Matrices

A matrix is a way of writing a set of numbers using rows + columns

2 columns

2 rows $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 3 & 2 \\ 1 & -5 \\ 7 & 2 \\ 5 & 4 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & -5 & 10 \\ 12 & 8 & 4 & 9 \\ 25 & -30 & 1 & -1 \end{bmatrix}$

2x2 matrix 4x2 matrix 3x4 matrix

Reading Information from a Matrix

- Cells in a matrix can be referenced in the form (row, column).

$$\begin{array}{c} \\ R1 \\ R2 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & & C3 & C4 \end{array} \\ \begin{bmatrix} 3 & -4 & 6 & \boxed{5} \\ 7 & 2 & \boxed{0} & 1 \\ 5 & 9 & 13 & -2 \\ -10 & -1 & 4 & 3 \end{bmatrix} \end{array}$$

State the value for each of the following

(2, 3) = 0 (1, 4) = 5

State the location of: ~~#1~~

−1 = (4, 2) 1 = (2, 4)

Matrix Addition & Subtraction

- To add matrices, they must be the same size.

$\begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 12 \\ 5 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 10 & 3 & 2 \\ -5 & 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 14 \end{bmatrix}$

* Add matching cells *

$= \begin{bmatrix} 2+3 & 1+12 \\ 4+5 & 7+(-2) \end{bmatrix}$ $= \begin{bmatrix} 1-10 & 2-3 & 3-2 \\ 4-(-5) & 5-1 & 6-(-1) \end{bmatrix}$

Cannot Add! (not the same size)

$= \begin{bmatrix} 5 & 13 \\ 9 & 5 \end{bmatrix}$ $= \begin{bmatrix} -9 & -1 & 1 \\ 9 & 4 & 7 \end{bmatrix}$

Scalar Multiplication & Division

$5 \times \begin{bmatrix} 2 & -7 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5(2) & 5(-7) \\ 5(3) & 5(4) \end{bmatrix}$ $\begin{bmatrix} 6 & -8 & 24 \\ 18 & 22 & -4 \end{bmatrix} \div 2 = \begin{bmatrix} 6/2 & -8/2 & 24/2 \\ 18/2 & 22/2 & -4/2 \end{bmatrix}$

* Multiply all * Cells by the Coefficient

$= \begin{bmatrix} 10 & -35 \\ 15 & 20 \end{bmatrix}$ $= \begin{bmatrix} 3 & -4 & 12 \\ 9 & 11 & -2 \end{bmatrix}$

Matrix Multiplication

- The number of columns ~~rows~~ in the 1st matrix must equal the number of rows ~~columns~~ in the 2nd matrix.

3 columns

$$\begin{bmatrix} 3 & 5 & 1 \\ 2 & 8 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Big\} \text{3 rows}$$

$$= \begin{bmatrix} 3(2)+5(3)+1(1) \\ 2(2)+8(3)+4(1) \end{bmatrix}$$

$$= \begin{bmatrix} 6+15+1 \\ 4+24+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 \\ 32 \end{bmatrix}$$

← Answer has the same # of rows as the 1st matrix and the same # of columns as the 2nd matrix.

1 column

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 & 5 & 1 \\ 2 & 8 & 4 \end{bmatrix} \text{2 rows}$$

Not Possible!

$$\begin{bmatrix} 2 & 8 \\ 3 & 4 \\ 1 & 0 \\ 6 & 5 \end{bmatrix} \times \begin{bmatrix} 10 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2(10)+8(2) & 2(3)+8(5) \\ 3(10)+4(2) & 3(3)+4(5) \\ 1(10)+0(2) & 1(3)+0(5) \\ 6(10)+5(2) & 6(3)+5(5) \end{bmatrix}$$

$$= \begin{bmatrix} 20+16 & 6+40 \\ 30+8 & 9+20 \\ 10+0 & 3+0 \\ 60+10 & 18+25 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 46 \\ 38 & 29 \\ 10 & 3 \\ 70 & 43 \end{bmatrix}$$

Zero Matrix

- All entries in the matrix are zero.

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 4+0 \\ 3+0 & 5+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

* Same as original *

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0(2)+0(5)+0(-1)+0(3) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}$$

* Equals Zero! *

Identity Matrix

- When multiplying by the identity matrix, the original matrix does not change!

