Section 4.2 – Arithmetic Sequences

Definition of Arithmetic Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence if there is a real number ***d*** such that for every positive integer ***k***,

$$a_{k+1} = a_k + d$$

The number $d = a_{k+1} - a_k$ is called the ***common difference*** of the sequence.

Example

Show that the sequence: 1, 4, 7, 10, … , $3n - 2$, … is arithmetic, and find the common difference.

<u>*Solution*</u>

If $a_n = 3n - 2$, then for every positive integer *k*,

$$\begin{aligned} a_{k+1} - a_k &= \left[3(k+1) - 2\right] - (3k - 2) \\ &= 3k + 3 - 2 - 3k + 2 \\ &= 3 \end{aligned}$$

Hence, the given sequence is arithmetic with common difference 3.

The **n*****th Term of an Arithmetic Sequence***: $\boxed{a_n = a_1 + (n-1)d}$

Example

The first three terms of an arithmetic sequence are 20, 16.5, and 13. Find the fifteenth term.

<u>*Solution*</u>

The common difference is: $a_2 - a_1 = 16.5 - 20 = -3.5$

Substituting $a_1 = 20, \quad d = -3.5, \quad n = 15$ in the formula:

$$\begin{aligned} a_{15} &= a_1 + (15 - 1)d \\ &= 20 + (15 - 1)(-3.5) \\ &= -29 \end{aligned}$$

Example

The fourth term of an arithmetic sequence is 5, and the ninth term is 20. Find the sixth term.

Solution

Given: $a_4 = 5 \quad a_9 = 20$

$$\begin{cases} a_4 = a_1 + (4-1)d \\ a_9 = a_1 + (9-1)d \end{cases} \Rightarrow \begin{cases} 5 = a_1 + 3d \\ 20 = a_1 + 8d \end{cases}$$

$$\begin{array}{r} 20 = a_1 + 8d \\ -\ \ 5 = a_1 + 3d \\ \hline 15 = 5d \end{array} \quad \Rightarrow \boxed{d = 3} \quad \textit{and} \quad \boxed{a_1 = 5 - 3d = 5 - 9 = -4}$$

$$\begin{aligned} a_6 &= a_1 + (6-1)d \\ &= -4 + (5)3 \\ &= 11 \end{aligned}$$

Theorem

Formulas for S_n

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference $\boldsymbol{d}$, then the nth partial sum S_n (that is, the sum of the first $\boldsymbol{n}$ terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad \textit{or} \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Proof

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ \ldots + \ a_n \\ &= a_1 + \left(a_1 + d\right) + \left(a_1 + 2d\right) + \ \ldots + \ \left(a_1 + (n-1)d\right) \\ &= a_1 + a_1 + \ \ldots + a_1 + \left[d + 2d + \ \ldots + (n-1)d\right] \\ &= na_1 + d\left[1 + 2 + \ \ldots + (n-1)\right] \\ &= na_1 + d\left[\frac{(n-1)n}{2}\right] \qquad \textit{Using the formula of sum: } S_n = \frac{n(n+1)}{2} \\ &= \frac{2na_1 + (n-1)nd}{2} \\ &= \frac{n}{2}\left[2a_1 + (n-1)d\right] \end{aligned}$$

Example

Find the sum of all even integers from 2 through 100.

Solution

The arithmetic sequence: 2, 4, 6, …, 2n, …

Substituting $n = 50,\quad a_1 = 2,\quad and\quad a_{50} = 100$ in the formula:

$$S_n = \frac{50}{2}(2+100) = 2550$$

Arithmetic Mean

The ***arithmetic mean*** of two numbers *a* and *b* is defined as $\frac{a+b}{2}$ (this is the average)

Example

Express in terms of summation notation: $\frac{1}{4}+\frac{2}{9}+\frac{3}{14}+\frac{4}{19}+\frac{5}{24}+\frac{6}{29}$

Solution

Numerators : $1, 2, 3, 4, 5$ *common difference* 1

Denominators : $4, 9, 14, 19, 24, 29$ *common difference* 5

Using the formula for *n*th term:

$$a_n = a_1 + (n-1)d = 1 + (n-1)1 = n$$

$$a_n = a_1 + (n-1)d = 4 + (n-1)5 = 4 + 5n - 5 = 5n - 1$$

Hence the *n*th term is:

$$\frac{1}{4}+\frac{2}{9}+\frac{3}{14}+\frac{4}{19}+\frac{5}{24}+\frac{6}{29} = \sum_{n=1}^{6} \frac{n}{5n-1}$$

