Big idea

Ever wonder where equations like $A = Pe^{rt}$ or $h = \frac{1}{2}gt^2 + v_0 t + h_0$ come from? These equations arise from solutions to differential equations. A **differential equation** is an equation that contains an unknown function and some of its derivatives.

Perhaps the most important of all the applications of calculus is differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. This is not surprising because in a real-world problem we often notice that changes occur and we want to predict future behavior on the basis of how current values change. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.

Feedback & Assessment of Your Success

			Finished assignment pages? **Made corrections?**	**Summarized notes in a journal?** **Added your own explanations?**	**How many extra practice questions did you try in each topic?**	**Tentative TEST date:** ____________________
Date	**Pages**	**Topics**				**Questions to ask the teacher:**
	2-5	Slope Fields (AB) Journal #1				
	6-9	Separable Differential Equations (AB) Journal #2				
0.5day	10-11	Euler's Method for Solving DE (BC) Journal #3				
	12-13	Integration by Parts (BC) Journal #4				
	14-17	Integration by Partial Fractions (BC) Journal #5				
0.5day	18-19	Trig Integrals (BC) Journal #6				
	20-23	Integration by Trig Sub (BC) Journal #7				
3days	24-32	Applications (AB) Journal #8				

ASSIGNMENT Slope Fields (AB)

1. A **Differential Equation** is an equation that contains an unknown function and one or more of its _______________________.

Ordinary DE is where the function(s) are in terms of ONE independent variable ___________

Partial DE involve functions of multiple independent variables and partial derivatives _________

The **order** of a differential equation is the order of the _________________ derivative that occurs in the equation.

Ex.1 $\frac{dy}{dx} = 6\sin x + 4x$

Ex.2 $y''' + \sin y = 7x + y$

Ex.3 $\frac{\partial u}{\partial x} + x = \frac{\partial^2 u}{\partial y^2}$

The **general solution** means _____________.

The **particular solution** requires an initial condition and means we have to ________________ .

A **separable DE** is one where you can separate ________________________.

Some can solve by inspection:

Ex.4 $y' = y$ Ex.5 $y'' = -y$

Antidifferentiation = Indefinite Integration = Solve DE

Many times, differential equations are NOT explicit functions of a single variable, and sometimes they are not even solvable by analytic methods like separation of variables or other methods. Fear not, there is a way to graphically solve such (and any) differential equation.

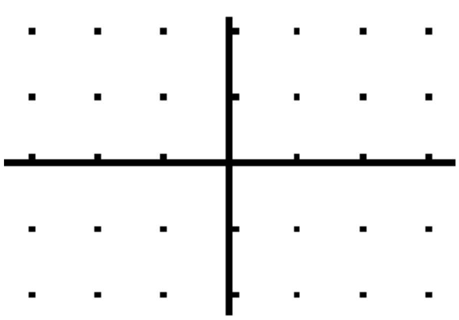
A **direction field** (or slope field or trajectory space or vector field or flow field) shows the ____________ at given points, and is a graphical general solution to a DE.

A solution is **unique** if no two curves in a trajectory space, or slope field ____________________. Imagine several functions whose only difference is a different C value, so that all the graphs are "parallel" and differ only by a vertical shift

Drawing a slope field:

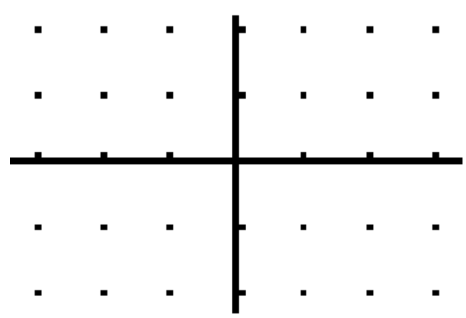
- To every point (x, y) in the domain of f, assign a small piece of a tangent line with a slope of $f'(x, y)$.
- Be sure your slopes of 0, 1, – 1 and ∞ are spot on. All other slopes must be at a steepness relative to these slopes and the others around it.
- When drawing a piece of the tangent line at a point, draw the line long enough to see, but not so long that it interferes with the other tangent lines.

