Exercises *Section* 1.1 – Rates of Change and Tangents to Curves

1. Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

2. Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

3. Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

4. Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

5. Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this P.

6. Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this *P*.

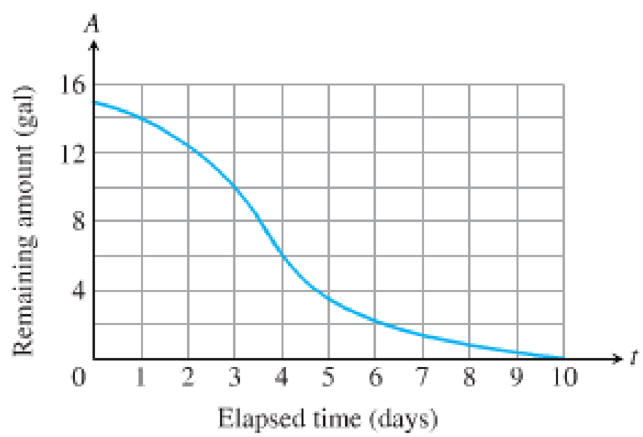
7. Make a table of values for the function $f(x) = \dfrac{x+2}{x-2}$ at the points

$x = 1.2,\quad x = \dfrac{11}{10},\quad x = \dfrac{101}{100},\quad x = \dfrac{1001}{1000},\quad x = \dfrac{10001}{10000},\quad and\ x = 1$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

8. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1,\ t = 4,\ and\ t = 8$

c) Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

Solution *Section* 1.1 – Rates of Change and Tangents to Curves

Exercise

Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2}$$

$$= \frac{3^3 + 1 - \left(2^3 + 1\right)}{1}$$

$$= 27 + 1 - (8 + 1)$$

$$= 19$$

Exercise

Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$= \frac{1^2 - (-1)^2}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

Exercise

Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$$= \frac{2 + \cos\pi - \left(2 + \cos(-\pi)\right)}{2\pi}$$

$$= \frac{2 - 1 - (2 - 1)}{2}$$

$$= 0$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 3 - (2^2 - 3)}{h}$$

$$= \frac{4 + 4h + h^2 - 3 - (4-3)}{h}$$

$$= \frac{4h + h^2}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$= 4 + h$$

As *h* approaches 0. Then the secant slope $h + 4 \to 4 = slope$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 4(x - 2)$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\boxed{y = 4x - 7}$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this P.

Solution

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^2 - 2(2+h) - 3 - \left(2^2 - 2(2) - 3\right)}{h} \\
&= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\
&= \frac{2h + h^2}{h} \\
&= 2 + h
\end{aligned}$$

As h approaches 0. Then the secant slope $2 + h \to 2 = slope$

$$y - y_1 = m\left(x - x_1\right)$$

$$y + 3 = 2(x - 2)$$

$$y = 2x - 4 - 3$$

$$\boxed{y = 2x - 7}$$

Exercise

Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this P.

Solution

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^3 - 2^3}{h} \\
&= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= 12 + 6h + h^2
\end{aligned}$$

As h approaches 0. Then $slope = 12$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 8 = 12(x - 2)$$

$$y = 12x - 24 + 8$$

$$\boxed{y = 12x - 16}$$

Exercise

Make a table of values for the function $f(x)=\dfrac{x+2}{x-2}$ at the points

$$x=1.2, \quad x=\frac{11}{10}, \quad x=\frac{101}{100}, \quad x=\frac{1001}{1000}, \quad x=\frac{10001}{10000}, \quad \textit{and } x=1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x\neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x=1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	-3

$$\frac{\Delta y}{\Delta x}=\frac{-4-(-3)}{1.2-1}=-5.0$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{4}-(-3)}{1.1-1}=-4.\overline{4}$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{04}-(-3)}{1.01-1}=-4.\overline{04}$$

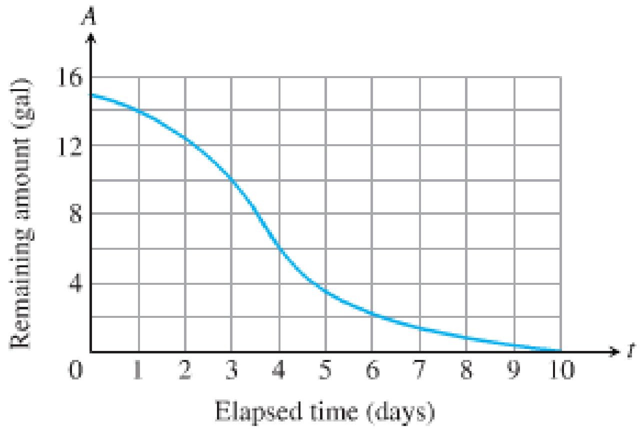
$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{004}-(-3)}{1.001-1}=-4.\overline{004}$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{0004}-(-3)}{1.0001-1}=-4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x=1$ is $\underline{-4}$

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ and\ t=8$

c) Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

Solution

a) Average rate of gasoline consumption over the time intervals:

$$[0,\ 3] \Rightarrow \quad Average\ Rate = \frac{10-15}{3-0} \approx = -1.67\ gal\,/\,day$$

$$[0,\ 5] \Rightarrow \quad Average\ Rate = \frac{3.9-15}{3-0} \approx -2.2\ gal\,/\,day$$

$$[7,\ 10] \Rightarrow \quad Average\ Rate = \frac{0-1.4}{10-7} \approx -0.5\ gal\,/\,day$$

b) At $t=1 \quad \rightarrow P(1,\ 14)$