Exercises Section 4.2 – Arithmetic Sequences

1. Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

2. Find the *n*th term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \ldots$

3. Find the *n*th term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

4. Find the *n*th term, and the tenth term of the arithmetic sequence: $-6,\ -4.5,\ -3,\ -1.5,\ \ldots$

5. Find the *n*th term, and the tenth term of the arithmetic sequence: $\ln 3,\ \ln 9,\ \ln 27,\ \ln 81,\ \ldots$

6. Find the common difference for the arithmetic sequence with the specified terms:

$a_4 = 14, \quad a_{11} = 35$

7. Find the specified term of the arithmetic sequence that has two given terms:

$a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

8. Find the specified term of the arithmetic sequence that has two given terms:

$a_1; \quad a_8 = 47, \quad a_9 = 53$

9. Find the specified term of the arithmetic sequence that has two given terms:

$a_{10}; \quad a_2 = 1, \quad a_{18} = 49$

10. Find the sum S_n of the arithmetic sequence that satisfies the conditions:

$a_1 = 40, \quad d = -3, \quad n = 30$

11. Find the sum S_n of the arithmetic sequence that satisfies the conditions:

$a_7 = \frac{7}{3}, \quad d = -\frac{2}{3}, \quad n = 15$

12. Find the sum: $\sum_{k=1}^{20} (3k-5)$

13. Find the sum: $\sum_{k=1}^{18} \left(\frac{1}{2}k + 7\right)$

14. Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

15. Express the sum in terms of summation notation: $4+11+18+\ldots+466$. (Answers are not unique)

16. Express the sum in terms of summation notation: $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$. (Answers are not unique)

17. Express the sum in terms of summation notation and find the sum $2+11+20+\ldots+16{,}058$.

18. Find the number of terms in the arithmetic sequence with the given conditions:

$$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$$

19. Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution   *Section* 4.2 – Arithmetic Sequences

Exercise

Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

Solution

We to show that $a_{k+1}-a_k$ equals to a constant.

$$\begin{aligned} a_{k+1}-a_k &= 4(k+1)-10-(4k-10) \\ &= 4k+4-10-4k+10 \\ &= 4 \end{aligned}$$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \ldots$

Solution

$d = 6-2 = 4$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 2+(n-1)4 \\ &= 2+4n-4 \\ &= 4n-2 \end{aligned}$$

$a_{10} = 4(10)-2 = 38$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

Solution

$d = 2.7-3 = -0.3$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 3+(n-1)(-0.3) \\ &= 3-0.3n+0.3 \\ &= 3.3-0.3n \end{aligned}$$

$a_{10} = 3.3-0.3(10) = 0.3$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $-6, \ -4.5, \ -3, \ -1.5, \ \ldots$

Solution

$$d = -4.5-(-6) = 1.5$$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -6+(n-1)(1.5) \\ &= -6+1.5n-1.5 \\ &= 1.5n-7.5 \end{aligned}$$

$$a_{10} = 1.5(10)-7.5 = 7.5$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $\ln 3, \ \ln 9, \ \ln 27, \ \ln 81, \ \ldots$

Solution

$$\ln 3, \ \ln 3^2, \ \ln 3^3, \ \ln 3^4, \ \ldots$$

$$\ln 3, \ 2\ln 3, \ 3\ln 3, \ 4\ln 3, \ \ldots$$

$$d = 2\ln 3 - \ln 3 = \ln 3$$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= \ln 3 + (n-1)\ln 3 \\ &= \ln 3 + n\ln 3 - \ln 3 \\ &= n\ln 3 \end{aligned}$$

$$a_{10} = 10\ln 3 = \ln 3^{10}$$

Exercise

Find the common difference for the arithmetic sequence with the specified terms: $a_4 = 14, \quad a_{11} = 35$

Solution

$$a_n = a_1 + (n-1)d$$

$$a_{11} = a_1 + 10d \ \rightarrow \ 35 = a_1 + 10d$$

$$a_4 = a_1 + 3d \ \rightarrow \ 14 = a_1 + 3d$$

$$21 = 7d \ \rightarrow \ \boxed{d = 3}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{12}; $a_1 = 9.1$, $a_2 = 7.5$

Solution

$$d = a_2 - a_1 = 7.5 - 9.1 = -1.6$$

$$a_n = a_1 + (n-1)d$$

$$\underline{|a_{12}} = 9.1 + (11)(-1.6) \; \underline{= -8.5|}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_1; $a_8 = 47$, $a_9 = 53$

