Chapter 5

Logistic Growth

Purpose: *To solve the differential equation for the logistic growth model and to apply the solution. This is only required by the AP Calculus BC exam.*

In the previous two chapters, we have discussed cases in which the rate of change of quantity P is either directly proportional to itself (P), or to its remaining room for growth $(K - P)$. **Logistic growth** deals with growth rates that are directly proportional to both of these quantities:

$$\frac{dP}{dt} = r'P\left(K - P\right) \tag{5.1}$$

Here r' is used because the logistic equation is more commonly written in this form:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{5.2}$$

(Differential equation for logistic growth)

where $r = r'K$. In the above equation, K is the same carrying capacity or equilibrium value as we discussed before. The constant r is called the **intrinsic growth rate**, that is, the growth rate in the absence of any limiting factors. The logistic equation is mostly used to provide a more realistic model for population growth (refer to Appendix A for a detailed derivation). The logistic equation is also frequently used to describe the spreading of diseases or rumors, autocatalytic chemical reactions, and other processes.

The logistic equation shows that if P is small relative to the carrying capacity K, the rate of its growth will be close to the constant rate r of the exponential growth model. As P nears K, the rate will shrink toward 0, resulting in an S-shaped curve (refer to Figures 5.1 and 5.2). According to this model, when P reaches K, the growth rate is 0, and the population will be stable. If P were to somehow exceed K, the rate would become negative and the population would decrease toward K.

In order to solve equation 5.2, we separate the variables first and integrate both sides:

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int \frac{KdP}{P\left(K - P\right)} = \int rdt$$

Separating the integrand by partial fractions we have

$$\frac{K}{P\left(K - P\right)} = \frac{1}{P} + \frac{1}{K - P}$$

Therefore,

$$\int \frac{dP}{P} + \int \frac{dP}{K - P} = \int rdt$$

$$\ln|P| - \ln|K - P| = rt + C$$

$$\ln\left|\frac{K-P}{P}\right| = -rt - C$$

If $\frac{K-P}{P} > 0$, we have:

$$\frac{K-P}{P} = C_1 e^{-rt} \quad (C_1 = e^{-C})$$

Assuming that $P = P_0$ when $t = 0$, then

$$C_1 = \frac{K-P_0}{P_0}$$

Therefore

$$\frac{K-P}{P} = \frac{K-P_0}{P_0} e^{-rt}$$

$$P = \frac{K}{1 + \frac{K-P_0}{P_0} e^{-rt}} = \frac{KP_0}{P_0 + (K-P_0)\, e^{-rt}}$$

If $\frac{K-P}{P} < 0$, we have:

$$\frac{P-K}{P} = C_1 e^{-rt} \quad (C_1 = e^{-C})$$

Assume that $P = P_0$ when $t = 0$, then

$$C_1 = \frac{P_0-K}{P_0}$$

$$\frac{P-K}{P} = \frac{P_0-K}{P_0} e^{-rt}$$

This will lead to the same solution as in the previous case. So the final solution is:

$$\boxed{P = \frac{KP_0}{P_0 + (K-P_0)\, e^{-rt}}} \tag{5.3}$$

(Logistic growth solution)

Example 5.1

A population of bacteria in a culture is 50 million, and is growing at a rate of 2 million per hour. Assume the carrying capacity is 1 billion. Use one million as a base unit.

a) Write the logistic differential equation using the data.

b) Use the model to predict the population in 2 hours, 5 hours, and a day from now.

c) Use the model to predict when the population will reach half the carrying capacity.

Solution:

a) Since the initial population is small compared to the carrying capacity, take the initial relative growth rate ($\frac{2}{50}$) to be an estimate of r. Substitute all the known numbers into equation 5.2:

$$\frac{dP}{dt} = \frac{2}{50} P\left(1 - \frac{P}{1000}\right) = 0.04P\left(1 - \frac{P}{1000}\right)$$

b) Using equation 5.3,

$$P = \frac{KP_0}{P_0 + (K - P_0)\, e^{-rt}} = \frac{(1000)\,(50)}{50 + (1000 - 50)\, e^{-0.04t}} = \frac{1000}{1 + 19e^{-0.04t}}$$

$$P\,(2) \approx 53.9 \text{ million}$$

$$P\,(5) \approx 60.4 \text{ million}$$

$$P\,(24) \approx 121 \text{ million}$$

c) Set $P = 500$ to solve for t:

