Calculus UNIT 1

LIMITS and RATES of Change (mostly MCV)– journal questions

Summarize everything you need to know about these topics. Use examples and concise (not long – but with enough detail) explanations. Include definitions and diagrams if necessary

1. RATES OF CHANGE (MHF)

Copy/Paste the following

Secant line of a curve is a line that (locally) intersects two points on the curve. The slope of the secant is the ________________ of all the slopes of the curve between the two points.

Tangent line to a curve at a given point is a line that "just touches" the curve at that point. The slope of the tangent is the ______________ as the slope of the curve at that point.

Slope formula from grade 9:

$$m=\frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Average Rate of Change AROC:

on the interval $x\in[a,b]$,

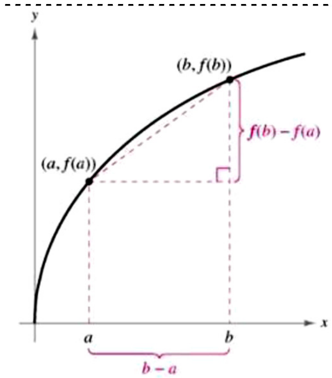
$$AROC=m_{\sec}=\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}$$

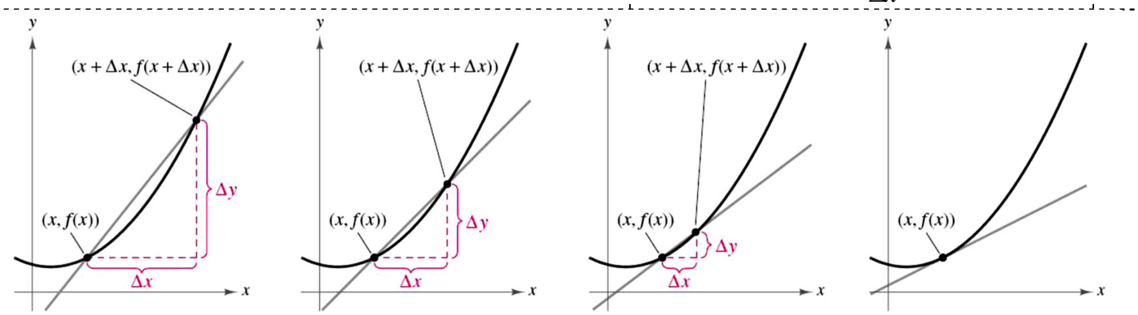
Instantaneous Rate of Change IROC using a DIFFERENCE QUOTIENT:

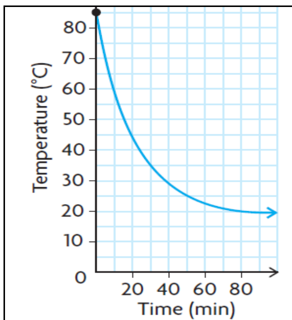
Let $b=a+\Delta x$ then the interval is $x\in[a,a+\Delta x]$,

$$m_{\tan}=\lim_{\Delta x\to 0}m_{\sec} \text{ or } IROC=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$IROC=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$$







Answer the following:
(use the Temperature-Time graph above on the right)
A cup of hot cocoa left on a desk in a classroom had its temperature measured once every minute. The graph shows the relationship between the temperature of the cocoa, in degrees Celsius, and time, in minutes.

a) Determine the slope of the secant line that passes through the points (5, 70) and (50, 25).
b) What does the answer to part a) mean in this context?
c) Estimate the slope of the tangent line at the point (30, 35).
d) What does the answer to part c) mean in this context?

2. INTRO TO LIMITS (MCV)

Copy/Paste the following:

What is a limit?

A limit is the intended height of a function. OR The value that a function or sequence "approaches" as the input or index approaches some value.

Limits are not concerned with what is going on at $x=a$ *. Limits are only concerned with what is going on around* $x=a$ *.*

Notation:

Instead of as $x\to a, f(x)\to L$. We now will use $\boxed{\lim_{x\to a}f(x)=L}$ which is read as "the limit of function $f(x)$, as x approaches a, equals L"

One sided limits:

Approaching from the RIGHT:
$\lim_{x\to a^+}f(x)=L$
and approaching from the LEFT:
$\lim_{x\to a^-}f(x)=L$

Limit exists at x=a if:

1. $\lim_{x\to a^+}f(x)$ exists (not infinite, DOES settle to some value)
2. $\lim_{x\to a^-}f(x)$ exists
3. $\lim_{x\to a^+}f(x)=\lim_{x\to a^-}f(x)$

Limit existing examples:

At any x in the domain of the continuous functions, the limit exists.