Solution

$$d = a_9 - a_8 = 53 - 47 = 6$$

$$a_n = a_1 + (n-1)d$$

$$a_8 = a_1 + (7)(6)$$

$$\underline{|a_1} = 47 - 42 \; \underline{= 5|}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_2 = 1$, $a_{18} = 49$

Solution

$$a_2 = a_1 + d \Rightarrow a_1 = a_2 - d$$

$$a_{18} = a_1 + (17)d = a_2 - d + 17d = a_2 + 16d$$

$$49 = 1 + 16d \Rightarrow 16d = 48 \Rightarrow \underline{|d} = \frac{48}{16} \; \underline{= 3|}$$

$$a_1 = a_2 - d = 1 - 3 = -2$$

$$\underline{|a_{10}} = -2 + 9(3) \; \underline{= 25|}$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_1 = 40, \quad d = -3, \quad n = 30$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$S_n = \frac{30}{2}\left[2(40) + (30-1)(-3)\right]$$

$$= -105$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_7 = \frac{7}{3}, \quad d = -\frac{2}{3}, \quad n = 15$

Solution

$$a_7 = a_1 + (6)\left(-\frac{2}{3}\right) = \frac{7}{3}$$

$$a_1 = \frac{7}{3} + 4 = \frac{19}{3}$$

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$S_n = \frac{15}{2}\left[2\left(\frac{19}{3}\right) + (15-1)\left(-\frac{2}{3}\right)\right]$$

$$= 25$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{20}(3k-5)$

Solution

$$a_1 = 3(1) - 5 = -2 \quad \text{and} \quad a_{20} = 3(20) - 5 = 55$$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\sum_{k=1}^{20}(3k-5) = \frac{20}{2}(-2+55) = 530$$

Exercise

Find the sum: $\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

Solution

$$a_1=\frac{1}{2}(1)+7=\frac{15}{2} \quad \textit{and} \quad a_{18}=\frac{1}{2}(18)+7=16$$

$$S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)=\frac{18}{2}\left(\frac{15}{2}+16\right)=\frac{423}{2}$$

Exercise

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

Number of terms: $n = 5$

Difference in terms: $d = 11 - 4 = 7$

$$a_n=a_1+(n-1)d$$

$$a_n=4+(n-1)7=4+7n-7=7n-3$$

$$\sum_{n=1}^{5}(7n-3)$$

Exercise

Express the sum in terms of summation notation: $4+11+18+\ldots+466$. (Answers are not unique)

Solution

Difference in terms: $d = 11 - 4 = 7$

Number of terms: $n=\frac{466-4}{7}+1=67$

$$a_n=a_1+(n-1)d$$

$$a_n=4+(n-1)7=4+7n-7=7n-3$$

$$\sum_{n=1}^{67}(7n-3)$$

Exercise

Express the sum in terms of summation notation: $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$. (Answers are not unique)

Solution

Number of terms: $n = 4$

Numerators : $5, 10, 15, 20$ *common difference* 5

Denominators : $13, 11, 9, 7$ *common difference* -2

Using the formula for *n*th term $a_n = a_1 + (n-1)d$:

Numerator: $a_n = 5 + (n-1)5 = 5 + 5n - 5 = 5n$

Denominator: $a_n = 13 + (n-1)(-2) = 13 - 2n + 2 = 15 - 2n$

Hence the *n*th term is:

$$\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}=\sum_{n=1}^{4}\frac{5n}{15-2n}$$

Exercise

Express the sum in terms of summation notation and find the sum $2+11+20+\ldots+16{,}058$.

Solution

Difference in terms: $d = 11 - 2 = 9$

Number of terms: $n=\frac{16058-2}{9}+1=1785$

$a_n = a_1 + (n-1)d$

$a_n = 2 + (n-1)(9) = 2 + 9n - 9 = 9n - 7$

Hence the *n*th term is: $\sum_{n=1}^{1785}(9n-7)$

$S_{1785}=\frac{1789}{2}(2+16058) = 14{,}333{,}550$.

Exercise

Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$21 = \frac{n}{2}\left[2(-2) + (n-1)\frac{1}{4}\right]$$

$$21 = -2n + \frac{1}{8}n(n-1)$$

$$(8)21 = -2n(8) + \frac{1}{8}n(n-1)(8)$$

$$168 = -16n + \left(n^2 - n\right)$$

$$0 = n^2 - 17n - 168$$

$$\boxed{n = 24} \qquad n = -7$$

Exercise

Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution

Number of terms: $n = \frac{390 - 36}{6} + 1 = 60$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{60}{2}(36 + 390)$$

$$= 12780$$