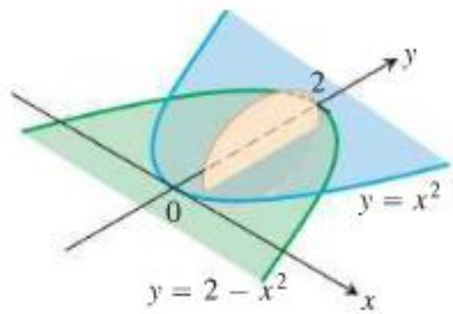
Exercises *Section* 1.1 – Volumes Using Cross-Sections

1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

2. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.



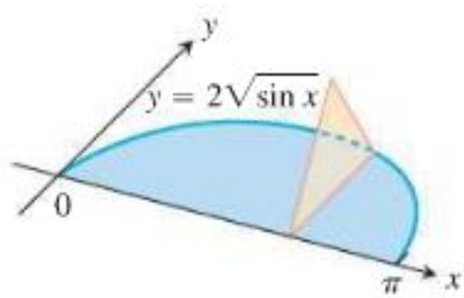
3. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. Find the volume of the solid.

4. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

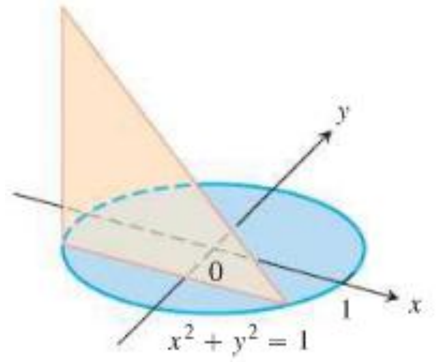
a) Equilateral triangles with bases running from the x-axis to the curve as shown

b) Squares with bases running from the x-axis to the curve.

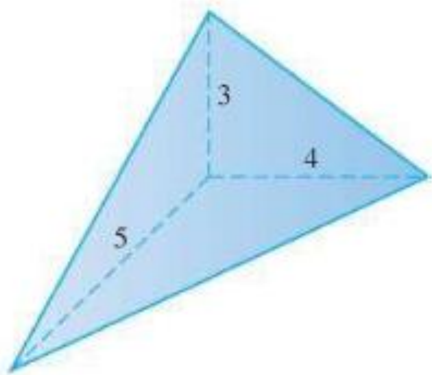
Find the volume of the solid.



5. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



6. Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)

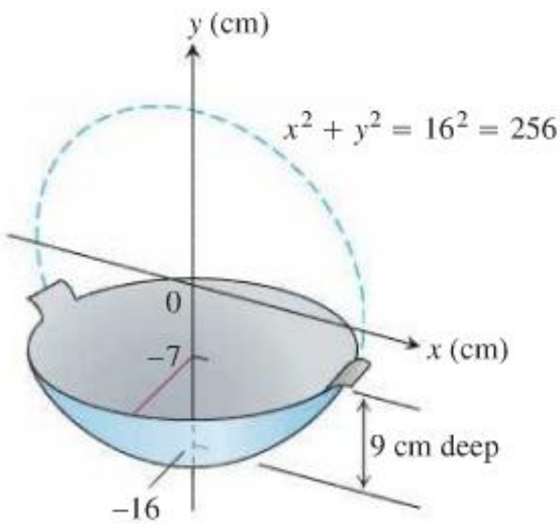


12. Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2\sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}$, $x = 0$ about the y-axis.

13. Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2$, $x = 0$ about the x-axis.

14. Find the volume of the solid generated by revolving the region bounded by $y = \sec x$, $y = \tan x$ and the lines $x = 0$, $x = 1$ about the x-axis.

