1. Use the Intermediate Value Theorem to show that there is a positive number $c$ such that $c^2 = 2$.

2. If $f(x) = x^3 - x^2 + x$, show that there is $c \in \mathbb{R}$ such that $f(c) = 10$.

3. If $g(x) = x^5 - 2x^3 + x^2 + 2$, show that there is $c \in \mathbb{R}$ such that $g(c) = -1$.

In problems 4–7, use the Intermediate Value Theorem to show that there is a root of the given equation in the given interval.

4. $x^3 - 3x + 1 = 0$, $(0,1)$

5. $x^5 - 2x^4 - x - 3 = 0$, $(2,3)$

6. $x^3 + 2x = x^2 + 1$, $(0,1)$
7. \( x^2 = \sqrt{x + 1}, \ (1, 2) \)

8. Let \( f \) be a continuous function on \([0, 1]\). Show that if \(-1 \leq f(x) \leq 1\) for all \( x \in [0, 1] \) then there is \( c \in [0, 1] \) such that \([f(c)]^2 = c\).
1. Use the Intermediate Value Theorem to show that there is a positive number $c$ such that $c^2 = 2$.

Solution: Let $f(x) = x^2$. Then $f$ is continuous and $f(0) = 0 < 2 < 4 = f(2)$. By the IVT there is $c \in (0, 2)$ such that $c^2 = f(c) = 2$.

2. If $f(x) = x^3 - x^2 + x$, show that there is $c \in \mathbb{R}$ such that $f(c) = 10$.

Solution: $f(0) = 0$ and $f(3) = 27 - 9 + 3 = 21$, so $f(0) < 10 < f(3)$. Since $f$ is continuous everywhere, there must be $c \in \mathbb{R}$ such that $f(c) = 10$.

3. If $g(x) = x^5 - 2x^3 + x^2 + 2$, show that there is $c \in \mathbb{R}$ such that $g(c) = -1$.

Solution: First, $g$ is continuous everywhere. Then $g(0) = 2$ and $g(-2) = -32 + 2(-8) + 4 + 2 = -32 + 22 = -10$. So $g(-2) < -1 < g(0)$. By IVT there is $c \in (-2, 0)$ such that $g(c) = -1$.

In problems 4–7, use the Intermediate Value Theorem to show that there is a root of the given equation in the given interval.

4. $x^3 - 3x + 1 = 0$, $(0, 1)$

Solution: Let $f(x) = x^3 - 3x + 1$. Then $f$ is continuous everywhere, $f(0) = 1$, and $f(1) = 1 - 3 + 1 = -1$. Therefore $f(1) < 0 < f(0)$ and by the IVT there is $x \in (0, 1)$ such that $f(x) = 0$.

5. $x^5 - 2x^4 - x - 3 = 0$, $(2, 3)$

Solution: Let $f(x) = x^5 - 2x^4 - x - 3$. Then $f$ is continuous everywhere, $f(2) = 32 - 32 - 2 - 3 = -5$, and $f(3) = 243 - 162 - 3 - 3 = 75$. Therefore $f(2) < 0 < f(3)$ and by the IVT there is $x \in (2, 3)$ such that $f(x) = 0$.

6. $x^3 + 2x = x^2 + 1$, $(0, 1)$

Solution: Let $f(x) = x^3 - x^2 + 2x - 1$. Then $f$ is continuous everywhere, $f(0) = -1$, and $f(1) = 1 - 1 + 2 - 1 = 1$. Therefore $f(0) < 0 < f(1)$ and by the IVT there is $x \in (0, 1)$ such that $f(x) = 0$. But then $x^3 - 2x^2 + 2x - 1 = 0$ or $x^3 + 2x = x^2 + 1$. 

7. \( x^2 = \sqrt{x + 1}, \ (1, 2) \)

*Solution:* Let \( f(x) = x^2 - \sqrt{x + 1} \). Then \( f \) is continuous for all \( x > -1 \), \( f(1) = 1 - \sqrt{2} \), and \( f(2) = 4 - \sqrt{3} \). Therefore \( f(1) < 0 < f(2) \) and by the IVT there is \( x \in (1, 2) \) such that \( f(x) = 0 \). But then \( x^2 - \sqrt{x + 1} = 0 \) or \( x^2 - \sqrt{x + 1} \).

8. Let \( f \) be a continuous function on \([0, 1]\). Show that if \(-1 \leq f(x) \leq 1\) for all \( x \in [0, 1] \) then there is \( c \in [0, 1] \) such that \( [f(c)]^2 = c \).

*Solution:* If \( f(x) \) is continuous on \([0, 1]\) then so is \( [f(x)]^2 \). Set \( g(x) = [f(x)]^2 - x \). Then \( g \) is also continuous on \([0, 1]\). Now \( g(0) = [f(0)]^2 - 0 = [f(0)]^2 \geq 0 \) and \( g(1) = [f(1)]^2 - 1 \leq 0 \), so by IVT there is \( c \in [0, 1] \) such that \( g(c) = 0 \). Then \( [f(c)]^2 - c = 0 \) or \( [f(c)]^2 = c \).