{36}\ times/\sec$

$r_1 = 15\ cm \quad r_2 = 8\ cm$

The angular velocity of the larger pulley is:

$$\omega = \frac{25}{36}\frac{times}{\sec}\frac{2\pi\ rad}{1\ time} = \frac{25\pi}{18}\ rad/\sec$$

The linear velocity of the larger pulley is:

$$v = r\omega = 15\left(\frac{25\pi}{18}\right) = \frac{125\pi}{6}\ cm/\sec$$

The angular velocity of the smaller pulley is:

$$\omega = \frac{v}{r} = \frac{1}{r}v$$

$$= \frac{1}{8}\frac{125\pi}{6}$$

$$= \frac{125\pi}{48}\ rad/\sec$$

Exercise

A thread is being pulled off a spool at the rate of 59.4 cm per sec. Find the radius of the spool if it makes 152 revolutions per min.

Solution

Given: $\omega = 152\ rev / \min$

$v = 59.4\ cm / \sec$

$$r = \frac{v}{\omega} = \frac{1}{\omega} v$$

$$= \frac{1}{152 \frac{rev}{\min}} 59.4 \frac{cm}{\sec}$$

$$= \left(\frac{1}{152} \frac{\min}{rev} \frac{60\ \sec}{1\ \min} \frac{1\ rev}{2\pi\ rad} \right) \left(59.4 \frac{cm}{\sec} \right)$$

$$\approx 3.7\ cm$$

Exercise

A railroad track is laid along the arc of a circle of radius 1800 ft. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30 mph to go around this portion of the track?

Solution

Given: $r = 1800$ ft.

$$\theta = 40^\circ = 40^\circ \frac{\pi}{180^\circ} = \frac{2\pi}{9} rad$$

$$v = 30\ mph$$

The arc length: $s = r\theta = 1800 \left(\frac{2\pi}{9} \right) = 400\pi\ ft$

$$v = \frac{s}{t} \Rightarrow t = \frac{s}{v}$$

$$t = \frac{400\pi\ ft}{30 \frac{mi}{hr}}$$

$$= \frac{40\pi}{3}\ ft \frac{hr}{mi} \frac{1 mi}{5280\ ft} \frac{3600 \sec}{1 hr}$$

$$\approx 29\ \sec$$

Exercise

A 90-horsepower outboard motor at full throttle will rotate it propeller at exactly 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

Solution

$$\omega = 5000 \frac{rev}{\min} \frac{2\pi}{1} \frac{rad}{rev} \frac{1}{60} \frac{\min}{\sec}$$

$$\approx 523.6\ rad / \sec$$

Exercise

The shoulder joint can rotate at 25 rad/min. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 ft, find the linear speed of the club head from the shoulder rotation.

Solution

Given: $\omega = 25\ rad / \min \quad r = 5\,ft$

$$v = r\omega = 5(25) = 125\ ft / \min$$