2. Draw the slope field $y' = -\frac{x}{y}$



Solve algebraically given point (1, 1)

3. Draw the slope field $y' = \frac{1}{2}y$

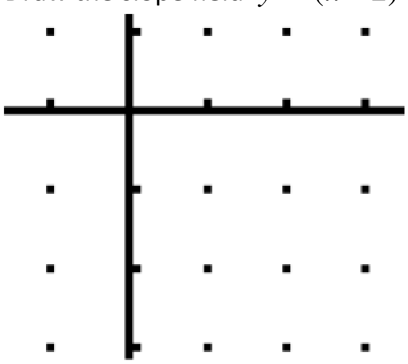


Solve algebraically given point (2, 1)

4. Verify that $y = e^{-x}$ satisfies the differential equation $3y' + 4y = e^{-x}$

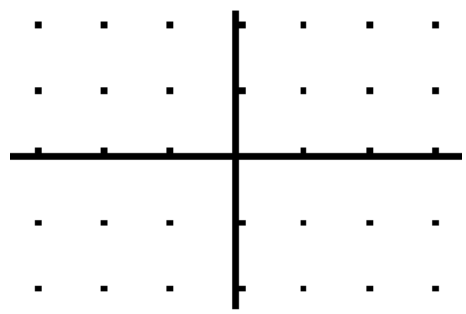
5. For what values of r is $y = e^{rt}$ a solution of $y'' + 2y' - 3y = 0$? where $y(t)$

6. Draw the slope field $y' = (x-2)^2(y+3)$



Solve algebraically given point (2, 1)

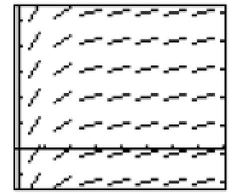
7. Sketch the graph of the solution of the initial value problem $y' = x + y,\ \ y(0) = 1$



Try solving algebraically to see impossible to separate variables.

8\.

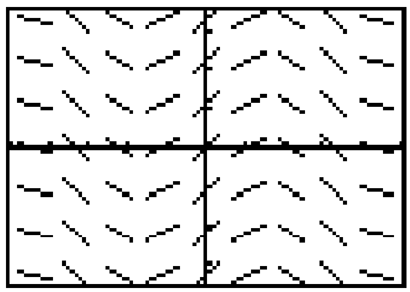
The slope field from a certain differential equation is shown at right. Which of the following could be a specific solution to that differential equation?



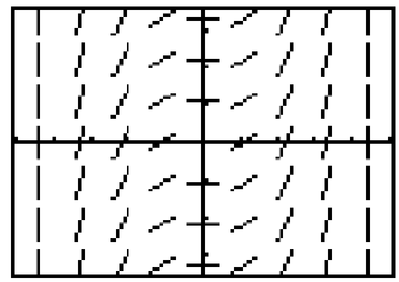
(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$ (D) $y = \cos x$ (E) $y = \ln x$

9\. Match each slope field with the equation of the general solution it could represent.

I.



II.

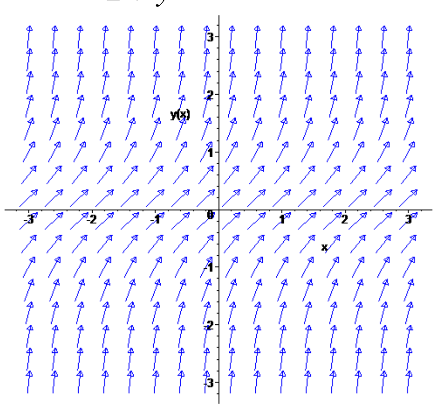


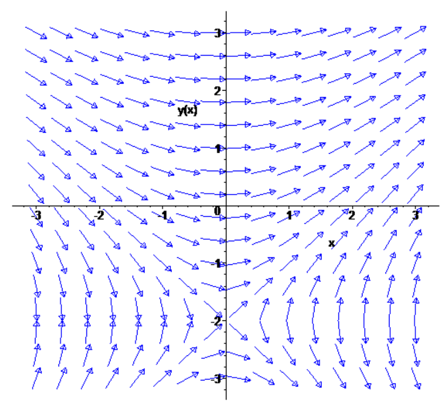
(A) $y = 1$ (B) $y = x$ (C) $y = x^2$ (D) $y = \dfrac{x^3}{6}$ (E) $y = \dfrac{1}{x^2}$ (F) $y = \sin x$

