Section 2.4 – Law of Sines and Cosines

Oblique Triangle

A triangle that is not a right triangle, either acute or obtuse.
The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known.

The Law of Sines

There are many relationships that exist between the sides and angles in a triangle.

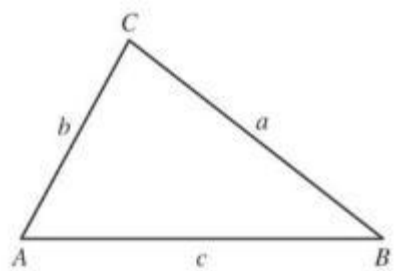
One such relation is called the law of sines.

Given triangle ABC

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

or, equivalently

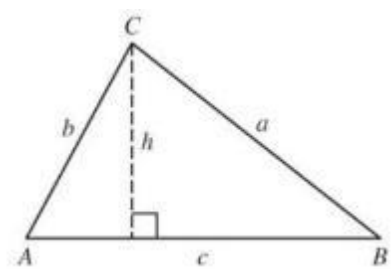
$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$



Proof

$$\sin A = \frac{h}{b} \Rightarrow h = b\sin A \quad (1)$$

$$\sin B = \frac{h}{a} \Rightarrow h = a\sin B \quad (2)$$



From (1) & (2)

$$h = h$$

$$b\sin A = a\sin B$$

$$\frac{b\sin A}{ab} = \frac{a\sin B}{ab}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Angle – Side - Angle (*ASA* or *AAS*)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

Example

In triangle *ABC*, $A = 30°$, $B = 70°$, and $a = 8.0\,cm$. Find the length of side c.

<u>*Solution*</u>

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 180° - 100° \\ &= 80° \end{aligned}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\begin{aligned} c &= \frac{a}{\sin A}\sin C \\ &= \frac{8}{\sin 30°}\sin 80° \\ &= 16\,cm \end{aligned}$$

Example

Find the missing parts of triangle *ABC* if $A = 32°$, $C = 81.8°$, , and $a = 42.9\,cm$.

<u>*Solution*</u>

$$\begin{aligned} B &= 180° - (B + C) \\ &= 180° - (32° + 81.8°) \\ &= 66.2° \end{aligned}$$

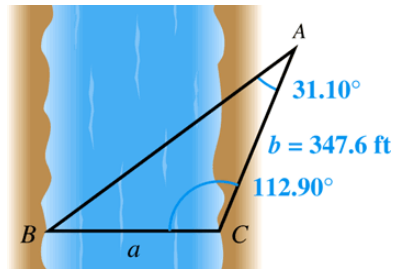
$\frac{a}{\sin A} = \frac{b}{\sin B}$ $b = \frac{a\sin B}{\sin A}$ $= \frac{42.9\sin 66.2°}{\sin 32°}$ $\approx 74.1\,cm$	$\frac{c}{\sin C} = \frac{a}{\sin A}$ $c = \frac{a\sin C}{\sin A}$ $= \frac{42.9\sin 81.8°}{\sin 32°}$ $\approx 80.1\,cm$

Example

You wish to measure the distance across a River. You determine that C = 112.90°, A = 31.10°, and $b = 347.6\ ft$. Find the distance a across the river.

<u>*Solution*</u>

$$B = 180° - A - C$$
$$= 180° - 31.10° - 112.90°$$
$$= 36°$$



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 31.1°} = \frac{347.6}{\sin 36°}$$

$$a = \frac{347.6}{\sin 36°}\sin 31.1°$$

$$\boxed{a = 305.5\ ft}$$

Example

Find distance x if a = 562 *ft.*, $B = 5.7°$ and $A = 85.3°$

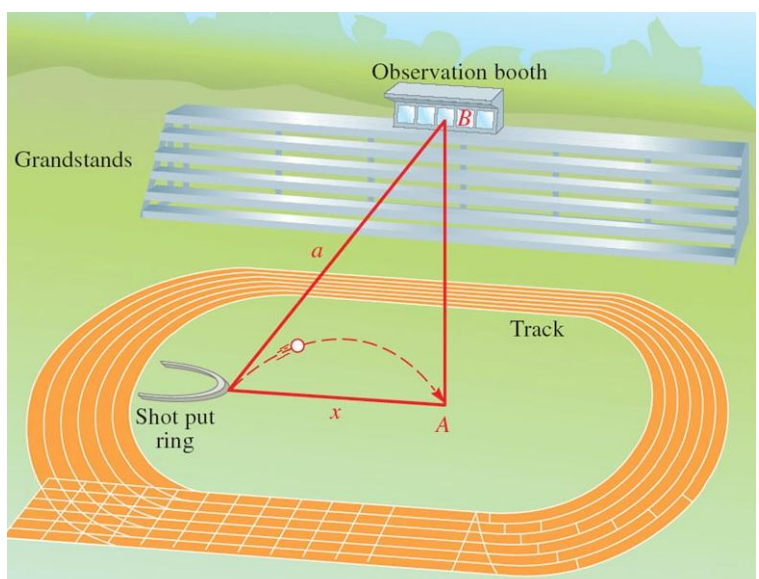
<u>*Solution*</u>

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$

$$x = \frac{a \sin B}{\sin A}$$

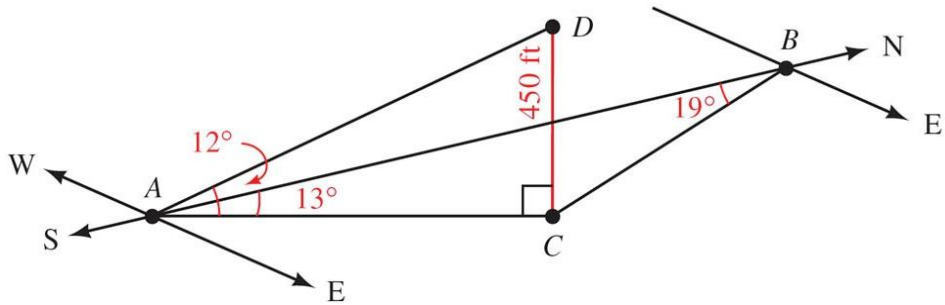
$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$= 56.0\ ft$$



Example

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 feet above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?



<u>*Solution*</u>

$$\tan 12° = \frac{DC}{AC}$$

$$AC = \frac{DC}{\tan 12°}$$

$$= \frac{450}{\tan 12°}$$

$$= 2{,}117 \ ft$$

$$\angle ACB = 180° - (13° + 19°)$$

$$= 148°$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$AB = \frac{2117 \sin 148°}{\sin 19°}$$

$$= 3{,}400 \ ft$$

Ambiguous Case

Side – Angle – Side (*SAS*)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

Example

Find angle B in triangle ABC if a = 2, b = 6, and $A = 30°$

<u>*Solution*</u>

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{6 \sin 30°}{2}$$

$$= 1.5 \qquad -1 \le \sin \alpha \le 1$$

Since $sinB > 1$ is impossible, no such triangle exists.

Example

Find the missing parts in triangle ABC if C = 35.4°, a = 205 ft., and c = 314 ft.

<u>*Solution*</u>

$$\sin A = \frac{a \sin C}{c}$$

$$= \frac{205 \sin 35.4°}{314}$$

$$= 0.3782$$

$$A = \sin^{-1}(0.3782)$$

$A = 22.2°$

$$A' = 180° - 22.2° = 157.8°$$

$$C + A' = 35.4° + 157.8°$$

$$= 193.2° \quad > 180°$$

$$B = 180° - (22.2° + 35.4°) = 122.4°$$

$$b = \frac{c \sin B}{\sin C}$$

$$= \frac{314 \sin 122.4°}{\sin 35.4°}$$

$$= 458 \ ft$$

Example

Find the missing parts in triangle ABC if a = 54 cm, b = 62 cm, and A = 40°.

