Calculus UNIT 2

DERIVATIVES (MCV) – journal questions

Summarize everything you need to know about these topics. Use examples and concise (not long – but with enough detail) explanations. Include definitions and diagrams if necessary

1. THE DERIVATIVE FUNCTION (MCV)
   a. Copy/Paste the following.

The **derivative of a function *f* at a number *a***, is

Direct Definition

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Alternate Definition

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

If $f'(a)$ exists, we say that ***f* is differentiable at *a*.**

The **derivative function $f'(x)$** is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists. A function that is differentiable at every point of its domain is a **differentiable function.**

Notation

y' ➔ "y prime"

$\frac{dy}{dx}$ ➔ "*dy* by *dx*" or "the derivative of *y* with respect to *x*

$\frac{df}{dx}$ ➔ "*df* by *dx*" or "the derivative of *f* with respect to *x*

$\frac{d}{dx}f(x)$ ➔ "*d* by *dx* of *f* at *x*" or "the derivative of *f* at *x*"

Equation of Tangent line at pt.(a,b)

$m=f'(a) \qquad y-b=m(x-a)$

Equation of Secant line at points $(a_1,b_1),(a_2,b_2)$

$m=\frac{b_2-b_1}{a_2-a_1} \qquad y-b_1=m(x-a_1)$

Equation of Normal line at pt.(a,b)

$m=-\frac{1}{f'(a)} \qquad y-b=m(x-a)$

A **critical number** of a function *f* is a number *c* in the domain of *f* such that either $f'(c)=0$ or $f'(c)$ does not exist.

A point *P* on a curve $y=f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or vice versa) at *P* and $f''(c)=0$ or DNE