At x value with a removable discontinuity (hole), the limit exists

Limits not existing examples:

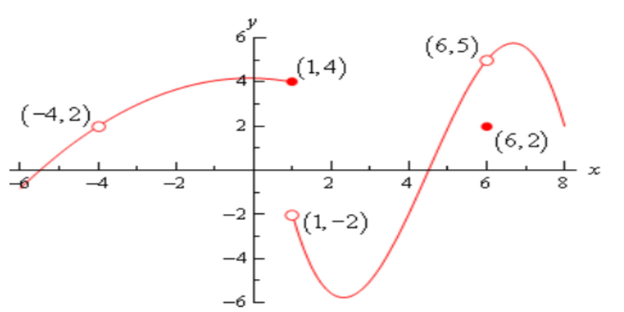
one sided graph

jump in the graph

oscillation

approaching ∞ because of a VA

3. LIMITS FROM GRAPHS. Find the following using the provided graph



(a) $f(-4)$

(b) $\lim\limits_{x \to -4^-} f(x)$

(c) $\lim\limits_{x \to -4^+} f(x)$

(d) $\lim\limits_{x \to -4} f(x)$

(e) $f(1)$

(f) $\lim\limits_{x \to 1^-} f(x)$

(g) $\lim\limits_{x \to 1^+} f(x)$

(h) $\lim\limits_{x \to 1} f(x)$

(i) $f(6)$

(j) $\lim\limits_{x \to 6^-} f(x)$

(k) $\lim\limits_{x \to 6^+} f(x)$

(l) $\lim\limits_{x \to 6} f(x)$

4. CONTINUITY(MCV)

a. Copy/Paste the following

A function f is **continuous** at a number a if the following conditions are met:

1. $f(a)$ exists
2. $\lim\limits_{x \to a} f(x)$ exists
3. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

NOTES: Don't confuse continuity and existence of a limit. Continuity implies the limit exists but not vise versa.

Continuous functions on their domains are:

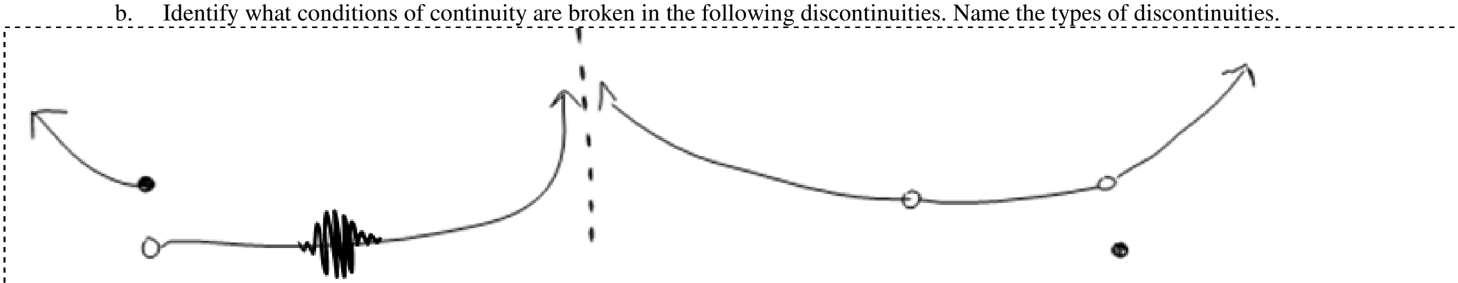
Polynomial, trig, rational, irrational, exponential, logarithmic. Any sum, difference or product of the above functions will also defined on their domains so you may use **DIRECT SUBSTITUTION** to find the limit for any x *in the domain*.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

CAUTION:

- Quotients will be defined provided we don't get division by zero upon evaluating the limit.
- For one sided graphs like root functions be careful in using the direct substitution, since it will look like it works but at the edge – only one sided limit exists.

b. Identify what conditions of continuity are broken in the following discontinuities. Name the types of discontinuities.



c.
Find the intervals of continuity of the following (ie. find the domain since all of these functions are continuous on their domains.)

$$f(x) = \sqrt{x^2 - 4} + \frac{1}{\sqrt{x+1}} - 3\cos^2 x$$

d.
Discuss the continuity of the following function. If the function is discontinuous at a point, state the kind of discontinuity.