<u>*Solution*</u>

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{62 \sin 40°}{54}$$

$$= 0.738$$

$$B = \sin^{-1}(0.738) = 48°$$

$$C = 180° - (40° + 48°)$$

$$= 92°$$

$$c = \frac{a \sin C}{\sin A}$$

$$= \frac{54 \sin 92°}{\sin 40°}$$

$$= 84\ cm$$

$$B = 180° - 48° = 132°$$

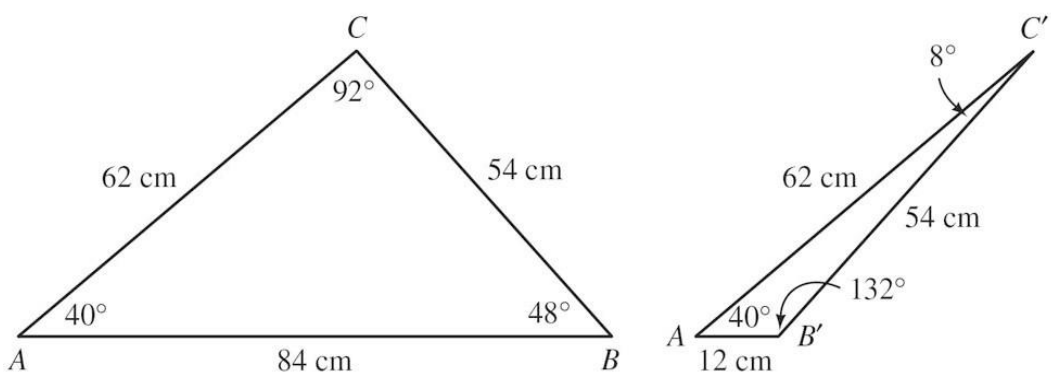
$$C' = 180° - (40° + 132°)$$

$$= 8°$$

$$c' = \frac{a \sin C'}{\sin A}$$

$$= \frac{54 \sin 8°}{\sin 40°}$$

$$= 12\ cm$$



Area of a Triangle (SAS)

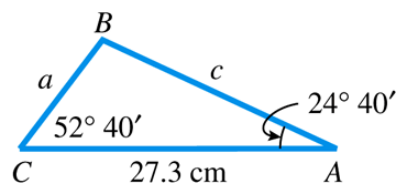
In any triangle *ABC*, the area A is given by the following formulas:

$$A=\frac{1}{2}bc\sin A \qquad A=\frac{1}{2}ac\sin B \qquad A=\frac{1}{2}ab\sin C$$

Example

Find the area of triangle ABC if $A=24°40',\ b=27.3\ cm,\ \ and\ \ C=52°40'$

<u>*Solution*</u>



$$B=180°-24°40'-52°40'$$
$$=180°-\left(24°+\frac{40°}{60}\right)-\left(52°+\frac{40°}{60}\right)$$
$$\approx 102.667°$$

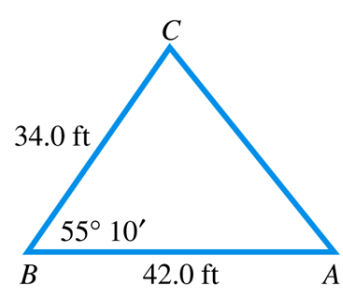
$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{a}{\sin(24°40')}=\frac{27.3}{\sin(102°40')}$$
$$a=\frac{27.3\sin(24°40')}{\sin(102°40')}$$
$$\approx 11.7\ cm$$

$$A=\frac{1}{2}ac\sin B$$
$$=\frac{1}{2}(11.7)(27.3)\sin(52°40')$$
$$\approx 127\ cm^2$$

Example

Find the area of triangle ABC.

<u>*Solution*</u>



$$A=\frac{1}{2}ac\sin B$$
$$=\frac{1}{2}(34.0)(42.0)\sin(55°10')$$
$$\approx 586\ ft^2$$

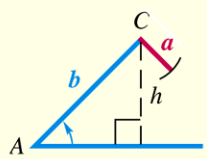
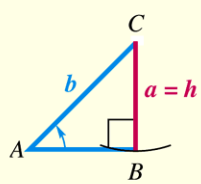
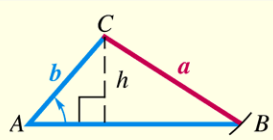
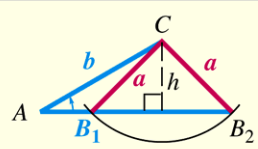
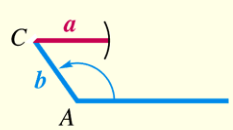
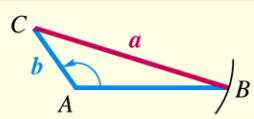
Number of Triangles Satisfying the Ambiguous Case (SSA)

Let sides a and b and angle A be given in triangle ABC. (The law of sines can be used to calculate the value of sin B.)

1. If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
2. If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
3. If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.

   a) If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use B_1 for B in the first triangle.

   b) Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use B_2 for B in the second triangle.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B > 1$, $a < h < b$
1		$\sin B = 1$, $a = h$ and $h < b$
1		$0 < \sin B < 1$, $a \geq b$
2		$0 < \sin B_2 < 1$, $h < a < b$
0		$\sin B \geq 1$, $a \leq b$
1		$0 < \sin B < 1$, $a > b$

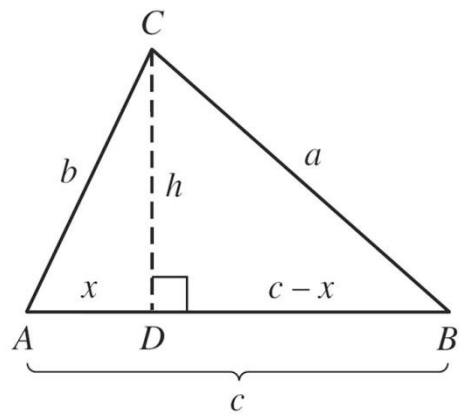
Law of Cosines (*SAS*)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Derivation



$$a^2 = (c-x)^2 + h^2$$

$$= c^2 - 2cx + x^2 + h^2 \qquad \textbf{(1)}$$

$$b^2 = x^2 + h^2 \qquad \textbf{(2)}$$

From **(2):**

(1) $a^2 = c^2 - 2cx + b^2$

$$a^2 = c^2 + b^2 - 2cx \qquad \textbf{(3)}$$

$$\cos A = \frac{x}{b}$$

$$b\cos A = x$$

(3) $\Rightarrow a^2 = c^2 + b^2 - 2cb\cos A$

Example

Find the missing parts in triangle ABC if $A = 60°$, $b = 20$ in, and $c = 30$ in.

<u>*Solution*</u>

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 20^2 + 30^2 - 2(20)(30)\cos 60° \\ &= 700 \end{aligned}$$

$$a \approx 26$$

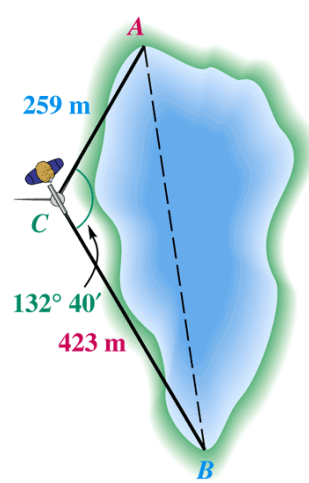
$$\begin{aligned} \sin B &= \frac{b\sin A}{a} \\ &= \frac{20\sin 60°}{26} \\ &= 0.6662 \end{aligned}$$

$$\begin{aligned} B &= \sin^{-1}(0.6662) \\ &= 42° \end{aligned}$$

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 60° - 42° \\ &= 78° \end{aligned}$$

Example

A surveyor wishes to find the distance between two inaccessible points A and B on opposite sides of a lake. While standing at point C, she finds that $AC = 259$ m, $BC = 423$ m, and angle $ACB = 132°40'$. Find the distance AB.



<u>*Solution*</u>

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2(AC)(BC)\cos C \\ &= 259^2 + 423^2 - 2(259)(423)\cos(132°40') \\ &= 394510 \end{aligned}$$

$$\boxed{AB \approx 628}$$

Law of Cosines (***SSS***) - Three Sides

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example

Solve triangle *ABC* if *a* = 34 km, *b* = 20 km, and c = 18 km

Solution

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{20^2 + 18^2 - 34^2}{2(20)(18)}$$

$$= -0.6$$

$$A = \cos^{-1}(-0.6)$$

$$= 127°$$

OR

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{34^2 + 20^2 - 18^2}{2(34)(20)}$$

$$= 0.91$$

$$C = \cos^{-1}(0.91)$$

$$= 25°$$

$$\sin C = \frac{c \sin A}{a}$$

$$= \frac{18 \sin 127°}{34}$$

$$= 0.4228$$

$$C = \sin^{-1}(0.4228)$$

$$= 25°$$

$$B = 180° - A - C$$

$$= 180° - 127° - 25°$$

$$= 28°$$

Example

A plane is flying with an airspeed of 185 miles per hour with heading 120°. The wind currents are running at a constant 32 miles per hour at 165° clockwise from due north. Find the true course and ground speed of the plane.

