8-4 Ellipses

What You'll Learn

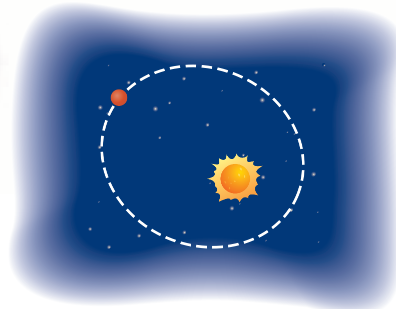
- Write equations of ellipses.
- Graph ellipses.

Vocabulary

- ellipse
- foci
- major axis
- minor axis
- center

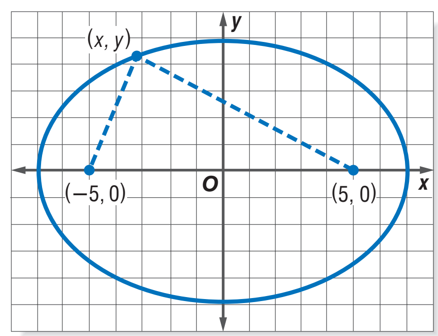
Why are ellipses important in the study of the solar system?

Fascination with the sky has caused people to wonder, observe, and make conjectures about the planets since the beginning of history. Since the early 1600s, the orbits of the planets have been known to be ellipses with the Sun at a focus.



EQUATIONS OF ELLIPSES As you discovered in the Algebra Activity on page 432, an **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the **foci** of the ellipse.

The ellipse at the right has foci at (5, 0) and (−5, 0). The distances from either of the x-intercepts to the foci are 2 units and 12 units, so the sum of the distances from any point with coordinates (x, y) on the ellipse to the foci is 14 units.



You can use the Distance Formula and the definition of an ellipse to find an equation of this ellipse.

$$\underbrace{\text{The distance between } (x, y) \text{ and } (-5, 0)} + \underbrace{\text{the distance between } (x, y) \text{ and } (5, 0)} = 14.$$

$$\sqrt{(x+5)^2 + y^2} + \sqrt{(x-5)^2 + y^2} = 14$$

$$\sqrt{(x+5)^2 + y^2} = 14 - \sqrt{(x-5)^2 + y^2} \quad \text{Isolate the radicals.}$$

$$(x+5)^2 + y^2 = 196 - 28\sqrt{(x-5)^2 + y^2} + (x-5)^2 + y^2 \quad \text{Square each side.}$$

$$x^2 + 10x + 25 + y^2 = 196 - 28\sqrt{(x-5)^2 + y^2} + x^2 - 10x + 25 + y^2$$

$$20x - 196 = -28\sqrt{(x-5)^2 + y^2} \quad \text{Simplify.}$$

$$5x - 49 = -7\sqrt{(x-5)^2 + y^2} \quad \text{Divide each side by 4.}$$

$$25x^2 - 490x + 2401 = 49[(x-5)^2 + y^2] \quad \text{Square each side.}$$

$$25x^2 - 490x + 2401 = 49x^2 - 490x + 1225 + 49y^2 \quad \text{Distributive Property}$$

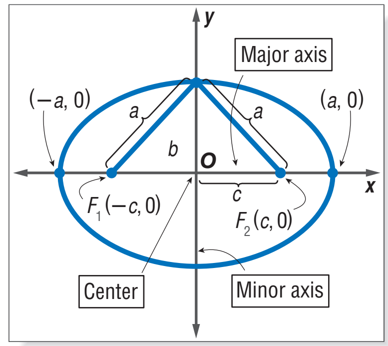
$$-24x^2 - 49y^2 = -1176 \quad \text{Simplify.}$$

$$\frac{x^2}{49} + \frac{y^2}{24} = 1 \quad \text{Divide each side by } -1176.$$

An equation for this ellipse is $\frac{x^2}{49} + \frac{y^2}{24} = 1$.

Every ellipse has two axes of symmetry. The points at which the ellipse intersects its axes of symmetry determine two segments with endpoints on the ellipse called the **major axis** and the **minor axis**. The axes intersect at the **center** of the ellipse. The foci of an ellipse always lie on the major axis.

Study the ellipse at the right. The sum of the distances from the foci to any point on the ellipse is the same as the length of the major axis, or $2a$ units. The distance from the center to either focus is c units. By the Pythagorean Theorem, a, b, and c are related by the equation $c^2 = a^2 - b^2$. Notice that the x- and y-intercepts, $(\pm a, 0)$ and $(0, \pm b)$, satisfy the quadratic equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is the standard form of the equation of an ellipse with its center at the origin and a horizontal major axis.