$$f(x) = \begin{cases} \dfrac{x}{x^2+1} & x \le 1 \\ \dfrac{1}{x-5} & x > 1 \end{cases}$$

e.
Find the value of the constant so that the function is continuous everywhere

$$f(x) = \begin{cases} x^2 - 7, & \text{if } x \le 2 \\ 4x^3 - 3kx + 2, & \text{if } x > 2 \end{cases}$$

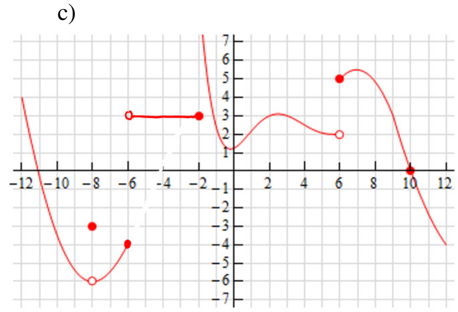
5. LIMIT LAWS find the given limits using the laws on the next page. (MCV)

a) $\lim\limits_{x \to \infty} \ln\left(\dfrac{1}{x}\right) =$

b) $f(x) = \begin{cases} 2x & \text{if } x \ne 1 \\ 5 & \text{if } x = 1 \end{cases}, g(x) = 1$

Find $\lim\limits_{x \to 0} f(g(x)) =$

c)



$\lim\limits_{x \to -8} f(f(x)) =$

$\lim\limits_{x \to 6^-} f(f(x)) =$

Copy/Paste the following.

LIMIT LAWS Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

- $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$
- $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
- If $f(x)$ is continuous then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$
- if $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = \begin{cases} f(b) & \text{if\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ \lim_{x\to b} f(x) & \text{if\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{cases}$$

<u>This last one can be confusing, so it may help to:</u>
Find the composition function sketch and look at the result
Or
Create a table and analyze input and output numbers and possibly small interval near the numbers in question

x	$g(x)$	$f(g(x))$

d) $\lim_{x\to 1} x^2 + \log x =$

e) $\lim_{x\to \pi} 7\cos x =$

f) $\lim_{x\to 0} e^x \sin(x - \frac{\pi}{2}) =$

g) $\lim_{x\to 1} \frac{3^x}{x-2} =$

h) $\lim_{x\to -1} (|x| - 4)^2 =$

6. INDETERMINATE FORMS(MCV)
   a. If $a \neq 0$ or $a \neq \pm\infty$ then group the following forms into the following categories by colour: i) *yellow* defined, ii) *green* undefined but can determine without more algebra, iii) *pink* indeterminate without further algebraic manipulation. For *yellow* and *green* record underneath what are the equivalent/approaching values?

$$\frac{0}{0},\ \frac{a}{0},\ \frac{0}{a}, \infty\text{-}\infty,\ a-\infty,\ \frac{\infty}{\infty},\ \frac{a}{\infty},\ \frac{\infty}{a},\ \frac{0}{\infty},\ \frac{\infty}{0},\ a\cdot 0,\ a\cdot\infty,\ 0\cdot\infty,\ \infty\cdot\infty,\ 1^{\infty},\ 0^{\infty},\ \infty^{\infty},\ \infty^{0},\ 0^{0},\ a^{0}$$

   b. Copy/Paste the following strategies to try, record key types of forms that work for the strategies

<u>Algebraic methods</u> - THINGS to try, if sketching it is not practical:
- Always try direct substitution first (this may be misleading if function is not continuous or is one sided at the point of interest so be careful for roots, logs and piecewise types)
- Factor or Expand to get the 'problem' to cancel
- LCD
- Change of Variable
- Rationalize or use Conjugate (for top AND bottom)
- For absolute value use $|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$ (be careful where the split occurs, if you need to use both pieces or just one)
- Divide by highest power and use the fact that $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for $n \geq 1$
- Compare rates of growth to see if numerator overtakes denominator in the long run (in order from slow to fast growth: constant, logs, roots $\sqrt[3]{x}$, polynomial x^3, exponential 3^x, factorial $x!$, aleph x^x)
- Use squeeze theorem
- Use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Identify the form, then apply the method to solve each limit

c. Factoring method $\lim_{x\to 1} \frac{x^3 - x^2 - x + 1}{x^3 - 1}$	d. LCD method $\lim_{x\to 2} \frac{\frac{1}{x^2-3} - 1}{x^2 - 4}$	e. Rationalizing method $\lim_{x\to 2^+} \frac{\sqrt{x-2}}{x^2-4}$
f. Change of variable $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4}$ what other method would have worked here?	g. Rewriting as piecewise $\lim_{x\to 2} \frac{\lvert x\rvert - 2}{x-2}$	h. Dividing by the highest power $\lim_{x\to -\infty} \frac{\sqrt{4x^2+1}}{x}$