15. You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\,L = 1{,}000\ cm^3\right)$



16. Find the volume of a solid ball having radius a.

Solution   *Section* 1.1 – Volumes Using Cross-Sections

Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

Solution

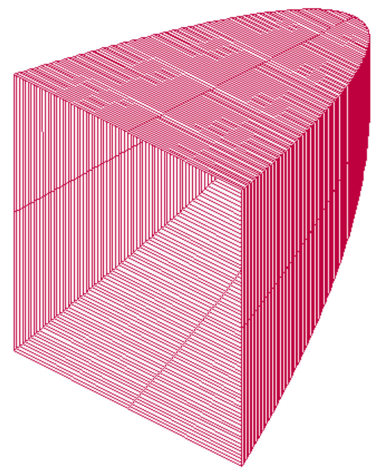
$$A(x) = \frac{1}{2}(diagonal)^2$$

$$= \frac{1}{2}\left(\sqrt{x} - \left(-\sqrt{x}\right)\right)^2$$

$$= \frac{1}{2}\left(2\sqrt{x}\right)^2$$

$$= \frac{1}{2}(4x)$$

$$= 2x \qquad a = 0, \quad b = 4;$$



$$V = \int_a^b A(x)\,dx$$

$$= \int_0^4 2x\,dx$$

$$= \left[x^2\right]_0^4$$

$$= 4^2 - 0$$

$$= 16$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

Solution

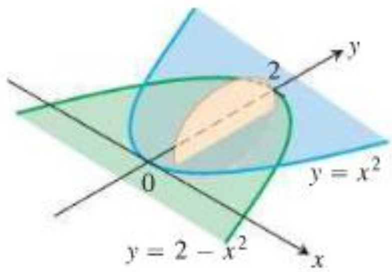
$$y = 2 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 2 \to x^2 = 1 \quad \Rightarrow \boxed{x = \pm 1}$$

$$A(x) = \frac{\pi}{4}(diameter)^2 = \frac{\pi}{4}\left(2 - x^2 - x^2\right)^2$$

$$= \frac{\pi}{4}\left(2\left(1 - x^2\right)\right)^2$$

$$= \frac{\pi}{4}4\left(1 - 2x^2 + x^4\right)$$

$$= \pi\left(1 - 2x^2 + x^4\right) \qquad a = -1, \quad b = 1;$$



$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} \pi\left(1 - 2x^2 + x^4\right)dx$$

$$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$$

$$= \pi\left[\left(1 - \frac{2}{3}(1)^3 + \frac{1}{5}(1)^5\right) - \left(-1 - \frac{2}{3}(-1)^3 + \frac{1}{5}(-1)^5\right)\right]$$

$$= \pi\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

$$= 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{16\pi}{15}$$

Exercise

The solid lies between planes perpendicular to the x-axis at x = –1 and x = 1. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$. Find the volume of the solid.

Solution

$$A(x) = (\text{base})^2$$

$$= \left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right)^2$$

$$= \left(2\sqrt{1-x^2}\right)^2$$

$$= 4(1-x^2) \qquad a = -1 \qquad b = 1$$

$$V = \int_a^b A(x)dx$$

$$= \int_{-1}^{1} 4(1-x^2)dx$$

$$= 4\int_{-1}^{1} 1 - x^2 dx$$

$$= 4\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= 4\left[1 - \frac{1}{3}(1)^3 - \left(-1 - \frac{1}{3}(-1)^3\right)\right]$$

$$= 4\left(2 - \frac{2}{3}\right)$$

$$= \frac{16}{3}$$

Exercise

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval [0, π] on the *x*-axis. The cross-sections perpendicular to the *x*-axis are

a) Equilateral triangles with bases running from the *x*-axis to the curve as shown

b) Squares with bases running from the *x*-axis to the curve.

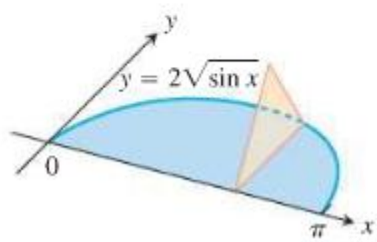
Find the volume of the solid.