<u>*Solution*</u>

$$\alpha = 180° - 120°$$
$$= 60°$$

$$\theta = 360° - 165° - \alpha$$
$$= 360° - 165° - 60°$$
$$= 135°$$

$$|V+W|^2 = |V|^2 + |W|^2 - 2|V|\cdot|W|\cos\theta$$
$$= 185^2 + 32^2 - 2(185)(32)\cos 135°$$
$$= 43{,}621$$

$$|V+W| = 210\ mph$$

$$\frac{\sin\beta}{32} = \frac{\sin\theta}{210}$$

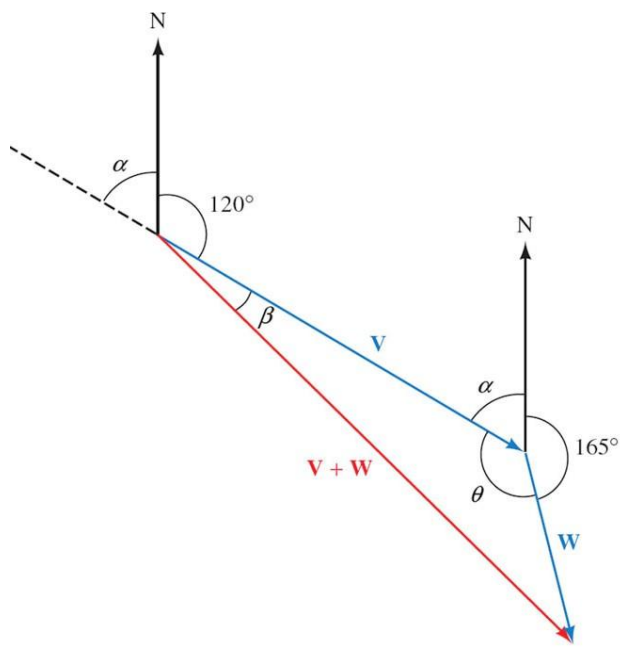
$$\sin\beta = \frac{32\sin 135°}{210}$$
$$= 0.1077$$

$$\beta = \sin^{-1}(0.1077) = 6°$$

The true course is:

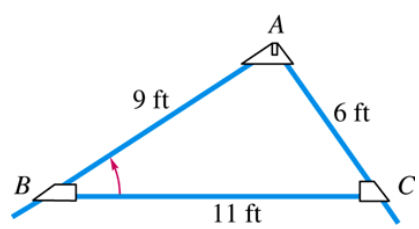
$$120° + \beta = 120° + 6° \ \underline{= 126°}.$$

The speed of the plane with respect to the ground is 210 mph.



Example

Find the measure of angle B in the figure of a roof truss.



<u>*Solution*</u>

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{11^2 + 9^2 - 6^2}{2(11)(9)}$$

$$\underline{B = \cos^{-1}\left(\frac{11^2 + 9^2 - 6^2}{2(11)(9)}\right) \approx 33°}$$

Heron's Area Formula (*SSS*)

If a triangle has sides of lengths a, b, and c, with semi-perimeter

$$s = \frac{1}{2}(a+b+c)$$

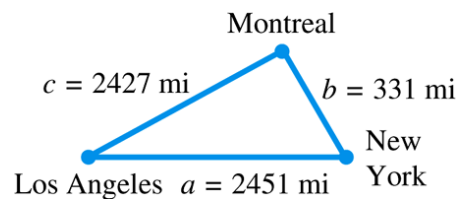
Then the area of the triangle is:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example

The distance "as the crow flies" from Los Angeles to New York is 2451 miles, from New York to Montreal is 331 miles, and from Montreal to Los Angeles is 2427 miles. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)

Solution



The semiperimeter s is:

$$s = \frac{1}{2}(a+b+c)$$

$$= \frac{1}{2}(2451+331+2427)$$

$$= 2604.5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{2604.5(2604.5-2451)(2604.5-331)(2604.5-2427)}$$

$$\approx 401{,}700\ mi^2$$

Exercises *Section* 2.4 – Law of Sines and Cosines

1. In triangle *ABC*, $B = 110°$, $C = 40°$ and $b = 18\,in$. Find the length of side *c*.

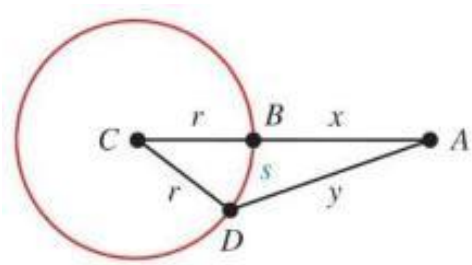
2. In triangle *ABC*, $A = 110.4°$, $C = 21.8°$ and $c = 246\,in$. Find all the missing parts.

3. Find the missing parts of triangle *ABC*, if $B = 34°$, $C = 82°$, and $a = 5.6\,cm$.

4. Solve triangle *ABC* if $B = 55°40'$, $b = 8.94$ m, and $a = 25.1$ m.

5. Solve triangle *ABC* if $A = 55.3°$, $a = 22.8$ *ft.*, and $b = 24.9$ ft.

6. Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in.



7. If $A = 26°,\ s = 22,\ and\ r = 19,$ find x

8. If $a = 13$ yd., $b = 14$ yd., and $c = 15$ yd., find the largest angle.

9. Solve triangle *ABC* if $b = 63.4$ km, and $c = 75.2$ km, $A = 124°\ 40'$

10. Solve triangle *ABC* if $A = 42.3°,\ b = 12.9m,\ and\ c = 15.4m$

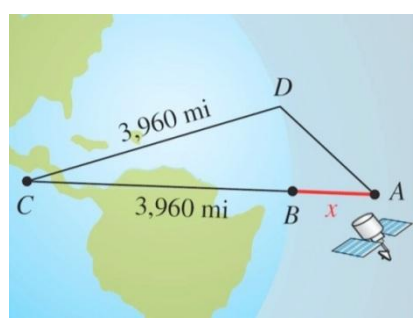
11. Solve triangle *ABC* if $a = 832$ ft., $b = 623$ ft., and c = 345 ft.

12. Solve triangle *ABC* if $a = 9.47\,ft,\ b = 15.9\,ft,\ and\ c = 21.1\,ft$

13. The diagonals of a parallelogram are 24.2 cm and 35.4 cm and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram

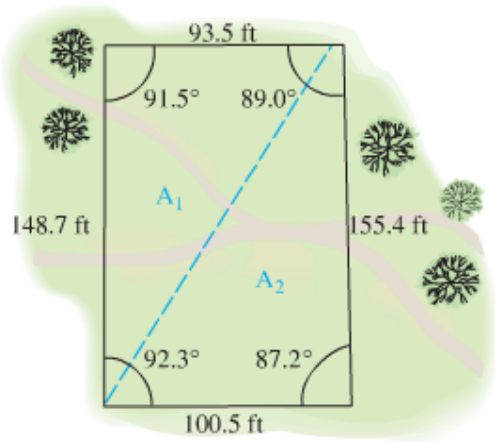
14. A man flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

15. A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 miles and the radius of the earth is 3,960 miles, how far above the earth is the satellite?

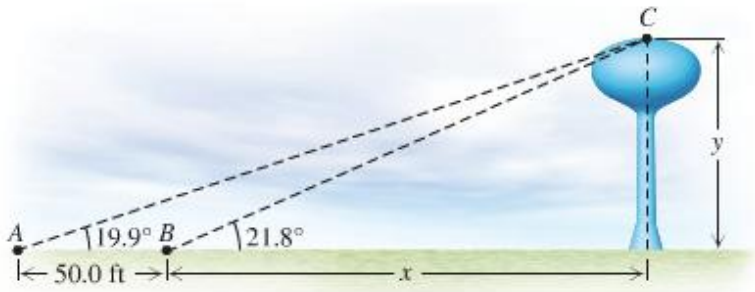


16. A pilot left Fairbanks in a light plane and flew 100 miles toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

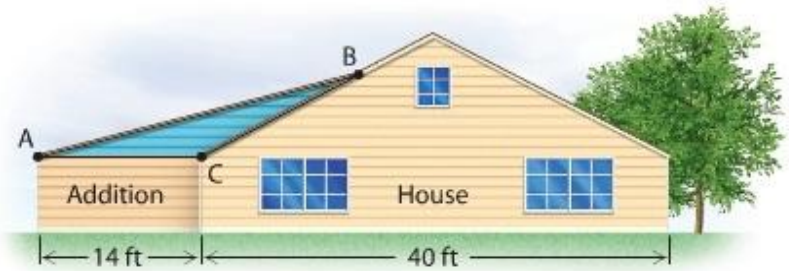
17. The dimensions of a land are given in the figure. Find the area of the property in square feet.



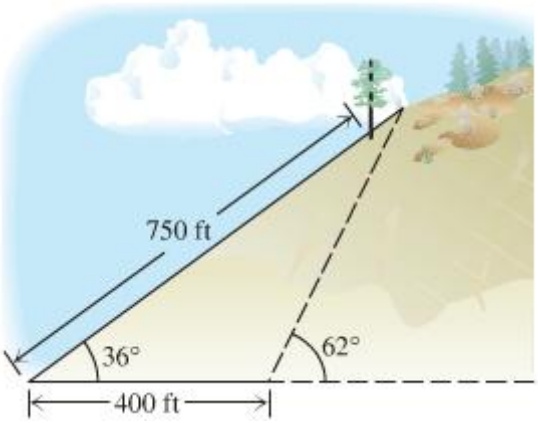
18. The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



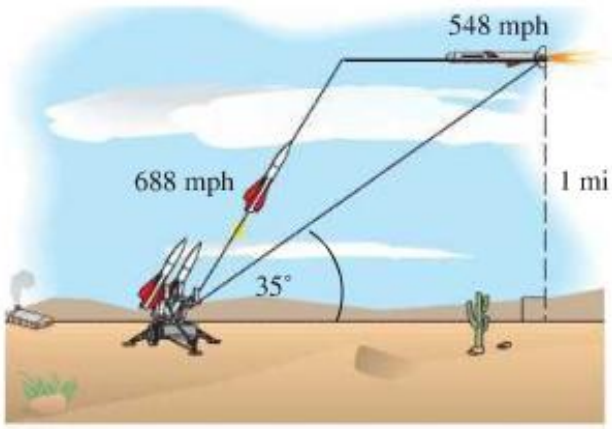
19. A 40-ft wide house has a roof with a 6-12 pitch (the roof rises 6 ft for a run of 12 ft). The owner plans a 14-ft wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$