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All other #s are zero

Numbers on the diagonal are all one

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+4(0) & 2(0)+4(1) \\ 3(1)+5(0) & 3(0)+5(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$ ← didn't change

Inverse Matrix

- When a matrix is multiplied by the inverse, the result is an identity matrix

Determine which of the following is the inverse of the matrix $\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$:

$$\begin{bmatrix} 8 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 8 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(8)+8(3) & 3(3)+8(1) \\ (1)(8)+3(3) & 1(3)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 24+24 & 9+8 \\ 8+9 & 3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 48 & 17 \\ 17 & 9 \end{bmatrix}$$

No!

$$= \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+8(-1) & 3(-8)+8(3) \\ 1(3)+3(-1) & 1(-8)+3(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -24+24 \\ 3-3 & -8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Yes!

Don't need to check.

- You can multiply/divide the rows of an equation matrix by a constant
- You can add/subtract rows to eliminate entries

Find the inverse of the following matrices:

$$\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$ ← Add identity matrix onto the end

Try to turn this side into identity matrix

$$\left[\begin{array}{cc|cc} 3 & 8 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \quad R2 \times 3$$

Try to turn into zero.

$$= \left[\begin{array}{cc|cc} 3 & 8 & 1 & 0 \\ 3 & 9 & 0 & 3 \end{array}\right] \quad R2 - R1$$

$$= \left[\begin{array}{cc|cc} 3 & 8 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad R2 \times 8$$

$$= \left[\begin{array}{cc|cc} 3 & 8 & 1 & 0 \\ 0 & 8 & -8 & 24 \end{array}\right] \quad R1 - R2$$

$$= \left[\begin{array}{cc|cc} 3 & 0 & 9 & -24 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

try to turn into a one.

$$= \left[\begin{array}{cc|cc} 3 & 0 & 9 & -24 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad R1 \div 3$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 3 & -8 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

identity matrix

$$\therefore \text{Inverse} = \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R1 \times 2 \\ R2 \times 3 \end{array}$$

$$= \left[\begin{array}{cc|cc} 12 & 6 & 2 & 0 \\ 12 & 3 & 0 & 3 \end{array}\right] \quad R2 - R1$$

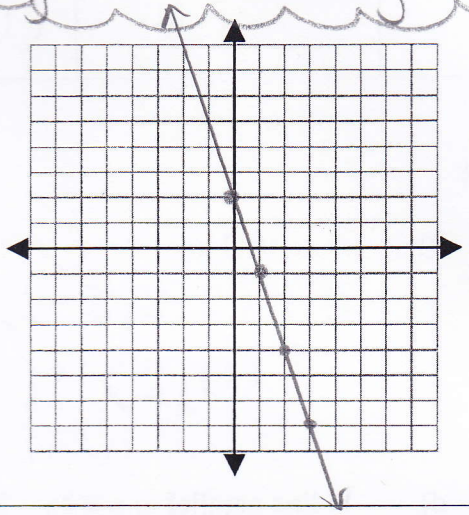
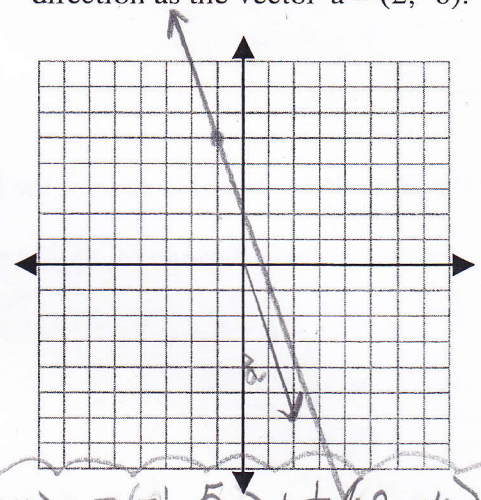
$$= \left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 0 & -3 & -2 & 3 \end{array}\right] \quad R1 + R2$$

$$= \left[\begin{array}{cc|cc} 6 & 0 & -1 & 3 \\ 0 & -3 & -2 & 3 \end{array}\right] \quad \begin{array}{l} R1 \div 6 \\ R2 \div (-3) \end{array}$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -1/6 & 1/2 \\ 0 & 1 & 2/3 & -1 \end{array}\right]$$

$$\therefore \text{Inverse} = \begin{bmatrix} -1/6 & 1/2 \\ 2/3 & -1 \end{bmatrix}$$