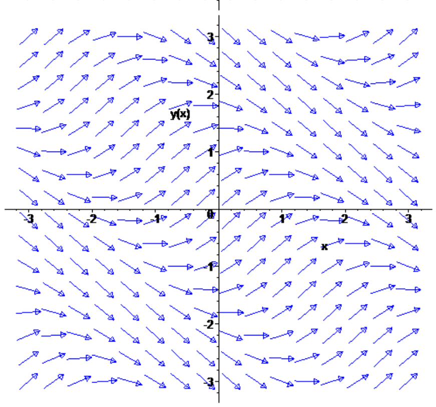
(G) $y = \cos x$ (H) $y = \ln|x|$

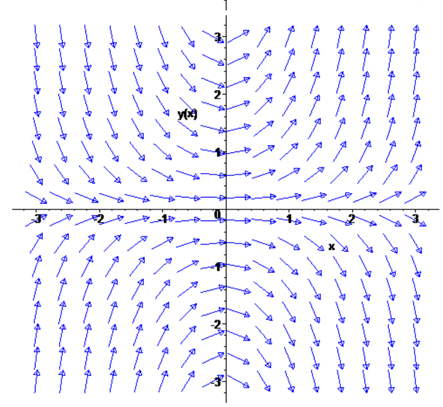
10\. Match each of the following differential equations with the corresponding direction field.

a) $y' = \dfrac{x}{2+y}$ b) $y' = \cos(x+y)$ c) $y' = 1 + y^2$ d) $y' = x\,y$







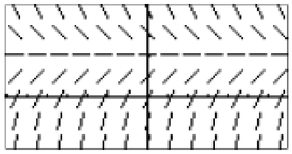


Things to look for:

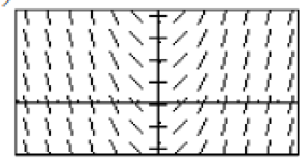
- Plug points in and check the resulting slope
- Does the function have an x and/or y dependency
- Look for x and/or y values that make slope 0 or undefined
- Is the function always inc/dec (ie,.slope always pos/neg)

11. Match the slope fields with their differential equations.

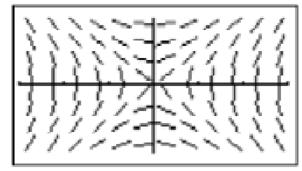
(A)



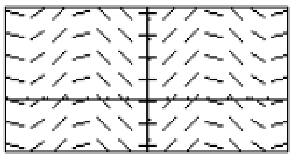
(B)



(C)

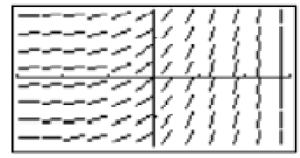


(D)



(E)

(F)



I. $\frac{dy}{dx}=e^x$ II. $\frac{dy}{dx}=\frac{x}{y}$ III. $\frac{dy}{dx}=2-y$ IV. $\frac{dy}{dx}=x$ V. $\frac{dy}{dx}=x-y$ VI. $\frac{dy}{dx}=\sin x$