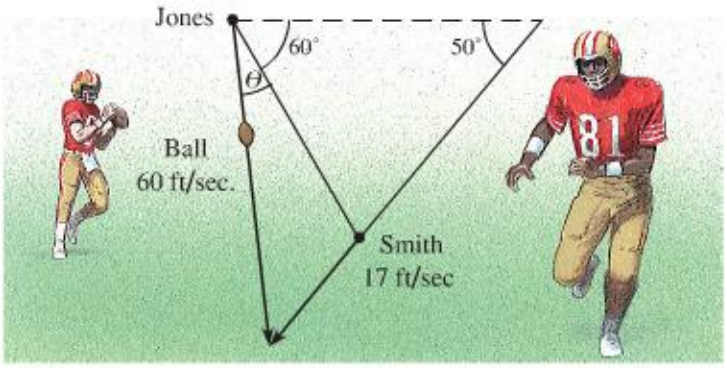
20. A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



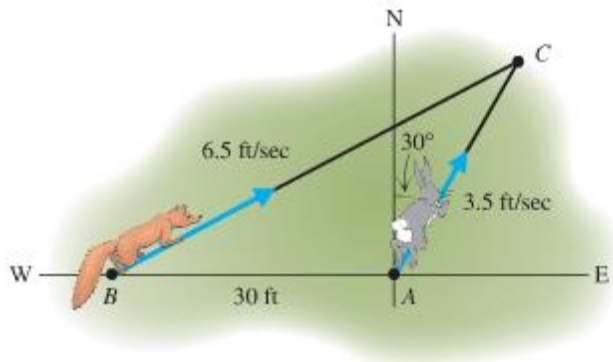
21. A cruise missile is traveling straight across the desert at 548 mph at an altitude of 1 mile. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?



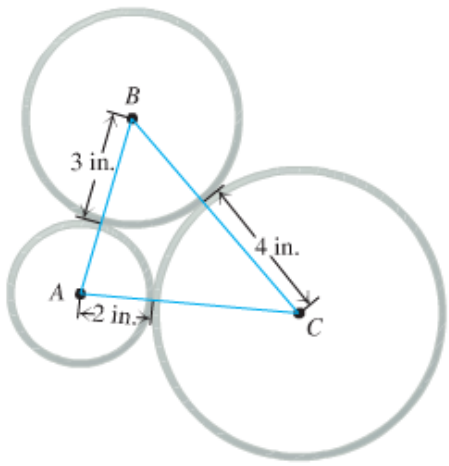
22. When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 ft/sec and Jones passes the ball at 60 ft/sec to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)



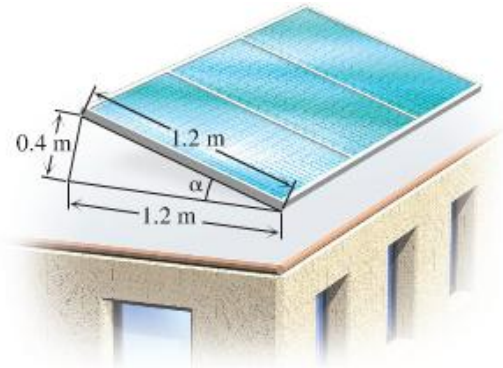
23. A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?



24. An engineer wants to position three pipes at the vertices of a triangle. If the pipes *A*, *B*, and *C* have radii 2 *in*, 3 *in*, and 4 *in*, respectively, then what are the measures of the angles of the triangle *ABC*?

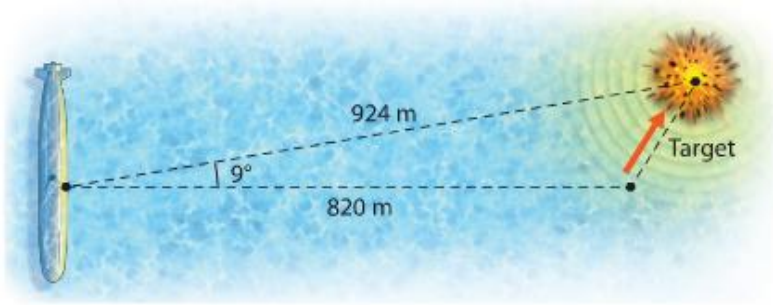


25. A solar panel with a width of 1.2 *m* is positioned on a flat roof. What is the angle of elevation α of the solar panel?

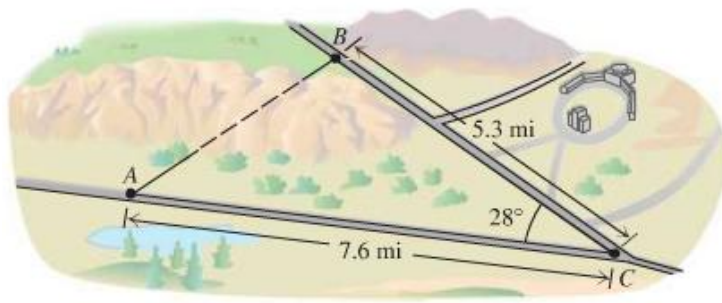


26. Andrea and Steve left the airport at the same time. Andrea flew at 180 *mph* on a course with bearing 80°, and Steve flew at 240 *mph* on a course with bearing 210°. How far apart were they after 3 hr.?

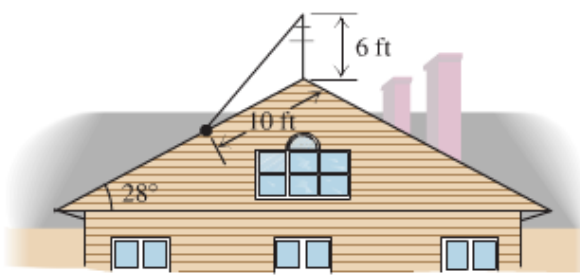
27. A submarine sights a moving target at a distance of 820 *m*. A torpedo is fired 9° ahead of the target, and travels 924 *m* in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



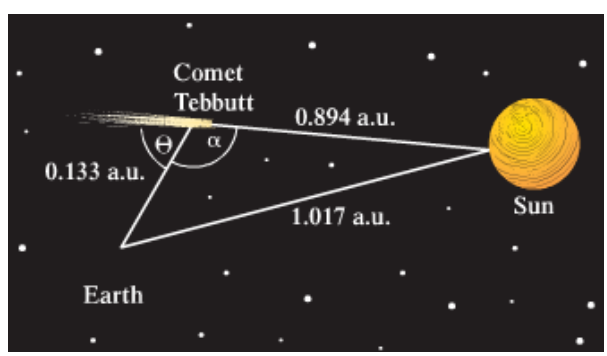
28. A tunnel is planned through a mountain to connect points A and B on two existing roads. If the angle between the roads at point C is 28°, what is the distance from point A to B? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.



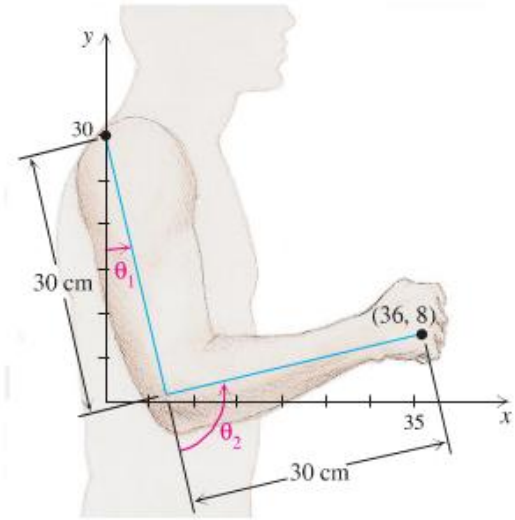
29. A 6-ft antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 ft down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



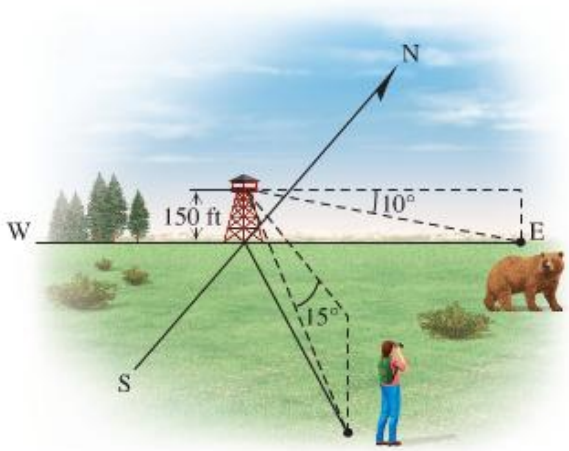
30. On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 a.u. from the earth, 0.894 a.u. from the sun, and the earth was 1.017 a.u. from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (a.u) is the average distance between the earth and the sub.)



