Chapter 2

Reduction to Separable Equations*

Purpose: *To learn how to convert several types of differential equations into separable equations and solve them. The material in this chapter is not covered on the AP Calculus exam.*

Separation of variables is one of the basic techniques for solving differential equations. In this chapter we are going to learn several types of differential equations that are not directly separable, but can be reduced to separable equations by simple mathematical manipulations. Although the content of this chapter is not a requirement of the AP Calculus exam, you are encouraged to read this chapter to enhance your skills of solving differential equations.

Homogeneous Equations

Homogeneous differential equations in the form of $\frac{dy}{dx} = f(x,y)$ have the property that $f(tx,ty) = f(x,y)$. For example, in the equation $\frac{dy}{dx} = \frac{x+y}{2y}$, $f(x,y) = \frac{x+y}{2y}$. Since $f(tx,ty) = \frac{tx+ty}{2ty} = f(x,y)$, the equation is homogeneous. A homogeneous equation can be transformed into a separable equation by making the substitution: $y = vx$, where v is a function of x. Thus,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

► TIP

A simple way to check whether an equation is homogeneous is to make sure that all the terms in $f(x,y)$ have the same degree.

Example 2.1

Solve the differential equation $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$.

Solution:

Since $f(tx,ty) = \frac{t^2x^2+t^2y^2}{2txty} = f(x,y)$, the equation is homogeneous (notice that all the terms in $\frac{x^2+y^2}{2xy}$ have degree 2). Make the substitution $y = vx$ then $v = \frac{y}{x}$, and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So the original equation becomes:

$$v + x\frac{dv}{dx} = \frac{x^2+x^2v^2}{2x^2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$$

The above equation can be solved by separating the variables v and x and integrating both sides:

$$\int \frac{2v\,dv}{1-v^2} = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + C$$

$$\left|1-v^2\right| = \frac{C_1}{|x|} \quad (C_1 = e^{-C})$$

To get rid of the absolute value signs on both sides of the equation, we need to assume there are two cases: $1 - v^2 = \frac{C_1}{x}$ and $1 - v^2 = -\frac{C_1}{x}$, therefore

$$1 - v^2 = \pm\frac{C_1}{x}$$

Substitute $v = \frac{y}{x}$ into the above equation:

$$1 - \frac{y^2}{x^2} = \pm\frac{C_1}{x}$$

$$y^2 = x^2 \mp C_1 x$$

To verify the solution, differentiate both sides of it with respect to x:

$$2y\frac{dy}{dx} = 2x \mp C_1$$

Therefore $\frac{dy}{dx} = \frac{2x \mp C_1}{2y}$ and $\frac{x^2+y^2}{2xy} = \frac{x^2+x^2 \mp C_1 x}{2xy} = \frac{2x \mp C_1}{2y} = \frac{dy}{dx}$

Linear Fractional Equations

A linear fractional equation has the form $\frac{dy}{dx} = \frac{a_1 x + b_1 x + c_1}{a_2 x + b_2 x + c_2}$, where a_1, b_1, a_2, and b_2 are non-zero constants. A special case of the equation is when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$. Under this condition, linear fractional equations can be reduced to separable equations by making the substitution $v = a_1 x + b_2 y$. Since $a_2 = \frac{a_1}{k}$ and $b_2 = \frac{b_1}{k}$, we have $a_2 x + b_2 x = \frac{1}{k}\left(a_1 x + b_1 x\right) = \frac{v}{k}$, and also $\frac{dv}{dx} = a_1 + b_1\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{1}{b_1}\left(\frac{dv}{dx} - a_1\right)$.

Example 2.2

Solve the differential equation $\frac{dy}{dx} = \frac{2x+3y+5}{4x+6y-3}$.

Solution:

Make the substitution $v = 2x + 3y$, then $\frac{dv}{dx} = 2 + 3\frac{dy}{dx}$, or $\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$. Substitute these into the original equation:

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v+5}{2v-3}$$

$$\frac{dv}{dx} = \frac{3\left(v+5\right)}{2v-3} + 2 = \frac{7v+9}{2v-3}$$

$$\int \frac{2v-3}{7v+9}dv = \int dx$$

Since $\frac{2v-3}{7v+9} = \frac{7(2v-3)}{7(7v+9)} = \frac{14v-21+(18-18)}{7(7v+9)} = \frac{(14v+18)-(21+18)}{7(7v+9)} = \frac{2}{7} - \frac{39}{7(7v+9)}$,

$$\int \left(\frac{2}{7} - \frac{39}{7(7v+9)} \right) dv = \int dx$$

$$\frac{2}{7}v - \frac{39}{49} \ln|7v+9| = x + C$$

Substitute $v = 2x + 3y$ back into the above equation to get an implicit solution of y:

$$14(2x+3y) - 39\ln|7(2x+3y)+9| = 49x + C_1 \quad (C_1 = 49C)$$

► **TIP**

Sometimes it is unnecessary or even impossible to find an explicit expression for the solution. An implicit solution is acceptable as long as it is reasonably simplified.