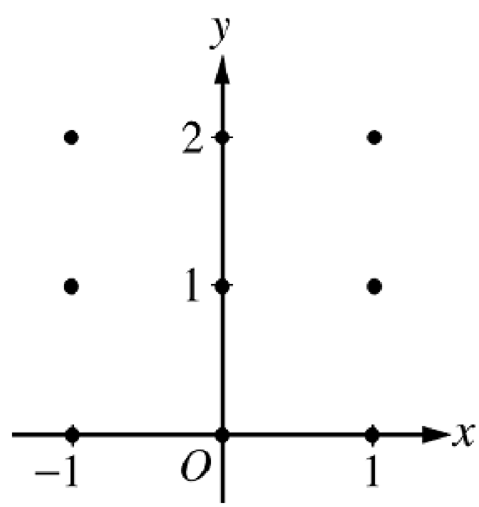
12. AP 2007B-5 Consider the differential equation $\frac{dy}{dx}=\frac{1}{2}x+y-1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y=f(x)$ be a particular solution to the differential equation with the initial condition $f(0)=1$. Does f have a relative minimum, a relative maximum, or neither at $x=0$? Justify your answer.

(d) Find the values of the constants m and b, for which $y=mx+b$ is a solution to the differential equation.



ASSIGNMENT Separable Differential Equations (AB)

1. A **separable DE** is one that can be written in the form $\frac{dy}{dx} = M(x)N(y)$ where M is a continuous function of x alone and N is a continuous function of y alone.

   Solve $y' = \frac{y}{x}$ given point y(2)=1

2. Solve $y' = x^2(y-2)$ given point y(0)=0

3. Solve $y' = \frac{2x}{y}$ given point y(3)=2

4. Solve $y' = 2y$ given point y(3)=5

5. Note there is more than one solution:

Solve $\frac{dy}{dx} = y^2 - 4$

6. Find the explicit solutions of the equation $y' = \dfrac{e^x}{1+y}$ having initial conditions $y(0)=1$ and $y(0)=-4$

7. Solve $yy' - 6\cos(\pi x) = 0$

8. Solve $\frac{\ln x}{y} - xy' = 0$

9. Find the equation of the graph that passes through the point and has the given slope.

(8,2), $y' = \frac{2y}{3x}$

10. A calf weighs 45 pounds at birth and gains weight according to $\frac{dw}{dt} = k(1100 - w)$ where w is the weight and t is the time in years. In 3 months, it weighs 180 pounds. Solve for w.

ASSIGNMENT Euler's Method (BC)

1. For differential equations that cannot be solved symbolically with ____________________________, a ______________________ can be a graphical solution to that differential equation. The problem with this approach is that this is only really good for getting general trends in solutions and for long-term behavior of solutions. There are times when we will need something more. Maybe we need to determine how a specific solution behaves, including some values that the solution will take. In these cases we must resort to numerical methods such as ________________________ that will allow us to approximate solutions to differential equations.

Euler's Method basically involves "walking out along a tightrope" from an initial point along it's tangent line. Instead of walking along the same line the whole time (as in a tangent line approximation), we change tangent lines with each step (of length Δx). This involves recalculating the point and slope after each step.

Recall that $m=\dfrac{\Delta y}{\Delta x}$ which rearranged becomes

$$\Delta y = m\Delta x$$

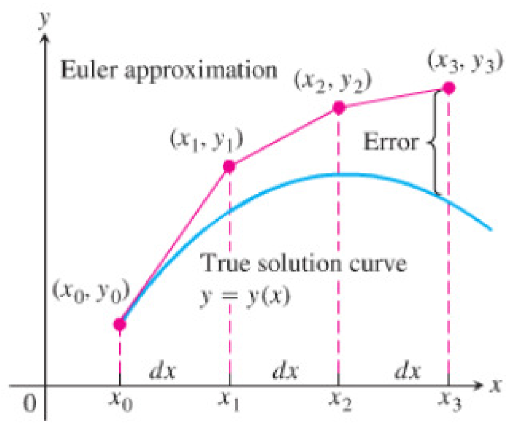
$$\Delta y = \left.\frac{dy}{dx}\right|_{(x_n, y_n)} \Delta x$$

Given $\dfrac{dy}{dx}$ and initial condition, y-value at $x=a$ you will be asked to find the y-value at $x=b$.