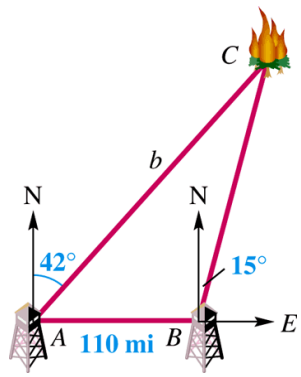
31. A human arm consists of an upper arm of 30 cm and a lower arm of 30 cm. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.



32. A forest ranger is 150 ft above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.



33. Two ranger stations are on an east-west line 110 mi apart. A forest fire is located on a bearing N 42° E from the western station at *A* and a bearing of N 15° E from the eastern station at *B*. How far is the fire from the western station?



Solution Section 2.4 – Law of Sines and Cosines

Exercise

In triangle ABC, $B = 110°$, $C = 40°$ and $b = 18\,in$. Find the length of side c.

Solution

$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - 110° - 40° \\ &= 30° \end{aligned}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 30°} = \frac{18}{\sin 110°}$$

$$a = \frac{18 \sin 30°}{\sin 110°}$$

$$\approx 9.6\ in$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 40°} = \frac{18}{\sin 110°}$$

$$c = \frac{18 \sin 40°}{\sin 110°}$$

$$\approx 12.3\ in$$

Exercise

In triangle ABC, $A = 110.4°$, $C = 21.8°$ and $c = 246\,in$. Find all the missing parts.

Solution

$$\begin{aligned} B &= 180° - A - C \\ &= 180° - 110.4° - 21.8° \\ &= 47.8° \end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 110.4°} = \frac{246}{\sin 21.8°}$$

$$a = \frac{246 \sin 110.4°}{\sin 21.8°}$$

$$\approx 621\ in$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{47.8} = \frac{246}{\sin 21.8°}$$

$$b = \frac{246 \sin 47.8°}{\sin 21.8°}$$

$$\approx 491\ in$$

Exercise

Find the missing parts of triangle *ABC* if $B = 34°$, $C = 82°$, and $a = 5.6\,cm$.

Solution

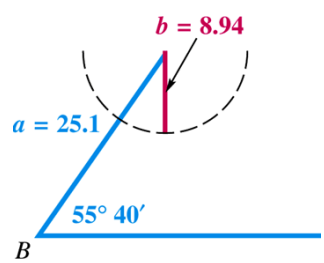
$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - (34° + 82°) \\ &= 180° - 116° \\ &= 64° \end{aligned}$$

$\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ $b = \dfrac{a \sin B}{\sin A}$ $= \dfrac{5.6 \sin 34°}{\sin 64°}$ $= 3.5\,cm$	$\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ $c = \dfrac{a \sin C}{\sin A}$ $= \dfrac{5.6 \sin 82°}{\sin 64°}$ $= 6.2\,cm$

Exercise

Solve triangle *ABC* if $B = 55°40'$, $b = 8.94$ m, and $a = 25.1$ m.

Solution



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{25.1} = \frac{\sin\left(55° + \frac{40°}{60}\right)}{8.94}$$

$$\sin A = \frac{25.1 \sin(55.667°)}{8.94} \approx 2.3184 > 1$$

Since $\sin A > 1$ is impossible, no such triangle exists.

Exercise

Solve triangle *ABC* if $A = 55.3°$, $a = 22.8$ *ft*, and $b = 24.9$ ft

Solution

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

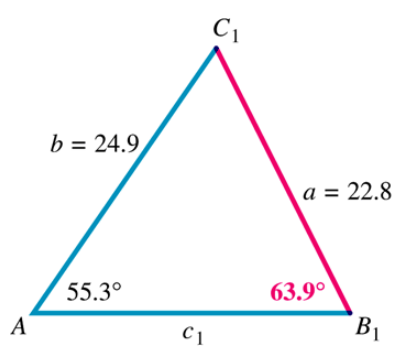
$$\sin B = \frac{24.9 \sin 55.3°}{22.8} \approx 0.89787$$

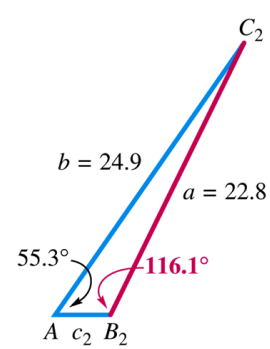
$$B = \sin^{-1}(0.89787)$$

$$B = 63.9° \quad \textit{and} \quad B = 180° - 63.9° = 116.1°$$

$$C = 180° - A - B$$

$C = 180° - 55.3° - 63.9°$ $C = 60.8°$ $c = \frac{a \sin C}{\sin A}$ $= \frac{22.8 \sin 60.8°}{\sin 55.3°}$ $= 24.2\ ft$	$C = 180° - 55.3° - 116.1°$ $C = 8.6°$ $c = \frac{a \sin C}{\sin A}$ $= \frac{22.8 \sin 8.6°}{\sin 55.3°}$ $= 4.15\ ft$





Exercise

Solve triangle *ABC* given *A* = 43.5°, *a* = 10.7 in., and *c* = 7.2 in

Solution

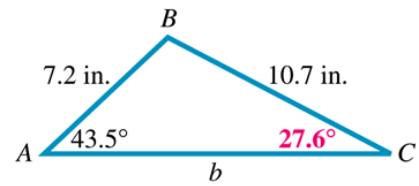
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \frac{7.2 \sin 43.5°}{10.7} \approx 0.4632$$

$$C = \sin^{-1}(0.4632)$$

$$C = 27.6° \quad and \quad C = 180° - 27.6° = 152.4°$$

$$B = 180° - A - C$$



$B = 180° - 43.5° - 27.6°$ $B = 108.9°$ $b = \frac{a \sin B}{\sin A}$ $= \frac{10.7 \sin 108.9°}{\sin 43.5°}$ $= 14.7\ in$	$B = 180° - 43.5° - 152.4°$ $B = -15.9°$ Is not possible

Exercise

If $A = 26°, s = 22, and\ r = 19$ find x

Solution

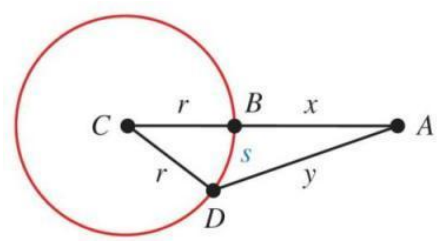
$$C = \theta = \frac{s}{r}\ rad = \frac{22}{19}\frac{180°}{\pi} \approx 66°$$

$$\boxed{D = 180 - A - C = 180° - 26° - 66° = 88°}$$

$$\frac{r + x}{\sin D} = \frac{r}{\sin A}$$

$$19 + x = \frac{19 \sin 88°}{\sin 26°}$$

$$\boxed{x = \frac{19 \sin 88°}{\sin 26°} - 19 \approx 24}$$



Exercise

If *a* = 13 *yd*, *b* = 14 *yd*, and c = 15 *yd*, find the largest angle.

Solution

$$C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab} = \cos^{-1}\left(\frac{13^2 + 14^2 - 15^2}{2(13)(14)}\right) \boxed{\approx 67°}$$

Exercise

Solve triangle *ABC* if b = 63.4 km, and c = 75.2 km, A = 124° 40′

Solution

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= (63.4)^2 + (75.2)^2 - 2(63.4)(75.2)\cos\left(124° + \frac{40°}{60}\right)$$
$$\approx 15098$$

$$a \approx 122.9\ km$$

$$\sin B = \frac{b\sin A}{a}$$
$$= \frac{63.4\sin 124.67°}{122.9}$$

$$B = \sin^{-1}\left(\frac{63.4\sin 124.67°}{122.9}\right) \approx 25.1°$$

$$C = 180° - A - B$$
$$= 180° - 124.67° - 25.1°$$
$$\approx 30.23°$$

Exercise

Solve triangle *ABC* if a = 832 ft, b = 623 ft, and c = 345 ft

Solution

$$C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$
$$= \cos^{-1}\left(\frac{832^2 + 623^2 - 345^2}{2(832)(623)}\right)$$
$$\approx 22°$$

$$\sin B = \frac{b\sin C}{c}$$
$$= \frac{623\sin 22°}{345}$$

$$B = \sin^{-1}\left(\frac{623\sin 22°}{345}\right)$$
$$\approx 43°$$

$$A = 180° - 22° - 43° = 115°$$

Exercise

Solve triangle ABC if $A = 42.3°,\ b = 12.9m,\ and\ c = 15.4m$

Solution

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 12.9^2 + 15.4^2 - 2(12.9)(15.4)\cos 42.3°$$
$$\approx 109.7$$

$$\boxed{a = 10.47\ m}$$

$$\sin B = \frac{b\sin A}{a} = \frac{12.9\sin 42.3°}{10.47}$$

$$B = \sin^{-1}\left(\frac{12.9\sin 42.3°}{10.47}\right) \approx 56.0°$$

$$C = 180° - 42.3° - 56° = 81.7°$$

Exercise

Solve triangle ABC if $a = 9.47\ ft,\ b = 15.9\ ft,\ and\ c = 21.1\ ft$

Solution

$$C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$
$$= \cos^{-1}\left(\frac{9.47^2 + 15.9^2 - 21.1^2}{2(9.47)(15.9)}\right)$$
$$\approx 109.9°$$