Equations of Lines in 2-Dimensions

Type	Scalar/Cartesian	Vector
Format	$y = mx + b$ $ax + by + c = 0$	$(x, y) = (x_1, y_1) + t\vec{m}$ $\vec{r} = (x_1, y_1) + t(\vec{m})$
Example	Find the equation of the line that has a slope of -3 and a y-intercept of 2. $y = -3x + 2$ or $3x + y - 2 = 0$ 	Find the equation of the line that passes through the point (-1, 5) and has the same direction as the vector $\vec{a} = (2, -6)$.  $(x, y) = (-1, 5) + t(2, -6)$

Type	Parametric	Symmetric
Format	$x = x_1 + t x_m$ $y = y_1 + t y_m$	$\frac{x - x_1}{x_m} = \frac{y - y_1}{y_m}$
Example	Find the equation of the line that passes through the point (-1, 5) and has the same direction as the vector $\vec{a} = (2, -6)$. $(x, y) = (-1, 5) + t(2, -6)$ $x = -1 + 2t$ $y = 5 - 6t$	Find the equation of the line that passes through the point (-1, 5) and has the same direction as the vector $\vec{a} = (2, -6)$. Isolate for t $x = -1 + 2t$ $\quad$ $y = 5 - 6t$ $x + 1 = 2t$ $\quad$ $y - 5 = -6t$ $\frac{x+1}{2} = t$ $\quad$ $\frac{y-5}{-6} = t$ $\frac{x+1}{2} = \frac{y-5}{-6}$

cross multiply

$\frac{x+1}{2} = \frac{y-5}{-6}$ $\rightarrow$ $-6(x+1) = 2(y-5)$

$-6x - 6 = 2y - 10$ $\rightarrow$ $-6x + 4 = 2y$

$-3x + 2 = y$

$3y = -2x + 12$

$y = -\frac{2}{3}x + 4$

State the **direction vector** and **one point** on each of the following lines.

Equation	$\vec{r} = (2,5) + t(-3,1)$	$x = 5 + 3t$ $y = 10 - 2t$	$\frac{x-3}{7} = \frac{y+5}{3}$	$2x + 3y = 12$
Direction Vector	$(-3,1)$	$(3,-2)$	$(7,3)$	$(3,-2)$
Point	$(2,5)$ or $\vec{r} = (2,5) + 2(-3,1)$ $\vec{r} = (2,5) + (-6,2)$ $\vec{r} = (-4,7)$	$(5,10)$	$(3,-5)$	Let $x = 0$ $2(0) + 3y = 12$ $3y = 12$ $y = 4$ $(0,4)$

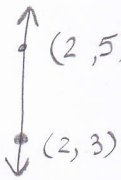
Find the **direction vector** for each of the following:

a) A line passing through (4, 5) and (-2, 1)

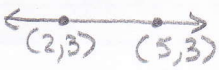
$\vec{m}_{AB} = (x_B - x_A, y_B - y_A)$
$= (-2-4, 1-5)$
$= (-6,-4)$

b) A vertical line

$\vec{m} = (2-2, 5-3)$
$= (0,2)$



c) A horizontal line



$\vec{m} = (5-2, 3-3)$
$= (3,0)$

d) A line parallel to y = 4x – 2

$m = \frac{4}{1}$ rise (y) / run (x)

$\vec{m} = (1,4)$

e) A line perpendicular to y = 4x – 2

$m = 4$ $\quad m_{perp} = -\frac{1}{4}$

$\vec{m} = (4,-1)$

f) A line with a normal vector $\vec{N} = (2,7)$ ← perp.

$\vec{m} = (+7,-2)$

$m = \frac{7}{2}$

$m_p = \frac{-2}{7}$

Does the point (1, -7) lie on the line $\vec{r} = (3, 6) + t(2, -5)$?

$\vec{r} = (3,6) + t(2,-5)$

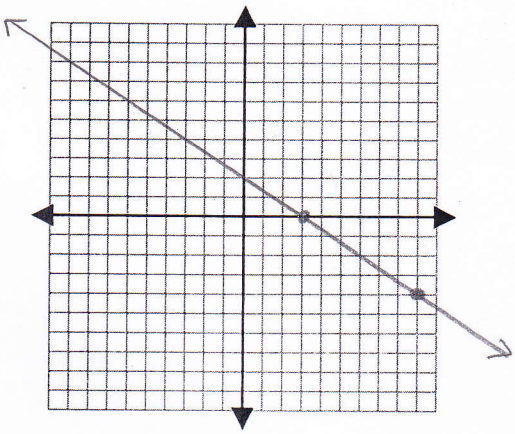
$(1,-7) = (3,6) + t(2,-5)$

$1 = 3 + 2t$
$-2 = 2t$
$\boxed{-1 = t}$

$-7 = 6 - 5t$
$-13 = -5t$
$\boxed{\frac{13}{5} = t}$

∴ No!