- Choose the number of steps, n,
- Find step size $\Delta x = \dfrac{b-a}{n}$ and $x_k = a + k\Delta x$
- Let $a = x_0$ as the initial condition $y(x_0) = y_0$ then

$$y_{n+1} = y_n + \Delta y$$

$$y_{n+1} = y_n + \Delta x \left.\frac{dy}{dx}\right|_{(x_n, y_n)}$$



2. Find the approximate value of the solution to the initial value problem $y' = y + x$, $y(0) = 1$ using Euler's method with $\Delta x = 0.25$ at $x = 1$

x_n	y_n	$m = \left.\dfrac{dy}{dx}\right\|_{(x_n, y_n)}$	$\Delta y = m\Delta x$	$y_{n+1} = y_n + \Delta y$

3.

Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

x_n	y_n	$m = \left.\frac{dy}{dx}\right\rvert_{(x_n, y_n)}$	$\Delta y = m\Delta x$	$y_{n+1} = y_n + \Delta y$

4.

Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2		

ASSIGNMENT Integration by Parts (BC)

1. Recall you can do the following

$\int x(x+1)dx$

$\int x\sin(x^2)dx$

What about these?

$\int x\cos x dx$

Product Rule of two functions of x rearranged:

Integration by Parts Formula:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

or, alternatively

$$\int udv = uv - \int vdu$$

Use the **L.I.P.E.T.** technique when wondering what to choose for **u**. Follow the order by letting u be:

L = natural logarithm
I = inverse trigonometric function
P = polynomial
E = exponential
T = trigonometric

2. $\int x\cos x dx$ try u=x

3. $\int x\cos x dx$ try u=cosx

4. $\int 2xe^{4x}dx$

5. $\int x\ln x dx$

6. $\int \ln x dx$

7. $\int \arctan x dx$

8. $\int x^2 \sin x dx$

Tabular method

We determine the elements in the "D" column by repeatedly **differentiating** and the elements in the "I" column by repeatedly **integrating** (we note **I(g)** as anti-derivative of **g(x)**). The operations preceding each row are alternating between "plusses" and "minuses".

	D		I
+	$f(x)$		$g(x)$
		↘	
−	$f'(x)$		$I(g)$
		↘	
+	$f''(x)$		$I^2(g)$
		↘	
⋮	⋮	⋮	⋮
$\pm\int$	$f^n(x)$	→	$I^n(g)$

This process is continued until one of the following conditions is met.

- The function on the left becomes zero. (it will always do this if f is a polynomial)
- The product on the bottom row can be easily integrated.
- The product on the bottom row is just a constant multiplier of the product of the top row.

9. Repeat questions 7. and 8. with tabular method

10. $\int t^4 e^{2t} dt$

11. **Circular example:**

$\int e^x \sin x\, dx$

12. If the rate of change of medication in the bloodstream is $\frac{dA}{dt} = t^2 \cdot e^{-t}$, what is the net change in the amount of medication from time $t = 0$ to $t = 1$?

ASSIGNMENT Integration by Partial Fractions (BC)

1. Review the method of integration to apply for each of the following rational functions integrands

$\int \frac{x^3}{x^2+2}dx$ $\int \frac{xdx}{x^2-4}$ $\int \frac{dx}{x^2+4x+4}$ $\int \frac{dx}{x^2+x+1}$ $\int \frac{dx}{x^2+2x-3}$

2. Review LCD

$\frac{5}{2x-1}-\frac{2}{x-3}$

3. $\int \frac{x-13}{2x^2-7x+3}dx$

4. Shortcut:

5. Use **Partial fractions** of the form

$$\frac{A_i}{(ax+b)^i} \text{ or } \frac{A_j x + B_j}{(ax^2+bx+c)^j}$$

Ex.