Study Tip

Vertices of Ellipses
The endpoints of each axis are called the *vertices* of the ellipse.

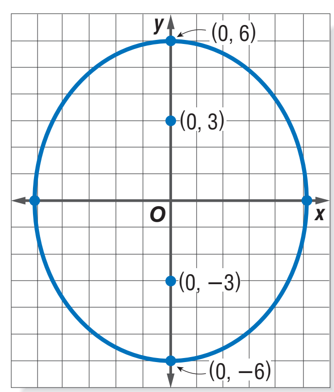
Key Concept — Equations of Ellipses with Centers at the Origin

Standard Form of Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

In either case, $a^2 \geq b^2$ and $c^2 = a^2 - b^2$. You can determine if the foci are on the x-axis or the y-axis by looking at the equation. If the x^2 term has the greater denominator, the foci are on the x-axis. If the y^2 term has the greater denominator, the foci are on the y-axis.

Example 1 Write an Equation for a Graph

Write an equation for the ellipse shown at the right.



In order to write the equation for the ellipse, we need to find the values of a and b for the ellipse. We know that the length of the major axis of any ellipse is $2a$ units. In this ellipse, the length of the major axis is the distance between the points at (0, 6) and (0, −6). This distance is 12 units.

$2a = 12$ Length of major axis = 12

$a = 6$ Divide each side by 2.

The foci are located at (0, 3) and (0, −3), so $c = 3$. We can use the relationship between a, b, and c to determine the value of b.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$9 = 36 - b^2$ $c = 3$ and $a = 6$

$b^2 = 27$ Solve for b^2.

Since the major axis is vertical, substitute 36 for a^2 and 27 for b^2 in the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. An equation of the ellipse is $\frac{y^2}{36} + \frac{x^2}{27} = 1$.

Example 2 Write an Equation Given the Lengths of the Axes

MUSEUMS In an ellipse, sound or light coming from one focus is reflected to the other focus. In a whispering gallery, a person can hear another person whisper from across the room if the two people are standing at the foci. The whispering gallery at the Museum of Science and Industry in Chicago has an elliptical cross section that is 13 feet 6 inches by 47 feet 4 inches.

a. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

The length of the major axis is $47\frac{1}{3}$ or $\frac{142}{3}$ feet.

$2a = \frac{142}{3}$ Length of major axis $= \frac{142}{3}$

$a = \frac{71}{3}$ Divide each side by 2.

The length of the minor axis is $13\frac{1}{2}$ or $\frac{27}{2}$ feet.

$2b = \frac{27}{2}$ Length of minor axis $= \frac{27}{2}$

$b = \frac{27}{4}$ Divide each side by 2.

Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$ into the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. An equation of the ellipse is $\frac{x^2}{\left(\frac{71}{3}\right)^2} + \frac{y^2}{\left(\frac{27}{4}\right)^2} = 1$.

b. How far apart are the points at which two people should stand to hear each other whisper?

People should stand at the two foci of the ellipse. The distance between the foci is $2c$ units.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$c = \sqrt{a^2 - b^2}$ Take the square root of each side.

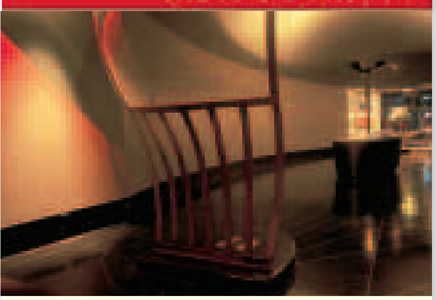
$2c = 2\sqrt{a^2 - b^2}$ Multiply each side by 2.

$2c = 2\sqrt{\left(\frac{71}{3}\right)^2 - \left(\frac{27}{4}\right)^2}$ Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$.

$2c \approx 45.37$ Use a calculator.

The points where two people should stand to hear each other whisper are about 45.37 feet or 45 feet 4 inches apart.

More About...



Museums

The whispering gallery at Chicago's Museum of Science and Industry has a parabolic dish at each focus to help collect sound.

Source: www.msichicago.org

GRAPH ELLIPSES As with circles, you can use completing the square, symmetry, and transformations to help graph ellipses. An ellipse with its center at the origin is represented by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. The ellipse could be translated h units to the right and k units up. This would move the center to the point (h, k). Such a move would be equivalent to replacing x with $x - h$ and replacing y with $y - k$.