$$500 = \frac{1000}{1 + 19e^{-0.04t}}$$

$$19e^{-0.04t} = 1$$

$$t \approx 73.6 \text{ hours}$$

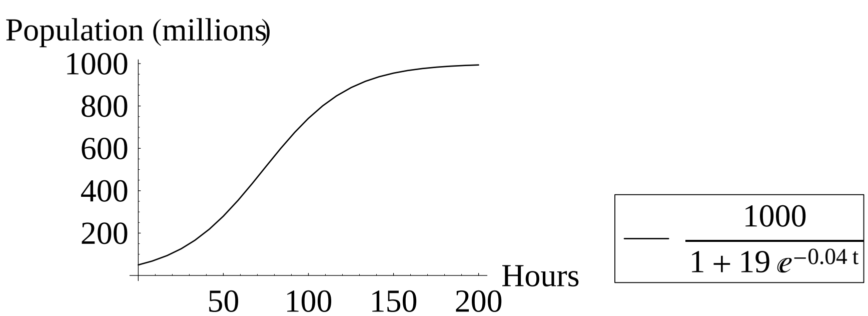

Figure 5.1: Graph for example 5.1

Example 5.2

A rumor is spreading in a city of 6000 people. Initially, three people know it; three days later 300 people have heard about it. Suppose the rumor spreads at a rate proportional to both the number of people knowing it and the number of people not knowing it. Find

a) the number of days for the rumor to spread to 50% of the people,

b) the approximate number of people knowing it after ten days.

Solution:

a) Use equation 5.3 to solve for e^{-r} with the known information $P_0 = 3$, $K = 6000$ and $P\,(3) = 300$.

$$300 = \frac{(6000)\,(3)}{3 + (6000 - 3)\, e^{-3r}}$$

$$3 + (6000 - 3)\, e^{-3r} = 60$$

$$1999e^{-3r} = 19$$

$$e^{-r} = \left(\frac{19}{1999}\right)^{\frac{1}{3}}$$

Now use the value for e^{-r} and $P = 3000$ to solve for t:

$$3000 = \frac{18000}{3 + 5997e^{-rt}}$$

$$3 + 5997e^{-rt} = 6$$

$$1999e^{-rt} = 1$$

$$\left(\frac{19}{1999}\right)^{\frac{t}{3}} = \frac{1}{1999}$$

$$t = \frac{3\ln\frac{1}{1999}}{\ln\frac{19}{1999}} \approx 4.9 \text{ days}$$

b) Letting $t = 10$, use equation 5.3 to solve for P:

$$P\left(10\right) = \frac{18000}{3 + 5997\left(\frac{19}{1999}\right)^{\frac{10}{3}}} \approx 5998 \text{ people}$$

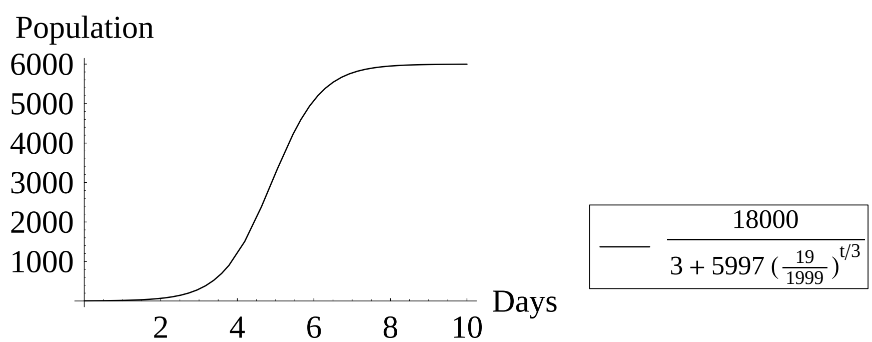

Figure 5.2: Graph for example 5.2

Practice problem set 5

Solve the following problems.

1. A certain population has 10,000 people. A disease is spreading through the population at a rate proportional to both the infected population and to the unaffected population. If it is known that 1,000 were infected two months ago, and 4,000 were infected last month, how many should be infected this month? How many months from now will 90% of the population be infected?

2. The number of people that hear a rumor follows logistic growth. In a school of 1500 students, 5 students start a rumor. After 2 hours, 120 students have heard about the rumor. Find the number of students to hear the rumor after one more hour.

3. Diseases sometimes spread according to logistic growth. If there were 100 people infected one week ago and 280 infected now out of an isolated population of 30,000, find the number of people that will be infected a week from today.

4. A salmon population of 1.5 million living off the coast of Alaska grows at a rate of $0.04P(t)$ per year, where $P(t)$ is the salmon population at time t. Suppose a group of predators moves into the waters of the salmon and starts to kill the salmon at a rate of $0.0002[P(t)]^2$ per year. Calculate how large the salmon population is after 5 years.