Linear First-Order Differential Equations

A first-order linear differential equation can be generally expressed as $\frac{dy}{dx} + p(x)y = q(x)$. This equation is not directly separable, but can be converted into a separable equation by multiplying both sides by an **integrating factor** $I(x)$. Then the equation becomes

$$I(x)y' + p(x)I(x)y = q(x)I(x)$$

To find $I(x)$, first notice that $\frac{d}{dx}(I(x)y) = I'(x)y + I(x)y'$, which resembles the left side of the previous equation. Let

$$I'(x)y + I(x)y' = I(x)y' + p(x)I(x)y$$

$$I'(x)y = p(x)I(x)y$$

$$\frac{d}{dx}I(x) = p(x)I(x)$$

$$\frac{d(I(x))}{I(x)} = p(x)dx$$

Integrating both sides,

$$\ln|I(x)| = \int p(x)dx$$

Since $I(x)$ is used as an integrating factor, there is no need to add a constant C here.

$$I(x) = e^{\int p(x)dx}$$

So the original equation with the integrating factor becomes

$$y'e^{\int p(x)dx} + p(x)ye^{\int p(x)dx} = q(x)e^{\int p(x)dx}$$

or

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q\left(x\right)e^{\int p(x)dx}$$

which can be separated and solved analytically to obtain

$$y = e^{-\int p(x)dx}\left(\int q\left(x\right)e^{\int p(x)dx}dx + C\right)$$

► NOTE

The purpose of multiplying the integrating factor $I\left(x\right)$ is to make the left side of the equation a derivative with respect to x. Although it is generally quite difficult or even impossible to find an integration factor for a differential equation, you do not have to struggle every time with a linear first-order differential equation; you can directly apply the general solution formula to solve it.

Example 2.3

Solve the differential equation $\frac{dy}{dx} + x^2y = x^2$

Solution:

In the above equation, $p\left(x\right) = q\left(x\right) = x^2$, and $I\left(x\right) = e^{\int x^2dx} = e^{\frac{x^3}{3}}$. So the solution can be directly calculated as

$$y = e^{-\frac{x^3}{3}}\left(\int x^2e^{\frac{x^3}{3}}dx + C\right)$$

$$y = e^{-\frac{x^3}{3}}\left(\int e^u du + C\right) \quad (u = \frac{x^3}{3} \text{ and } du = x^2dx)$$

$$y = e^{-\frac{x^3}{3}}\left(e^{\frac{x^3}{3}} + C\right) = 1 + Ce^{-\frac{x^3}{3}}$$

Example 2.4

Solve the differential equation $\frac{dy}{dx} + 2y\cot x + \sin 2x = 0$.

Solution:

In the above equation, $p\left(x\right) = 2\cot x$, $q\left(x\right) = -\sin 2x$, and $I\left(x\right) = e^{\int 2\cot xdx}$. Letting $u = \sin x$ and $du = \cos xdx$, $I\left(x\right) = e^{2\int \frac{1}{u}du} = e^{2\ln|\sin x|} = \sin^2 x$. So the solution is

$$y = \frac{1}{\sin^2 x}\left(\int -\sin 2x\sin^2 xdx + C\right)$$

$$y = \frac{1}{\sin^2 x}\left(\int -2\sin x\cos x\sin^2 xdx + C\right)$$

$$y = \frac{1}{\sin^2 x}\left(-\int udu + C\right) \quad (u = \sin^2 x \text{ and } du = 2\sin x\cos xdx)$$

$$y = \frac{1}{\sin^2 x}\left(-\frac{\sin^4 x}{2} + C\right) = -\frac{\sin^2 x}{2} + \frac{C}{\sin^2 x}$$

Practice problem set 2

Solve the following differential equations.