Key Concept — Equations of Ellipses with Centers at (h, k)

Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

www.algebra2.com/extra_examples

Example 3 Graph an Equation in Standard Form

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then graph the ellipse.

The center of this ellipse is at (0, 0).

Since $a^2 = 16$, $a = 4$. Since $b^2 = 4$, $b = 2$.

The length of the major axis is 2(4) or 8 units, and the length of the minor axis is 2(2) or 4 units. Since the x^2 term has the greater denominator, the major axis is horizontal.

$c^2 = a^2 - b^2$ — Equation relating a, b, and c

$c^2 = 4^2 - 2^2$ or 12 — $a = 4, b = 2$

$c = \sqrt{12}$ or $2\sqrt{3}$ — Take the square root of each side.

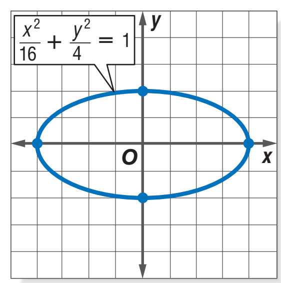
The foci are at $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the ellipse is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (−4, 0), (−3, 1.3), (−2, 1.7), and (−1, 1.9) lie on the graph.

x	y
0	2.0
1	1.9
2	1.7
3	1.3
4	0.0

The ellipse is also symmetric about the x-axis, so the points at (−3, −1.3), (−2, −1.7), (−1, −1.9), (0, −2), (1, −1.9), (2, −1.7), and (3, −1.3) lie on the graph.

Graph the intercepts, (−4, 0), (4, 0), (0, 2), and (0, −2), and draw the ellipse that passes through them and the other points.



Study Tip

Graphing Calculator

You can graph an ellipse on a graphing calculator by first solving for y. Then graph the two equations that result on the same screen.

If you are given an equation of an ellipse that is not in standard form, write it in standard form first. This will make graphing the ellipse easier.

Example 4 Graph an Equation Not in Standard Form

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 + 4x - 24y + 24 = 0$. Then graph the ellipse.

Complete the square for each variable to write this equation in standard form.

$x^2 + 4y^2 + 4x - 24y + 24 = 0$ — Original equation

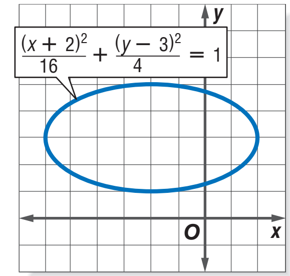
$(x^2 + 4x + ■) + 4(y^2 - 6y + ■) = -24 + ■ + 4(■)$ — Complete the squares.

$(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -24 + 4 + 4(9)$ — $\left(\frac{4}{2}\right)^2 = 4$, $\left(\frac{-6}{2}\right)^2 = 9$

$(x + 2)^2 + 4(y - 3)^2 = 16$ — Write the trinomials as perfect squares.

$\frac{(x + 2)^2}{16} + \frac{(y - 3)^2}{4} = 1$ — Divide each side by 16.

The graph of this ellipse is the graph from Example 3 translated 2 units to the left and up 3 units. The center is at $(-2, 3)$ and the foci are at $\left(-2 + 2\sqrt{3}, 0\right)$ and $\left(-2 - 2\sqrt{3}, 0\right)$. The length of the major axis is still 8 units, and the length of the minor axis is still 4 units.

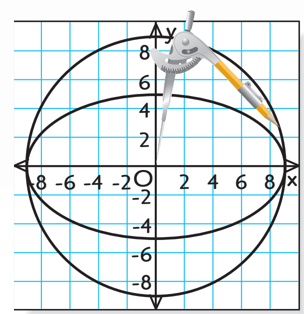


You can use a circle to locate the foci on the graph of a given ellipse.

Algebra Activity

Locating Foci

Step 1 Graph an ellipse so that its center is at the origin. Let the endpoints of the major axis be at $(-9, 0)$ and $(9, 0)$, and let the endpoints of the minor axis be at $(0, -5)$ and $(0, 5)$.



Step 2 Use a compass to draw a circle with center at $(0, 0)$ and radius 9 units.

Step 3 Draw the line with equation $y = 5$ and mark the points at which the line intersects the circle.

Step 4 Draw perpendicular lines from the points of intersection to the x-axis. The foci of the ellipse are located at the points where the perpendicular lines intersect the x-axis.

Make a Conjecture

Draw another ellipse and locate its foci. Why does this method work?