Solution

a) $A(x) = \frac{1}{2}(side)(side) \cdot \sin\frac{\pi}{3}$ ***Equilateral triangle*** $\theta = \frac{\pi}{3}$

$$= \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}\sin x \qquad a = 0, \quad b = \pi;$$



$$V = \sqrt{3}\int_0^{\pi} \sin x dx$$

$$= \sqrt{3}\left[-\cos x\right]_0^{\pi}$$

$$= -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$= 2\sqrt{3}$$

b) $A(x) = (side)^2 = \left(2\sqrt{\sin x}\right)^2 = 4\sin x \qquad a = 0, \quad b = \pi;$

$$V = 4\int_0^{\pi} \sin x dx$$

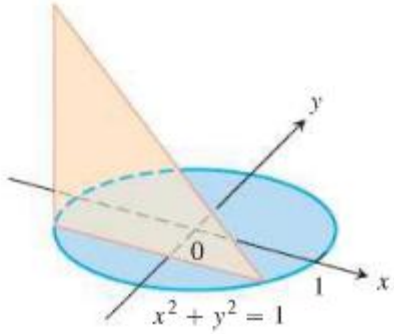
$$= 4\left[-\cos x\right]_0^{\pi}$$

$$= -4\left[\cos\pi - \cos 0\right]$$

$$= 8$$

Exercise

The base of the solid is the disk $x^2+y^2\le 1$. The cross-sections by planes perpendicular to the y-axis between $y=-1$ and $y=1$ are isosceles right triangles with one leg in the disk.



Solution

$$x^2+y^2=1 \quad\to\quad x^2=1-y^2 \Rightarrow \boxed{x=\pm\sqrt{1-y^2}}$$

$$A(x)=\tfrac{1}{2}(leg)(leg)=\tfrac{1}{2}\left[\sqrt{1-y^2}-\left(-\sqrt{1-y^2}\right)\right]^2$$

$$=\tfrac{1}{2}\left[2\sqrt{1-y^2}\right]^2$$

$$=2\left(1-y^2\right) \qquad c=-1,\quad d=1;$$

$$V=\int_c^d A(y)\,dy$$

$$=\int_{-1}^{1} 2\left(1-y^2\right)dy$$

$$=2\left[y-\tfrac{1}{3}y^3\right]_{-1}^{1}$$

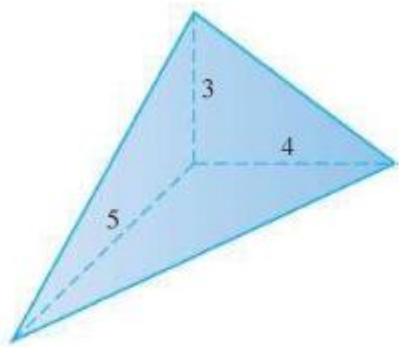
$$=2\left[\left(1-\tfrac{1}{3}1^3\right)-\left((-1)-\tfrac{1}{3}(-1)^3\right)\right]$$

$$=4\left(1-\tfrac{1}{3}\right)$$

$$=\frac{8}{3}$$

Exercise

Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)



Solution

Let consider the slices perpendicular to edge labeled 5 are triangles.

By similar triangles, we have: $\frac{height}{base} = \frac{h}{b} = \frac{3}{4} \quad \Rightarrow \quad h = \frac{3}{4}b$

The equation of the line through (5, 0) and (0, 4) is: $y = \frac{4-0}{0-5}(x-0)+4 \quad \to \quad y = -\frac{4}{5}x+4$

Therefore, the length of the base: $b = -\frac{4}{5}x+4$

$$h = \frac{3}{4}b = \frac{3}{4}\left(-\frac{4}{5}x+4\right) = -\frac{3}{5}x+3$$

$$A(x) = \frac{1}{2}(base)\cdot(height) = \frac{1}{2}\left(-\frac{4}{5}x+4\right)\left(-\frac{3}{5}x+3\right)$$

$$= \frac{1}{2}\left(\frac{12}{25}x^2 - \frac{24}{5}x + 12\right)$$

$$= \frac{6}{25}x^2 - \frac{12}{5}x + 5$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_0^5 \left(\frac{6}{25}x^2 - \frac{12}{5}x + 5\right)dx$$

$$= \left[\frac{2}{25}x^3 - \frac{6}{5}x^2 + 5x\right]_0^5$$

$$= \left[\frac{2}{25}(5)^3 - \frac{6}{5}(5)^2 + 5(5)\right]$$

$$= 10$$