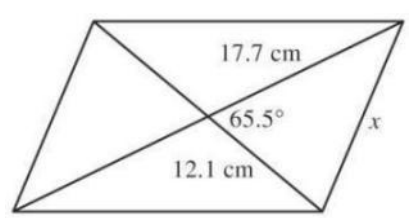
$$\sin B = \frac{b\sin C}{c} = \frac{15.9\sin 109.9°}{21.1}$$

$$B = \sin^{-1}\left(\frac{15.9\sin 109.9°}{21.1}\right) \approx 25.0°$$

$$A = 180° - 25° - 109.9° = 45.1°$$

Exercise

The diagonals of a parallelogram are 24.2 cm and 35.4 cm and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram



Solution

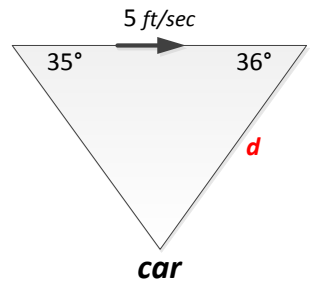
$$x^2 = 17.7^2 + 12.1^2 - 2(17.7)(12.1)\cos 65.5° = 282.07$$

$$\boxed{x = 16.8\ cm}$$

Exercise

A man flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

Solution



$$\angle car = 180° - 35° - 36° = 109°$$

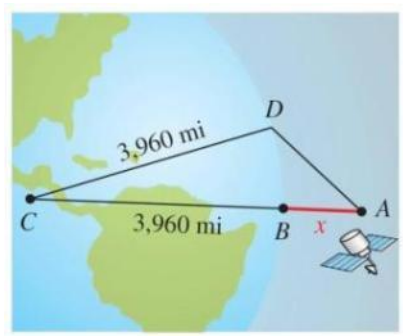
$$\frac{d}{\sin 35°} = \frac{450}{\sin 109°}$$

$$d = \frac{450 \sin 35°}{\sin 109°} \approx 273\ ft$$

Exercise

A satellite is circling above the earth. When the satellite is directly above point B, angle A is 75.4°. If the distance between points B and D on the circumference of the earth is 910 miles and the radius of the earth is 3,960 miles, how far above the earth is the satellite?

Solution



$$\theta = \frac{s}{r}$$

$C = arc\ length\ BD$ divides by $radius$

$$C = \frac{910}{3960} rad = \frac{910}{3960} \frac{180°}{\pi} = 13.2°$$

$$D = 180° - (A + C)$$

$$= 180° - (75.4° + 13.2°)$$

$$= 91.4°$$

$$\frac{CA}{\sin D} = \frac{3960}{\sin A}$$

$$\frac{x + 3960}{\sin 91.4°} = \frac{3960}{\sin 75.4°}$$

$$x + 3960 = \frac{3960 \sin 91.4°}{\sin 75.4°}$$

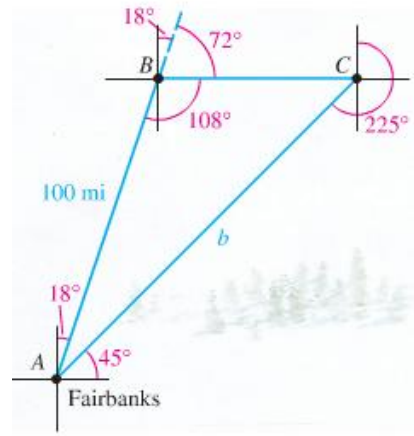
$$x = \frac{3960 \sin 91.4°}{\sin 75.4°} - 3960$$

$$x = 130\ mi$$

Exercise

A pilot left Fairbanks in a light plane and flew 100 miles toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

Solution



From the triangle ABC:

$\angle ABC = 90° + 18° = 108°$

$\angle ACB = 360° - 225° - 90° = 45°$

$\angle BAC = 90° - 18° - 45° = 27°$

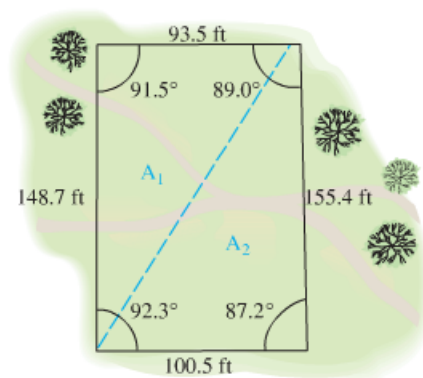
The length AC is the maximum distance from Fairbanks:

$$\frac{b}{\sin 108°} = \frac{100}{\sin 45°}$$

$$b = \frac{100 \sin 108°}{\sin 45°} \approx 134.5 \text{ miles}$$

Exercise

The dimensions of a land are given in the figure. Find the area of the property in square feet.



Solution

$$A_1 = \tfrac{1}{2}(148.7)(93.5)\sin 91.5° \approx 6949.3\ ft^2$$

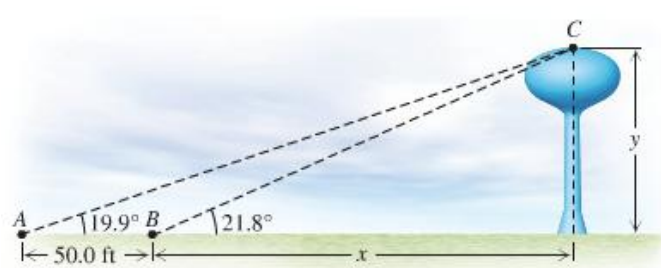
$$A_2 = \tfrac{1}{2}(100.5)(155.4)\sin 87.2° \approx 7799.5\ ft^2$$

The total area = $A_1 + A_2 = 6949.3 + 7799.5 = 14,748.8\ ft^2$

Exercise

The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?

Solution



$\angle ABC = 180° - 21.8° = 158.2°$

$\angle ACB = 180° - 19.9° - 158.2° = 1.9°$

Apply the law of sines in triangle ABC:

$$\frac{BC}{\sin 19.9°} = \frac{50}{\sin 1.9°}$$

$$\Rightarrow BC = \frac{50 \sin 19.9°}{\sin 1.9°} \approx 513.3$$

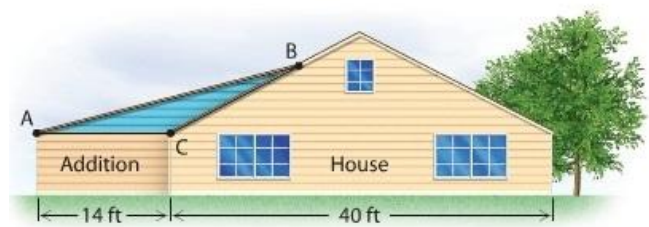
Using the right triangle: $\sin 21.8° = \dfrac{y}{BC}$

$\boxed{y = 513.3\sin 21.8° \approx 191\ ft}$

Exercise

A 40-ft wide house has a roof with a 6-12 pitch (the roof rises 6 ft for a run of 12 ft). The owner plans a 14-ft wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$.

Solution

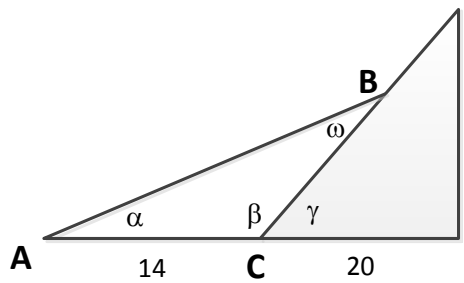


$$\tan\gamma = \frac{6}{12} \Rightarrow \gamma = \tan^{-1}\left(\frac{6}{12}\right) = 26.565°$$

$$\tan\alpha = \frac{3}{12} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{12}\right) = 14.036°$$

$$\beta = 180° - \gamma = 180° - 26.565° = 153.435°$$

$$\omega = 180° - 14.036° - 153.435° = 12.529°$$



$$\frac{AB}{\sin 153.435°} = \frac{14}{\sin 12.529°}$$

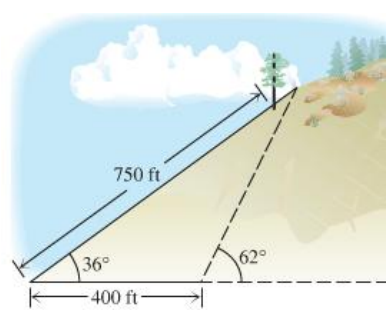
$$\Rightarrow \boxed{AB = \frac{14\sin 153.435°}{\sin 12.529°} \approx 28.9\ ft}$$

$$\frac{BC}{\sin 14.036°} = \frac{14}{\sin 12.529°}$$

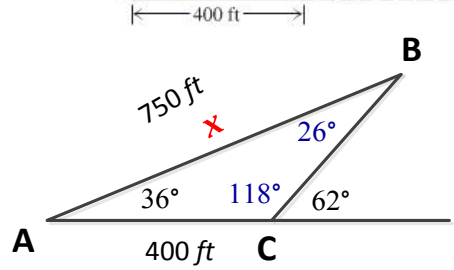
$$\Rightarrow \boxed{BC = \frac{14\sin 14.036°}{\sin 12.529°} \approx 15.7\ ft}$$

Exercise

A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution



$$\angle ACB = 180° - 62° = 118°$$

$$\angle ABC = 180° - 118° - 36° = 26°$$

Using the law of sines:

$$\frac{x}{\sin 118°} = \frac{400}{\sin 26°}$$

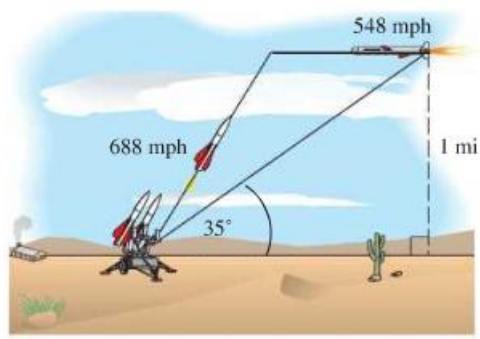
$$x = \frac{400\sin 118°}{\sin 26°} \approx 805.7\ ft$$

Yes, the tree will have to be excavated.