5. The rate of formation of a certain chemical X in the second order chemical reaction $A + B \to X$ is known to be governed by the equation

$$\frac{dx}{dt} = r(a - x)(b - x) \qquad (r > 0,\, a > b > 0)$$

   where x is the amount (concentration) of chemical X present at time t, and a, b are the initial concentrations of A and B, respectively. If $x = \frac{1}{2}(a + b)$ when $t = 0$, find x as a function of t and determine $\lim_{t\to\infty} x(t)$. Hint: use Example 1.4.

6. A rumor is spreading in a population of 800. Assume that each person meets four people each day (who may or may not know the rumor). Initially one person knows the rumor.

   a) When will 200 people know it?

   b) When will 799 people know it?

   Hint: the intrinsic growth rate is $r = 4$.

7. It is known that the enzyme pepsin digests protein in the stomach and is formed from the cleavage of a short segment of amino acids from its predecessor, pepsinogen. Under conditions of extremely low pH (the pH of the stomach is around 2), pepsin can also convert pepsinogen into pepsin, thus setting up an autocatalytic reaction. Assuming that there is 1.5 M pepsin in the stomach prior to a meal, and 5 M pepsinogen is secreted, find when 99% of the newly available pepsinogen will be converted, if after 10 seconds, there exists 2 M pepsin. Hint: use the logistic equation with $P_0 = 1.5$.

8. There are two islands, A and B. Initially, there are 1000 people on island A and no one on island B. If the rate of emigration is proportional to the difference between the population of A and the population of B, and after 5 years there are 250 people on island B and 750 people on island A, how many years does it take for island B to have 400 people? Hint: is this logistic growth?

Practice Problem Set 5 (Chapter 5)

1. $P(t) = \frac{10000}{1+9(1/6)^t}$; $P(2) = 8000$; $t \approx 0.45$ months
2. $\frac{1500}{1+299\left(\frac{23}{598}\right)^{\frac{3}{2}}} \approx 203$ students
3. $\frac{3000000}{100+(29900)\left(\frac{743}{2093}\right)^{\frac{14}{7}}} \approx 776$ people
4. $\frac{dP}{dt} = 0.04P - 0.0002P^2$ or $\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{200}\right)$; $P(5) \approx 1.83$ million
5. $x = b + \frac{a-b}{1+e^{r(a-b)t}}$; $\lim_{t \to +\infty} x(t) = b$
6. a) 1.40 days; b) 3.34 days
7. $2 = \frac{(6.5)(1.5)}{1.5+(6.5-1.5)e^{-10r}}$; $e^{-r} = 0.675^{\frac{1}{10}}$ $1.5 + (5)(0.99) = \frac{(6.5)(1.5)}{1.5+(6.5-1.5)0.675^{\frac{t}{10}}}$, $t \approx 154$ s
8. $\frac{dP}{dt} = r(1000 - 2P)$; $e^{-r} = \left(\frac{1}{2}\right)^{\frac{1}{10}}$, $t \approx 11.6$ years

Practice Problem Set 6 (Chapter 6)

1. ≈ 6.05 billion
2. ≈ 90 squirrels
3. $\frac{dx}{dt} = r(a-x)^2$; $r = 0.5$, in six hours
4. $\frac{0.2-0.06}{0.1-0.06} = \frac{0.2-0}{0.1-0}e^{(0.2-0.1)20k}$; $k \approx 0.28\frac{mol}{L \cdot min}$

Practice Problem Set 7 (Chapter 7)

1. $e^{-r} = \left(\frac{1}{1999}\right)^{1/2}$, $P(t) = \frac{2000}{1+1999\left(\frac{1}{1999}\right)^{\frac{t}{2}}}$
2. a) in 5.22 weeks; b) in 3.10 weeks
3. $\frac{1}{t(3a-2b)} \ln \frac{b(a-2x)}{a(b-3x)}$
4. Const. solutions at $P = a$ and $P = b$. If $P_0 < a$, func. increasing, concave down. If $a < P_0 < \frac{a+b}{2}$, func. decreasing, concave up. If $\frac{a+b}{2} < P_0 < b$, func. decreasing, concave down. If $P_0 > b$, func. increasing, concave up.

Practice Problem Set 8 (Chapter 10)

1. $\frac{31}{48} \approx 0.6458$
2. ≈ 3594 bacteria
3. $P = 100e^{2.3(3)} \approx 99227$ bacteria (Difference of $3594 - 99227 = -95633$)