SKETCHING DERIVATIVES

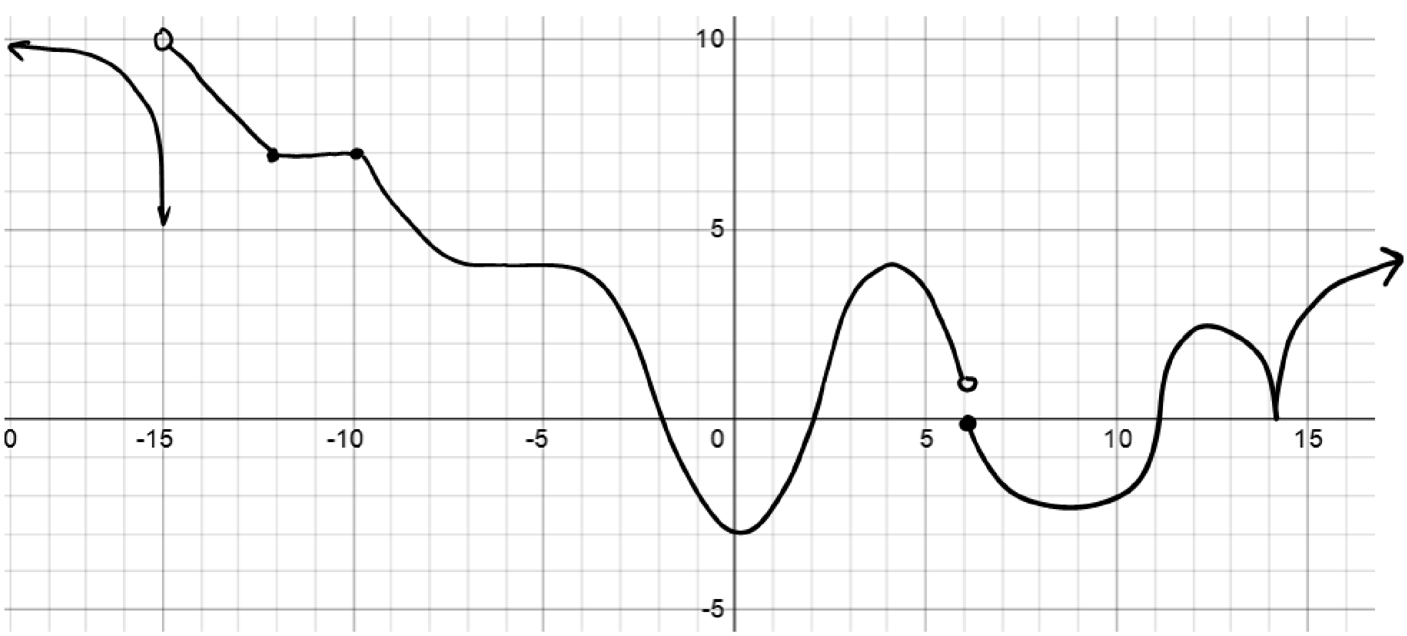
Drawing f' from f

- Split domain at the following x values: all the critical points (turning points, vertical tangents, cusps, sharp points, discontinuities) and inflection points.
- At each point visualize the tangent line and ask yourself what is the slope there, that value you'd plot as output value on the derivative graph for the same x value
- Repeat for points on either side of the above points.
- Connect. Do not worry too much about the curvature, you'll get good marks if you plot on/above/below x-axis correctly

b. Show how to find the derivative by direct definition for $f(x)=x^3$

c. Show how to find the derivative by alternate definition for $f(x)=\sqrt{x}$ at x=4

d. SKETCH the derivative graph f'(x) from f(x) graph below, Explain steps or show an organized characteristics list.



2. DERIVATIVE RULES (MCV)

a. Copy/Paste the following

Derivative of a Constant Function	$\frac{d}{dx}(c)=0$
The Power Rule (General Version) If n is any real number, then	$\frac{d}{dx}(x^n)=nx^{n-1}$
The Constant Multiple Rule If c is a constant and f is a differentiable function, then	$\frac{d}{dx}(c\cdot f(x))=c\cdot\frac{d}{dx}f(x)$
The Sum/Difference Rule If f and g are both differentiable, then	$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$

b. Discuss the reasons for each of the above laws. (see readings online if needed)

c. Find the derivative using the shortcut rules, explain the steps in words (what would you say to yourself?)

$$f(x)=x^3-4+2x^{-4}-5\sqrt{x}+x^{\sqrt{2}}$$

3. DIFFERENTIABILITY (MCV)

a. Copy/Paste the following

A function f is **differentiable** at a number a if it is continuous and smooth at a, which means these conditions must be true

$\boxed{1.}$ for continuity:

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$$

$\boxed{2.}$ for smoothness:

$$\lim_{x\to a^-} f'(x)=\lim_{x\to a^+} f'(x)$$

Differentiability Implies Local Linearity

Differentiability Implies Continuity

Theorem: If f is differentiable at a, then f is continuous at a

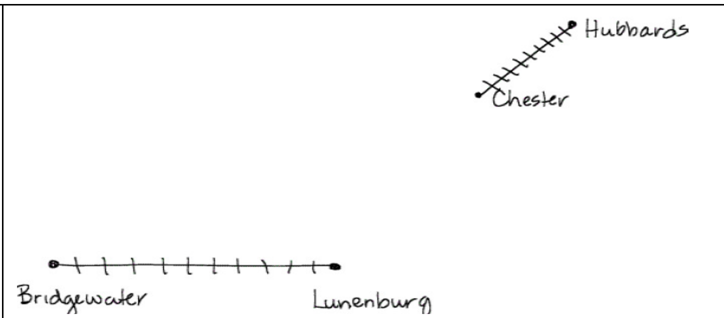
Not Differentiable

- corner: where one-sided derivatives differ. ($|x|$)
- cusp: where the slopes of the tangent lines approach ∞ from one side and -∞ from the other side (an extreme corner). ($\sqrt[3]{x^2}$)
- vertical tangent: where the slopes of the tangent lines go to either ∞ or -∞ from both sides. ($\sqrt[3]{x}$)
- any discontinuity: jump or VA or hole (rational or piecewise)

b. Is the converse of the theorem true? ie. if f is continuous at a, then f is differentiable at a? Explain, use an example.

c. The Clickety-Clack Railroad Company has laid out a track from Bridgewater to Lunenburg and also has a track in operation from Chester to Hubbards, as shown. Place origin at Lunenburg.

The railroad company wishes to connect the path smoothly. Assume that Lunenburg is due east of Bridgewater, Chester is 50 km from Lunenburg in a N30ºE direction, and Hubbards is located N45ºE from Chester.



d. Prove using limits that the function $f(x)=|x|$ is not differentiable at x=0, but differentiable at x=3.