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}$$

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6. $$\int\left(\frac{5x+3}{x^3-2x^2-3x}\right)dx$$

7. $$\int \frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)}\,dx$$

8. $$\int_2^3 \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}\,dx$$

ASSIGNMENT Trig Integrals (BC)

1. Use u- sub
$\int \cos x \sin^8 x \, dx$

2. Use trig identity
$\int \cos^3 x \sin^8 x \, dx$

3. Use trig identity and u-sub
$\int \sin^5 x \sqrt{\cos x} \, dx$

4. What if both powers are odd?
$\int \cos^3 x \cdot \sin^5 x \, dx$

5. What if both powers are even?
$\int \cos^2 x \sin^2 x \, dx$

6. What if trig function other than sine or cosine?
$\int \tan x \sec^3 x dx$

7. $\int \tan^2 x \sec^4 x dx$

8. $\int \cos^5 x \, dx$

9. $\int_0^{1/2} \sin^2 x \, dx$

10. $\int \tan^4 x dx$

11. $\int_0^{\pi/4} \sqrt{1+\cos 4x} \, dx$

ASSIGNMENT Integration by Trig Sub (BC)

1. $$\int \frac{x}{\sqrt{4-x^2}}\,dx$$

2.

Expression	$\sqrt{a^2-x^2}$	$\sqrt{a^2+x^2}$	$\sqrt{x^2-a^2}$
Identity			
Let x=			
Triangle			

3. $$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

4. $$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$$

5. $$\int \frac{\sqrt{x^2 - 16}}{x^6} dx$$

6. $\int \sqrt{25 - x^2}\, dx$

7. $\int \frac{x}{\sqrt{x^2 - 9}}\, dx$

8. $$\int_0^{2\sqrt{3}} \frac{x^3 dx}{\sqrt{16-x^2}}$$

ASSIGNMENT Applications of DE (AB)

1. **Exponential Decay and Growth**- the derivative is proportional to the function of time.

$$y' = ky$$

Show the general solution to this type of DE

2. Suppose that a rumour spreads according to $\frac{dP}{dt} = 1.1P$. If 3 people know the rumour initially, how many people will know the rumour as a function of time?

3. Recall in PreCalculus you solved exponential growth/decay problems using $y = ab^{\frac{x}{p}}$ and $A = Pe^{rt}$ for continuous rates. The 2nd equation is same as $y = Ce^{kx}$ we developed in question #1.

a) Solve the following using both equations $y = ab^{\frac{x}{p}}$ and $y = Ce^{kx}$.

"Polinomium has a half-life of 37 years. We now have 100 grams. How much is left after 20 years?"

b) Which method do you prefer?
c) What is the continuous rate of decay for this question?

4. Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level?

5. **Newton's Law Of Cooling** states that the rate of cooling of an object is proportional to the temperature difference between the object (medium) and its surroundings(air), provided this difference is not too large.
Write diff eq:

Solve for the general sol'n:

6. A boy stuck his tongue on a piece of metal inside (45^o) and went out in the 20^o cold. In 2 minutes the metal was 38^o. When will the metal stick to the boys tongue (it will stick at 30^o)?

7. A can of beer at 40 ^{o}F is placed into a room when the temperature is 70 ^{o}F. After 10 minutes the temperature of the beer is 50 ^{o}F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

8. At 10:07 pm, you find a secret agent murdered. Next to him is a martini that got shaken before the secret agent could stir it. Room temperature is 70^{o}F. The martini warms from 60^{o}F to 61^{o}F in the 2 minutes from 10:07 pm to 10:09 pm. If the secret agent's martinis are always served at 40^{o}F, what was the time of death?

Motion Problems Newton's 2nd law of motion for net force $F = ma$	To get velocity take antiderivative of acceleration and to get position take antiderivative of velocity $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

9. Suppose you drop a brick from the top of a building that is 250 m high. The brick has a mass of 2 kg, and the resistance force is given by $R = -4v$ where v is velocity of the brick. How long will it take the brick to reach the ground? What will be its velocity at that time?

10. A car begins to decelerate at a rate of $2m/s^2$. If its speed was 15m/s at t=0 and it decelerates at a constant rate when will it come to a full stop, and how far has it travelled?

11. Given that a sled leaves the top of a hill with a velocity of 4m/s and accelerates for 10sec at a rate of $2t\ m/s^2$. Determine the sled's velocity when it reaches a point 96m down the course.