Find the scalar, vector, parametric, and symmetric equations of the line that passes through the points A (3, 0) and B (9, -4).



Vector

pt = (3,0)

$\vec{m} = (9-3, -4-0)$

$= (6, -4)$

$\boxed{(x,y) = (3,0) + t(6,-4)}$

parametric

$\boxed{\begin{aligned} x &= 3+6t \\ y &= 0-4t \end{aligned}}$

Symmetric

$x = 3+6t$ $\quad$ $y = 0-4t$

$x-3 = 6t$ $\quad$ $y-0 = -4t$

$\frac{x-3}{6} = t$ $\quad$ $\frac{y}{-4} = t$

$\boxed{\frac{x-3}{6} = \frac{y}{-4}}$

Scalar

$\frac{x-3}{6} = \frac{y}{-4}$

$-4x+12 = 6y$

$\boxed{0 = 4x+6y+12}$

Determine whether the following vectors are parallel, perpendicular, or neither:

$x_1 = 6 + 2t$ $\quad$ $x_2 = 10 + 9s$

$y_1 = 10 - 3t$ $\quad$ $y_2 = -2 + 6s$

$\vec{m}_1 = (2,-3)$ $\quad$ $\vec{m}_2 = (9,6)$

$= 3(3,2)$

$m_1 = \frac{-3}{2}$ $\quad$ $m_2 = \frac{2}{3}$

$\therefore$ perpendicular

Equations of Lines in 3 Dimensions

Scalar	Vector
Not Possible	point, direction vector: $\vec{r} = (x_1, y_1, z_1) + t(\vec{m})$
Parametric	**Symmetric**
$x = x_1 + t(x_m)$ $y = y_1 + t(y_m)$ $z = z_1 + t(z_m)$	$\frac{x - x_1}{x_m} = \frac{y - y_1}{y_m} = \frac{z - z_1}{z_m}$

1. Find the vector, parametric, and symmetric equations of the line that passes through (2, 1, -3) and (5, -7, 4).

$\vec{m} = (5-2, -7-1, 4-(-3))$
$= (3, -8, 7)$

$x = 2 + 3t \quad y = 1 - 8t \quad z = -3 + 7t$
$x - 2 = 3t \quad y - 1 = -8t \quad z + 3 = 7t$
$\frac{x-2}{3} = t \quad \frac{y-1}{-8} = t \quad \frac{z+3}{7} = t$

vector
$\vec{r} = (2, 1, -3) + t(3, -8, 7)$

parametric
$x = 2 + 3t$
$y = 1 - 8t$
$z = -3 + 7t$

symmetric
$\frac{x-2}{3} = \frac{y-1}{-8} = \frac{z+3}{7}$

2. Find the parametric equation of the line that passes through (8, -10, 5) and is parallel to the line $\vec{r} = (5, -7, 4) + t(2, 1, -1)$.

same $\vec{m}$

$m = (2, 1, -1)$
$pt = (8, -10, 5)$
(vector) $\vec{r} = (2, 1, -1) + t(8, -10, 5)$

$x = 2 + 8t$
$y = 1 - 10t$
$z = -1 + 5t$

3. Find the symmetric equation of the line that passes through (2, -1, 5) and has a direction vector of (12, 4, -1).

(vector) $\vec{r} = (2, -1, 5) + t(12, 4, -1)$

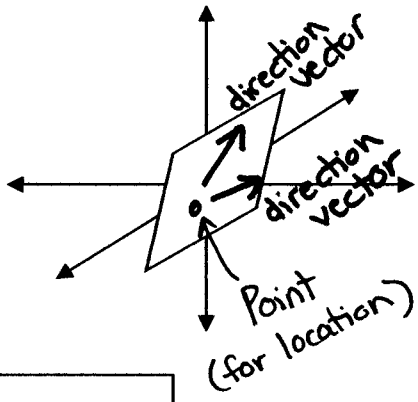
(parametric)
$x = 2 + 12t$
$y = -1 + 4t$
$z = 5 - 1t$

$\frac{x-2}{12} = \frac{y+1}{4} = \frac{z-5}{-1}$

OR

$\frac{x-12}{12} = \frac{y+1}{4} = \frac{-z+5}{1}$ } mult top + bottom by (-1)

Vector Equation of a Plane



A plane is a flat surface that extends infinitely in all directions.