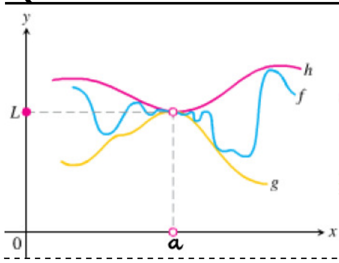
i. Divide by highest power	j. Compare rates of growth	k. Multiplying by the Conjugate
$\lim\limits_{x\to-\infty} \frac{1-x}{5+x^2} - 2$	$\lim\limits_{x\to\infty} \frac{\ln x}{x^2}$	$\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

7. TRIG LIMITS (AP)

a. Using the Sandwich Theorem	b. Using $\lim\limits_{x\to0}\frac{\sin x}{x}=1$	c. Using Trig identities
$\lim\limits_{x\to\infty}\frac{\sin x}{x}$	$\lim\limits_{x\to0}\frac{1-\cos x}{x}$	$\lim\limits_{\theta\to\frac{\pi}{4}}\frac{\cos\theta-\sin\theta}{\cos 2\theta}$

d. Copy/Paste the following

SQUEEZE/SANDWICH THEOREM



$$g(x) \le f(x) \le h(x)$$

and also suppose that:

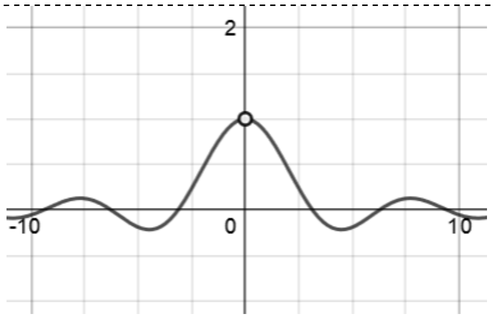
$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim\limits_{x\to a} f(x) = L.$

Useful Trig limit to use

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

Proof is available in readings online, but the truth of this fact is easily seen from the graph.



e. Discuss why sandwich theorem doesn't apply to the following problem and show why you can't use the theorem. Solve the limit using compound trig identity and the useful trig limit along with part 7b.

$$\lim_{x\to0}\frac{\sin\left[\frac{\pi}{6}+x\right]-\frac{1}{2}}{x}$$

f. Explain what is the mistake is in the following manipulation and correct it. Make a note that input of trig functions is 'untouchable'. Show a corrected solution to this

$$\lim_{x\to0}\frac{\sin 6x}{12x}$$
$$=\lim_{x\to0}\frac{\sin 6x}{12x}\cdot\frac{2}{2}$$
$$=\lim_{x\to0}\frac{\sin 12x}{12x}\cdot\frac{1}{2}$$
$$=(1)\cdot\frac{1}{2}$$
$$=\frac{1}{2}$$

8. DIFFERENCE QUOTIENT (MHF)

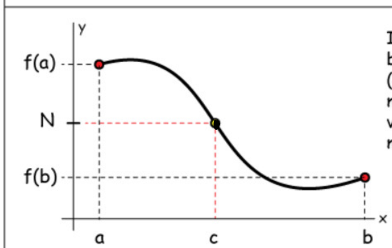
Use the difference quotient to find where the turning points are for $y=x^3-4x$. Use this information to sketch the polynomial accurately.

9. INTERMEDIATE VALUE THEOREM (AP)

a. Copy/Paste the following

The Intermediate Value Theorem

If f is a continuous function on the interval [a, b], and N is any number between f(a) and f(b), where f(a) ≠ f(b), then there is a number c in (a, b) such that f(c) = N.



If the graph of a function can be traced continuously from (a, f(a)) to (b, f(b)),then it must take on every intermediate value from f(a) to f(b) —possibly more than once — along the way.

b. For the function $f(x)=\begin{cases}5, & 4<x\le10\\ (x-2)^2, & x=4\end{cases}$. Find $f(4)$ and $f(10)$. Does the IVT guarantee a y-value u on $4\le x\le10$ such that $f(4)<u<f(10)$? Why or why not. Sketch the graph of $f(x)$ for added visual proof.

c. Use IVT to answer the question. "Is there any real number that is exactly one less than its cube?"

d. Prove that the equation $e^{-x} + 2 = x$ has at least one real solution.