12. The brakes of a car travelling 100kph decelerate the car at a constant rate of $6.7 m/s^2$.
    a) Find displacement function for time since brakes were applied.
    b) What is the car's speed when it hits the barrier 55m from when the brakes were applied?

13. A powerful model rocket is launched so that its acceleration for $0 \leq t \leq 20$ is described by $t^{\frac{3}{2}}$ m/s^2. If its velocity after 1sec is 20m/s and after 1sec it is 30m above the ground, determine its height after 12sec.

Mixing Problems

If $V(t)$ represents the total volume of water at time t with some impurity, like salt, dissolved in it and $A(t)$ represents the amount of impurity. If solution of certain concentration C_{in} (kg/L) or (% as decimal) of impurity is pumped in at a rate R_{in} (L/min), mixed well with contents of the tank to a new concentration $C_{out} = \frac{A(t)}{V(t)}$ and allowed to flow out at R_{out} then volume is given by:

$$\boxed{V(t) = V_0 + (R_{in} - R_{out})t}$$

and the rate of change of the amount of impurity is given by:

$$\frac{dA}{dt} = [\text{inFlowRateOf}A] - [\text{outFlowRateOf}A] \quad \text{(kg/min)}$$

$$\frac{dA}{dt} = [C_{in} \times \text{TotalVolRate}_{in}] - [C_{out} \times \text{TotalVolRate}_{out}]$$

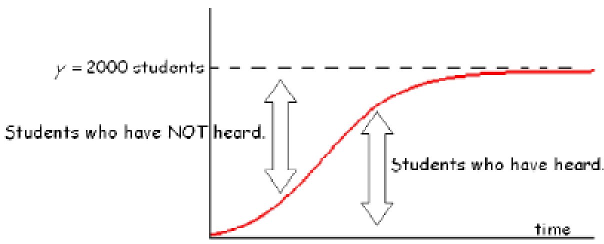
$$\boxed{\frac{dA}{dt} = C_{in}R_{in} - \frac{A}{V}R_{out}}$$

15. A room measuring 10m×5m×3m initially contains air that is free of carbon monoxide. At time t = 0, air containing 3% carbon monoxide enters the room at a rate of 1 m^3/minute, and the well-circulated air in the room leaves at the same rate.
(a) Set up, and then solve, the initial-value problem for the amount of carbon monoxide in the room at time t minutes.
(b) Find the time when the carbon monoxide concentration in the room reaches 2%

14. A tank has pure water flowing into it at 15L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10L/min. Initially, the tank contains 20kg of salt in 200L of water.
   a) Find the equation for volume
   b) Find the equation for amount of salt
   c) If the tank has a capacity of 1000L, what is the left over amount of salt in the tank at the time it starts to overflow?

16. Many things that grow exponentially cannot continue to do so indefinitely. After a while, things that start off growing exponentially begin to compete for resources like ___________________. The growth begins to taper off as it approaches some _________________________ of the system. This type of curve is called **Logistic Growth**.

Imagine a rumor spreading throughout a school of 2000 students. The rate at which the rumor spreads is directly proportional to BOTH the students who have heard the rumor AND the students who have yet to hear the rumor as the number of people hearing the rumor approaches 2000.



For quantities, y, that grow logistically with a carrying capacity of y = L , we can state the relation mathematically the following way:

$$\frac{dy}{dt} = ky\left(L - y\right)$$

17. Solve the Logistic DE in general

18. Suppose that a population develops according to the logistic differential equation $\frac{dP}{dt} = 0.2P - 0.002P^2$, where t is measured in weeks, $t \geq 0$.

If $P(0) = 5$, what is $\lim_{t\to\infty} P(t)$?

(BC)

19. The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.008P(100 - P)$, where t measured in years.

a) What is the carrying capacity for bears in this wildlife preserve?
b) What is the bear population when the population is growing the fastest?
c) What is the rate of change of population when it is growing the fastest?

20. The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

a) If $P(0) = 50$, solve for P as a function of t.
b) Use your solution to a) to find the size of the population when $t = 2$ days.
c) Use your solution to a) to find the number of days that have occurred when the flu is spreading the fastest.