4. CHAIN RULE (MCV)

a. Copy/Paste the following

The Chain Rule: If f and g are both differentiable and $F=f\circ g$ is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x)=f'(g(x))g'(x)$$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable functions, then

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

b. Fill in the table. Also explain the process in words of how to go from the question in 1st column to the answer in the last column without intermediate steps.

$y=f(g(x))$	$u=g(x)$	$y=f(u)$	$\frac{dy}{du}\frac{du}{dx}$ in terms of u and x	$\frac{dy}{dx}$ in terms of x only	Explanation in general
i. $y=(6x-5)^4$					
ii. $y=\sqrt{x^2-1}$					

c.

If $y = u\sqrt{1-u}$ and $u = 3x - x^2$

determine $\left.\dfrac{dy}{dx}\right|_{x=-1}$

5d Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$ (b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$ (d) $f(x) = [g(x)]^3$

5. PRODUCT + QUOTIENT RULES (MCV)
   a. Copy/Paste the following

The Product Rule: If f and g are both differentiable, then	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] = fg' + gf'$
The Quotient Rule If f and g are both differentiable, then	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} = \frac{gf' - fg'}{g^2}$

   b. Provide a proof for product and quotient rules for unknown functions f and g (hint look at the readings provided online)
   c. Find the derivative using both product rule and quotient rule. $y = (1-x)(1+x^2)^{-1} = \dfrac{1-x}{1+x^2}$. Explain in words – what would you say to yourself? how to simplify?
6. TRIG DERIVATIVES(MCV)
   a. Copy/Paste the following

Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

   b. Provide a 1st principles proof for derivative of sin(x) (hint look at the readings provided online)
   c. Explain why the above rules will not be true if x was not in radians. Ie. give the derivative of sin(x) if x is in degrees for comparison.
   d. Show that you do not have to memorize any of the above rules except the first two. Ie. show how one finds the derivative of tan(x) and csc(x) without looking up the derivative answer – by using only sine and cosine.
7. IMPLICIT DIFFERENTIATION (AP)
   a. What is the difference between implicit and explicit equations?
   b. Find derivative of *y* with respect to *x*, isolate the *y'* and record in proper form $\sqrt{x+y} + \sqrt{x-y} = 6$
   c. Find derivative of *y* with respect to *x*, $\sin(x^2y^2) = x$
   d. Find derivative of *x* with respect to *y*, $\sin(x^2y^2) = x$
8. EXPONENTIAL & LOG DERIVATIVES (MCV)

Find the derivatives:

a. $y = 2^x$ b. $y = x^2$ c. $y = e^{\pi}$ d. $y = e^{\sin x}$

e. $y = 4^{3-x}$ f. $y = \ln(\tan x)$ Simplify g. $y = \log_5(3x^2 - 1)$ h. $y = x^x$ HINT: Take ln of both sides to separate base from exponent since none of the laws apply as is. Explain this one in detail

Derivatives of Exponentials and Logarithms

$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\frac{d}{dx}a^x = a^x \ln a$	$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Definition: ***e*** is the number such that $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$

   i. Discuss the significance of the number *e* in relation to derivatives and show a proof of the derivatives of a^x and e^x.
   j. Show the proofs for derivatives of *log*'s and *ln*'s
9. INVERSE TRIG DERIVATIVES (AP)
   a. Copy/Paste the following

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

Derivatives of Inverse Functions in General at a POINT

If $g(x) = f^{-1}(x)$ then point (a,b) is on f and point (b,a) is on g

$$g'(b) = \left(\frac{dx}{dy}\right)_{\text{at b}} = \frac{1}{\left(\frac{dy}{dx}\right)_{\text{at a}}} = \frac{1}{f'(a)}$$

b. Find $g'(2)$ where g is the inverse function of $f(x) = x^5 - x^3 + 2x$. Explain.

c. Sketch all 6 trig inverse functions. Then use your knowledge of how to sketch derivatives from question 1 to do derivative sketches. How do the sketches relate to formulas in #9a? Check using Desmos.com ☺

d. Provide proofs using triangle for $\frac{d}{dx}(\cos^{-1} x)$ and $\frac{d}{dx}(\csc^{-1} x)$

e. Show how to find the derivative of $\frac{d}{dx}(\cos^{-1}(3x^2))$ using the formula directly without the triangle, recall what you'd say to yourself with chain rule.