1. $y^2 dx - x^2 dy = 0$
2. $\frac{dy}{dx} = \frac{2x-y}{x}$
3. $(x^3 + y^3)\, dx - 3xy^2 dy = 0$
4. $xdy - ydx - \sqrt{x^2 - y^2}dx = 0$
5. $\frac{dy}{dx} = \frac{2x+6y+3}{x+3y-9}$
6. $\frac{dy}{dx} + 2xy = 6x$
7. $(x-2)\frac{dy}{dx} = y + 4(x-2)^3$
8. $\frac{dy}{dx} + 2xy = 2x^3$; $y(0) = 1$
9. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$; when $x = \frac{\pi}{2}$, $y = -4$
10. $xy' = y(1 - x\tan x) + 2x^2\cos x$

Answers to Practice Problems

Practice Problem Set 1 (Chapter 1)

1. $y = C\left(1 + x^2\right)$
2. $y^{\frac{3}{2}} = 9x^{\frac{1}{2}} + C$
3. $(x-1)^2 - (y+1)^2 + 2\ln\left|\frac{x+1}{y-1}\right| = C$
4. $\sin^2 y = C\frac{x-1}{x+1}$
5. $t^3 y^2 = Ce^y$
6. $\sin x + y^2 = 1$
7. $y^2 + 2\ln|y| = x^2 - 4x + 5$
8. $2e^{x^2} + y^4 - 4y = 10$
9. $y = \ln\left|\frac{(x-3)^2(x+1)}{9}\right|$
10. If $y > a$ or $y < 0$, $y = \frac{Cae^{ax}}{Ce^{ax}-1}$;
    If $0 < y < a$, $y = \frac{Cae^{ax}}{Ce^{ax}+1}$

Practice Problem Set 2 (Chapter 2)

1. $y = x + Cxy$
2. $y = x - \frac{C}{x}$
3. $x^3 - 2y^3 = Cx$
4. $Cx = e^{\arcsin\frac{y}{x}}$
5. $(x + 3y) - 9\ln|x + 3y| = 7x + C$
6. $y = 3 + Ce^{-x^2}$
7. $y = 2(x-2)^3 + C(x-2)$
8. $y = 2e^{-x^2} + x^2 - 1$
9. $y\sin x + 5e^{\cos x} = 1$
10. $y = 2x^2\cos x + Cx\cos x$

Practice Problem Set 3 (Chapter 3)

1. $120\left(\frac{1}{2}\right)^{\frac{50}{74}} \approx 75$ grams
2. $3\ln\frac{25}{13}/\ln\frac{13}{6} \approx 2.54$ hours
3. $400(2)^5 = 12800$ cells
4. $\frac{10\ln 0.5}{\ln 0.32} \approx 6.08$ grams
5. $\ln 2/0.0525 \approx 13.2$ years
6. $100\left(\frac{1}{2}\right)^{\frac{3400}{5730}} \approx 66.3$ %
7. $20(0.7)^{\frac{20}{3}} \approx 1.86$ candelas
8. $\ln 2/0.05 \approx 13.9$ (In the 14th year)
9. $\frac{5\ln 2}{\ln 2.5} \approx 3.78$ (In the fourth year)
10. $2.5 = 5e^{-t/20}$, $t \approx 13.9$ minutes
11. 1.15 seconds
12. $I = 10e^{-0.4t}$, $I \approx 0.15A$

Practice Problem Set 4 (Chapter 4)

1. $\frac{12\ln\frac{1}{3}}{\ln\frac{13}{15}} \approx 92$ days
2. $50\left(1 - \left(\frac{3}{5}\right)^{\frac{30}{15}}\right) = 32$ words
3. $27 - 20\left(\frac{5}{6}\right)^{\frac{10}{5}} \approx 13$°C
4. $45 = K\left(1 - e^{-5r}\right)$,
    $80 = K\left(1 - e^{-10r}\right)$, 202.5 m/s
5. $\frac{5\ln 0.001}{\ln 12 - \ln 35} \approx 32.3$ minutes
6. $\frac{5\ln 0.75}{\ln 0.8} \approx 6.45$ minutes
7. $3\ln\left(\frac{98.6-65}{72-65}\right)/\ln\left(\frac{7}{15}\right) \approx -6.17$ hours
    Therefore, 6.17 hours ago from now is approximately 8:49 AM
8. $\frac{dP}{dt} = (0.097 - 0.047)P - 30000$
9. $v(t) = 4\left(1 - e^{-2.45t}\right)$ m/s
10. $Q(t) = 80 - 78e^{-\frac{t}{20}}$ lbs.
11. $143\ \mu$C
12. 21.2 seconds