Questions

1. State the power and constant of variation for the function \( f(x) = -2x^{-3} \). Graph the function and analyze it (domain, range, asymptotes, extrema, etc.).

2. The power \( P \) (in watts) produced by a windmill is proportional to the cube of the wind speed \( v \) (in mph). If a wind of 10 mph generates 15 watts of power, how much power is generated by winds of 20, 40, and 80 mph? Make a table and explain the pattern.

3. Match the function to its graph

   (i) \( y = -2x^4 \)  (ii) \( y = -2x^{1/4} \)  (iii) \( y = x^{1/4} \)  (iv) \( y = x^{-2} \)  (v) \( y = -\frac{1}{4}x^{-2} \)  (vi) \( y = x^3 \)

4. Is the function \( f(x) = \frac{1}{3}x^{-3} \) even, odd, or neither? Is it continuous?

5. Is the function \( f(x) = x^{1/3} \) even, odd, or neither?

6. Evaluate the following limits by thinking of what the sketch of the function looks like.

   (a) \( \lim_{x \to -\infty} x^4 = \)
   (b) \( \lim_{x \to -\infty} x^3 = \)
   (c) \( \lim_{x \to -\infty} x^{1/2} = \)
   (d) \( \lim_{x \to -\infty} x^3 = \)
Solutions

1. State the power and constant of variation for the function $f(x) = -2x^{-3}$. Graph the function and analyze it.

The function has power $-3$. Since the power is less than zero, this is an inverse variation function. The constant of variation is $-2$.

Sketch (from the sketch of the basic function $y = x^{-1}$, so sketched by hand, not with a computer, although I used a computer to draw the sketches).

Domain: $x \in (-\infty, 0) \cup (0, \infty)$.
Range: $y \in (-\infty, 0) \cup (0, \infty)$.
Continuous: the function is discontinuous at $x = 0$.
Increasing/Decreasing: The function is increasing for $x \in (-\infty, 0)$, and increasing for $x \in (0, \infty)$.
Symmetric: The function is odd ($f(-x) = f(x)$).
Boundedness: The function is not bounded above or below.
Extrema: none.
Asymptotes: The function has a horizontal asymptote of $y = 0$, and a vertical asymptote at $x = 0$.
Vertical Asymptotes: $\lim_{x \to 0^+} (-2x^{-3}) = -\infty$, $\lim_{x \to 0^-} (-2x^{-3}) = \infty$.
End Behaviour: $\lim_{x \to -\infty} (-2x^{-3}) = 0$ and $\lim_{x \to \infty} (-2x^{-3}) = 0$.

2. The power $P$ (in watts) produced by a windmill is proportional to the cube of the wind speed $v$ (in mph). If a wind of 10 mph generates 15 watts of power, how much power is generated by winds of 20, 40, and 80 mph? Make a table and explain the pattern.

We are told the power is proportional to the cube of the wind speed. Converting that to mathematics leads us to write

$$P \propto v^3$$

where $P$ is the power in watts and $v$ is the wind speed in mph. The symbol between them indicates that they are proportional to each other.

We need to know a relation for when they are equal, and we can get that by inserting a proportionality constant $k$:

$$P = kv^3$$

You could have started your solution with this relation. We don’t know the value of $k$ yet, but we can find it using some of the information given.
We are told that when \( v = 10 \text{ mph} \), \( P = 15 \text{ watts} \), and we can use this do determine the constant \( k \):

\[
P = kv^3
\]

\[
\frac{15}{1000} = k
\]

\[
k = \frac{3}{200}
\]

and we can write the relation between wind speed in mph and power in watts as:

\[
P = \frac{3}{200}v^3.
\]

Now we can construct the table that is asked for:

<table>
<thead>
<tr>
<th>( v ) (mph)</th>
<th>( P = \frac{3}{200}v^3 ) (watts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15 (this data was given)</td>
</tr>
<tr>
<td>20</td>
<td>120</td>
</tr>
<tr>
<td>40</td>
<td>960</td>
</tr>
<tr>
<td>80</td>
<td>7680</td>
</tr>
</tbody>
</table>

The values for \( P \) at wind speeds of 20, 40, 80 mph were calculated using the relation. As the wind speed increases, the power grows rapidly, by a cubic relation. The proportionality constant \( k = 3/200 \) scales the cubic growth.

3. Match the function to its graph

(a) (i) \( y = -2x^4 \)

(b) (ii) \( y = -2x^{1/4} \)

(c) (iii) \( y = x^{1/4} \)

(d) (iv) \( y = x^{-2} \)

(e) (v) \( y = -\frac{1}{4}x^{-2} \)

(f) (vi) \( y = x^3 \)
4. Is the function \( f(x) = \frac{1}{3}x^{-3} \) even, odd, or neither? Is it continuous?

To check even/odd/neither, we evaluate \( f(-x) \) and see what it simplifies to.

\[
f(-x) = \frac{1}{3}(-x)^{-3} = \frac{1}{3} \cdot \frac{1}{(-x)^3} = -\frac{1}{3} \cdot \frac{1}{x^3} = -\frac{1}{3}x^{-3} = -f(x)
\]

So, since \( f(-x) = -f(x) \), we have that \( f \) is odd. Since it is not defined at \( x = 0 \), it is not continuous at \( x = 0 \).

5. Is the function \( f(x) = x^{1/3} \) even, odd, or neither?

\[
f(-x) = (-x)^{1/3} = -x^{1/3} = -f(x)
\]

So, since \( f(-x) = -f(x) \), we have that \( f \) is odd.

6. Evaluate the following limits by thinking of what the sketch of the function looks like.

   (a) \( \lim_{x \to -\infty} x^4 = \infty \)
   (b) \( \lim_{x \to -\infty} x^3 = -\infty \)
   (c) \( \lim_{x \to -\infty} x^{1/2} \) does not exist, since the domain of \( x^{1/2} = \sqrt{x} \) is \( x > 0 \)
   (d) \( \lim_{x \to \infty} x^3 = \infty \)