Exercise

A cruise missile is traveling straight across the desert at 548 mph at an altitude of 1 mile. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?

Solution



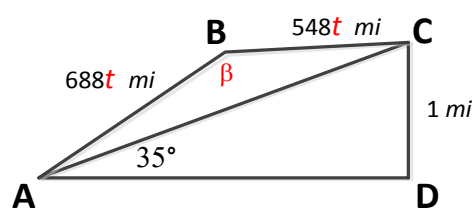
$$\angle ACB = 35°$$
$$\angle BAC = 180° - 35° - \beta$$

After ***t*** seconds;

The cruise missile distance: $548\frac{t}{3600}\ miles$

The Projectile distance: $688\frac{t}{3600}\ miles$

Using the law of sines:



$$\frac{\frac{548t}{3600}}{\sin(145° - \beta)} = \frac{\frac{688t}{3600}}{\sin 35°}$$

$$\frac{548t}{3600}\sin 35° = \frac{688t}{3600}\sin(145° - \beta)$$

$$548\sin 35° = 688\sin(145° - \beta)$$

$$\sin(145° - \beta) = \frac{548}{688}\sin 35°$$

$$145° - \beta = \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$

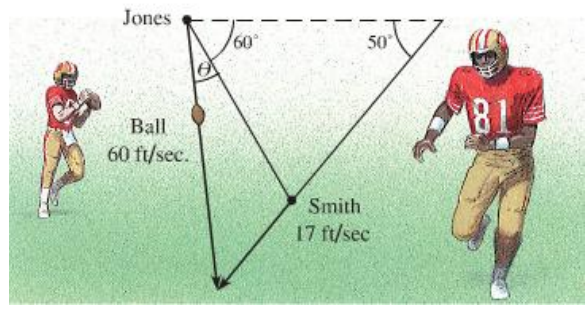
$$\beta = 145° - \sin^{-1}\left(\frac{548}{688}\sin 35°\right) \approx 117.8°$$

$$\Rightarrow \angle BAC = 180° - 35° - 117.8° = 27.2°$$

The angle of elevation of the projectile must be $(= 35° + 27.2°)$ 62.2°

Exercise

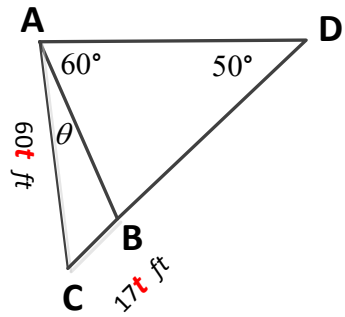
When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 ft/sec and Jones passes the ball at 60 ft/sec to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)



Solution

$$\angle ABD = 180° - 60° - 50° = 70°$$

$$\angle ABC = 180° - 70° = 110°$$



Using the law of sines:

$$\frac{17t}{\sin\theta} = \frac{60t}{\sin 110°}$$

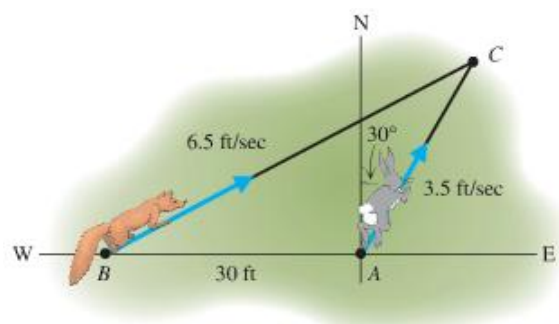
$$\frac{17}{\sin\theta} = \frac{60}{\sin 110°}$$

$$\sin\theta = \frac{17\sin 110°}{60}$$

$$\theta = \sin^{-1}\left(\frac{17\sin 110°}{60}\right) = 15.4°$$

Exercise

A rabbit starts running from point A in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?



Solution

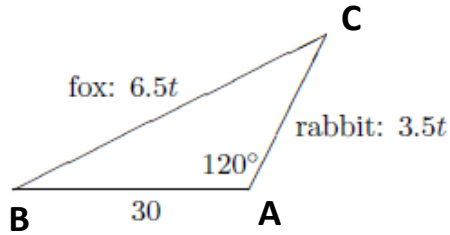
$$\angle BAC = 90° + 30° = 120°$$

$$\frac{6.5t}{\sin 120°} = \frac{3.5t}{\sin B}$$

$$\frac{6.5}{\sin 120°} = \frac{3.5}{\sin B}$$

$$\sin B = \frac{3.5\sin 120°}{6.5}$$

$$B = \sin^{-1}\left(\frac{3.5\sin 120°}{6.5}\right) \approx 28°$$



$C = 180° - 120° - 28° = 32°$

$$\frac{3.5t}{\sin 28} = \frac{30}{\sin 32°}$$

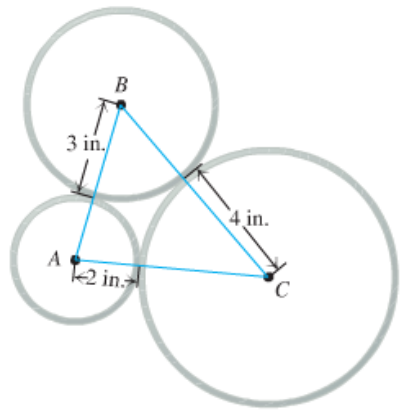
$$\boxed{t = \frac{30 \sin 28}{3.5 \sin 32°} \approx 7.6 \text{ sec}}$$

It will take 7.6 sec. to catch the rabbit.

Exercise

An engineer wants to position three pipes at the vertices of a triangle. If the pipes A, B, and C have radii 2 in, 3 in, and 4 in, respectively, then what are the measures of the angles of the triangle ABC?

Solution



$AC = 6 \quad AB = 5 \quad BC = 7$

$$\boxed{A = \cos^{-1}\left(\frac{5^2 + 6^2 - 7^2}{2(5)(6)}\right) \approx 78.5°}$$

$$\frac{6}{\sin B} = \frac{7}{\sin 78.5°}$$

$$\sin B = \frac{6 \sin 78.5°}{7}$$

$$\boxed{B = \sin^{-1}\left(\frac{6 \sin 78.5°}{7}\right) \approx 57.1°}$$

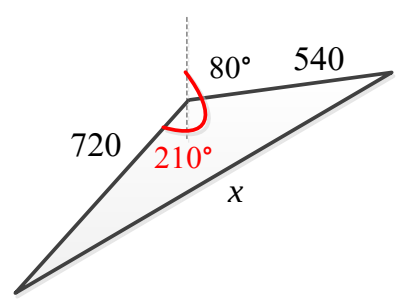
$$\boxed{C = 180° - 78.5° - 57.1° = 44.4°}$$

Exercise

Andrea and Steve left the airport at the same time. Andrea flew at 180 mph on a course with bearing 80°, and Steve flew at 240 mph on a course with bearing 210°. How far apart were they after 3 hr.?

Solution

After 3 hrs., Steve flew: 3(240) = 720 mph
Andrea flew: 3(180) = 540 mph

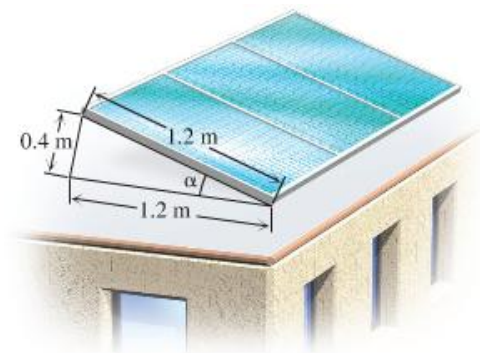


$$x^2 = 720^2 + 540^2 - 2(720)(540)\cos 210°$$

$$\boxed{x = \sqrt{720^2 + 540^2 - 2(720)(540)\cos 210°} \approx 1144.5 \text{ miles}}$$

Exercise

A solar panel with a width of 1.2 m is positioned on a flat roof. What is the angle of elevation α of the solar panel?



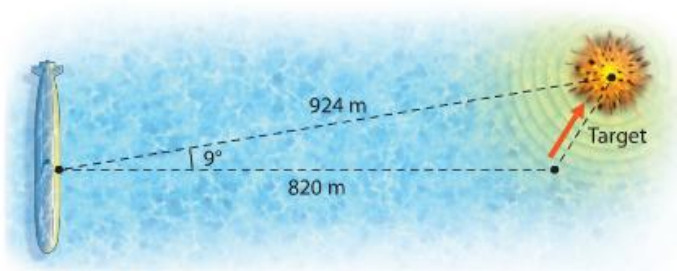
Solution

$$\alpha = \cos^{-1}\frac{1.2^2 + 1.2^2 - 0.4^2}{2(1.2)(1.2)} \approx 19.2°$$

or $\alpha = \frac{s}{r} = \frac{0.4}{1.2}$

Exercise

A submarine sights a moving target at a distance of 820 m. A torpedo is fired 9° ahead of the target, and travels 924 m in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



Solution

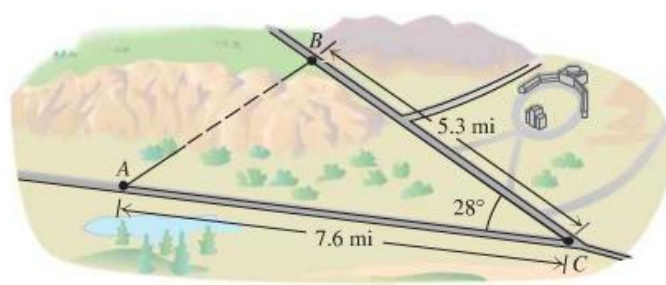
$$x = \sqrt{820^2 + 924^2 - 2(820)(924)\cos 9°} \approx 171.7 \text{ m}$$

Exercise

A tunnel is planned through a mountain to connect points *A* and *B* on two existing roads. If the angle between the roads at point *C* is 28°, what is the distance from point *A* to *B*? Find ∠*CBA* and ∠*CAB* to the nearest tenth of a degree.

Solution

By the cosine law,



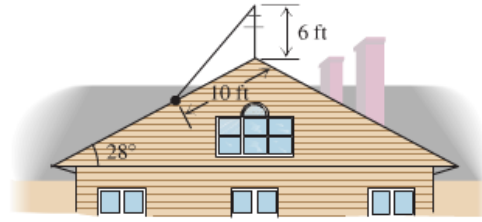
$$AB = \sqrt{5.3^2 + 7.6^2 - 2(5.3)(7.6)\cos 28°} \approx 3.8$$

$$\angle CBA = \cos^{-1}\frac{3.8^2 + 5.3^2 - 7.6^2}{2(3.8)(5.3)} \approx 112°$$

$$\angle BAC = 180° - 112° - 28° \approx 40°$$

Exercise

A 6-ft antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 ft down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



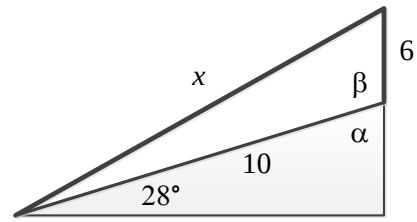
Solution

$$\alpha = 90° - 28° = 62°$$

$$\beta = 180° - 62° = 118°$$

By the cosine law,

$$x = \sqrt{6^2 + 100^2 - 2(6)(10)\cos 118°} \approx 13.9\ ft$$



Exercise

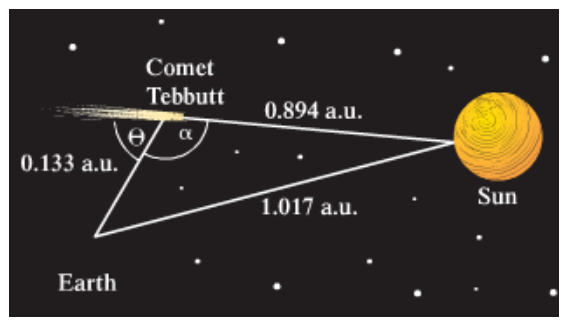
On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 a.u. from the earth, 0.894 a.u. from the sun, and the earth was 1.017 a.u. from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (a.u) is the average distance between the earth and the sub.)

Solution

By the cosine law:

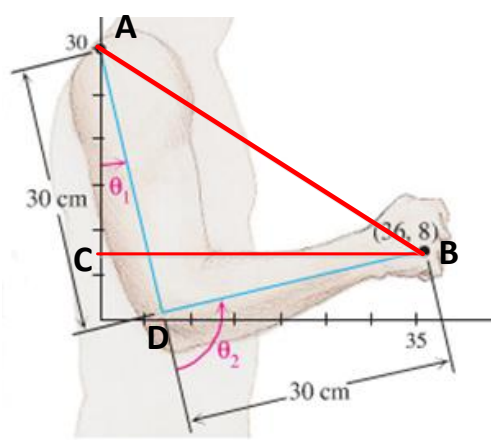
$$\alpha = \cos^{-1}\frac{0.133^2 + 0.894^2 - 1.017^2}{2(0.133)(0.891)} \approx 156°$$

$$\theta = 180° - \alpha = 180° - 156° = 24°$$



Exercise

A human arm consists of an upper arm of 30 cm and a lower arm of 30 cm. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.



Solution

$$AC = 30 - 8 = 22 \qquad BC = 36$$

$$AB = \sqrt{AC^2 + CB^2}$$
$$= \sqrt{22^2 + 36^2}$$
$$\approx 42.19$$

By the cosine law:

$$\angle ADB = \cos^{-1}\frac{AD^2 + DB^2 - AB^2}{2(AD)(DB)}$$

$$\angle ADB = \cos^{-1}\frac{30^2 + 30^2 - 42.19^2}{2(30)(30)} \quad \angle ADB \approx 89.4°$$

$$\theta_2 = 180° - 89.4° = 90.6°$$

$$\tan(\angle CAB) = \frac{BC}{AC} = \frac{36}{22}$$

$$\Rightarrow \angle CAB = \tan^{-1}\frac{36}{22} \approx 58.57°$$

$$\frac{\sin DAB}{30} = \frac{\sin 89.4°}{42.19} \Rightarrow \sin DAB = \frac{30\sin 89.4°}{42.19}$$

$$\angle DAB = \sin^{-1}\frac{30\sin 89.4°}{42.19} \approx 45.32°$$

$$\theta_1 = \angle CAB - \angle DAB$$

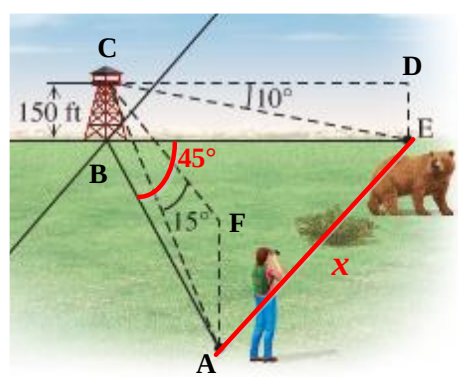
$$= 58.57° - 45.32°$$

$$= 13.25°$$

Exercise

A forest ranger is 150 ft above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.

Solution



$$\angle BEC = \angle ECD = 10°$$

From triangle EBC: $\tan 10° = \frac{150}{BE}$

$$\Rightarrow BE = \frac{150}{\tan 10°} \approx 850.692$$

$$\angle BAC = \angle ACF = 15°$$

From triangle ABC:

$$\tan 15° = \frac{150}{AB}$$

$$\Rightarrow AB = \frac{150}{\tan 15°} \approx 559.808$$

$$x = \sqrt{AB^2 + BE^2 - 2(AB)(BE)\cos 45°}$$

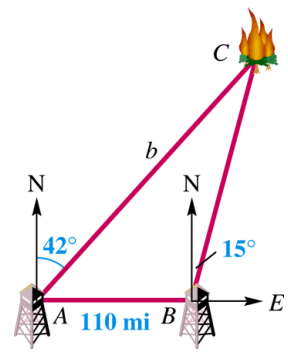
$$= \sqrt{559.808^2 + 850.692^2 - 2(559.808)(850.692)\cos 45°}$$

$$\approx 603\,ft$$

Exercise

Two ranger stations are on an east-west line 110 mi apart. A forest fire is located on a bearing N 42° E from the western station at A and a bearing of N 15° E from the eastern station at B. How far is the fire from the western station?

<u>*Solution*</u>



$$\angle BAC = 90° - 42° = 48°$$

$$\angle ABC = 90° + 15° = 105°$$

$$\angle C = 180° - 105° - 48° = 27°$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 105°} = \frac{110}{\sin 27°}$$

$$b = \frac{110 \sin 105°}{\sin 27°}$$

$$\boxed{b \approx 234\ mi}$$

The fire is about 234 miles from the western station.