Vector Equation of a Plane

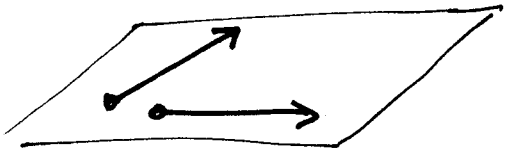
$$\vec{r} = (x_1, y_1, z_1) + s(\vec{m_1}) + t(\vec{m_2})$$

point — 2 direction vectors

To find the equation of a plane we need:

- One Point on the plane
- 2 different direction vectors

Write the vector, parametric and symmetric equations of the plane that contains the lines $\vec{r} = (3,10,-4) + t(4,0,1)$ and $\vec{r} = (-6,6,7) + t(11,-3,-2)$.



pt: $(3,10,-4)$ or $(-6,6,7)$

$\vec{m_1} = (4,0,1)$

$\vec{m_2} = (11,-3,-2)$

$\vec{r} = (3,10,-4) + s(4,0,1) + t(11,-3,-2)$ Vector

$x = 3 + 4s + 11t$

$y = 10 + 0s - 3t$

or

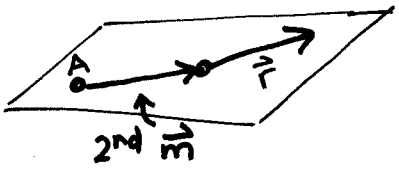
$y = 10 - 3t$

$z = -4 + 1s - 2t$

Parametric

Since there are 2 parameters (s and t), you cannot isolate for both at the same time. There is no symmetric equation of a plane.

Write the vector and parametric equations of the plane that passes through the point A(2, 5, 3) and contains the line $\vec{r} = (-2,4,7) + t(1,4,-5)$.



$pt = (2,5,3)$ or $(-2,4,7)$

$\vec{m}_1 = (1,4,-5)$

$\vec{m}_2 = (-2-2, 4-5, 7-3)$

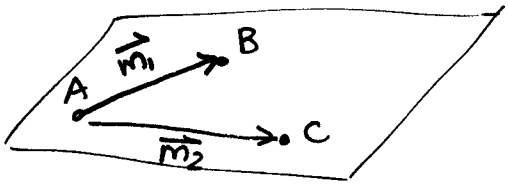
$= (-4,-1,3)$

vector

$\pi = (2,5,3) + s(1,4,-5) + t(-4,-1,3)$

$$\left.\begin{aligned} x &= 2+s-4t \\ y &= 5+4s-t \\ z &= 3-5s+3t \end{aligned}\right\} \text{parametric}$$

Find the vector equation of a plane that passes through the points A(1, 2, 1), B(4, 5, 1) and C(4, 0, 1).



$\vec{m}_1 = \vec{AB}$

$= (4-1, 5-2, 1-1)$

$= (3,3,0)$

or

$= (1,1,0)$ ÷ by 3

$\vec{m}_2 = \vec{AC}$

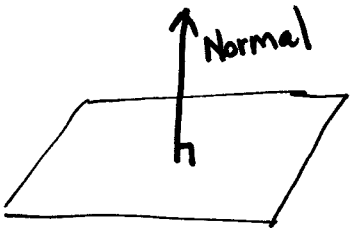
$= (4-1, 0-2, 1-1)$

$= (3,-2,0)$

$\therefore \pi = (1,2,1) + s(1,1,0) + t(3,-2,0)$

Scalar Equations of Planes

$$\boxed{\begin{array}{c} \mathbf{0 = Ax + By + Cz + D} \\ \text{Normal} = (A, B, C) \end{array}}$$



$$\begin{array}{c} 1 \times 0 \\ 4 \times -4 \\ 3 \times 1 \\ 1 \times 0 \end{array}$$

Find the scalar equation of the plane $\vec{r} = \underline{(2, -1, 7)} + t(3, 1, 4) + s(1, 0, -4)$.

$\vec{N} = \vec{m_1} \times \vec{m_2}$

$\vec{N} = (3,1,4) \times (1,0,-4)$

$\vec{N} = (-4-0, 4-(-12), 0-1)$

$\vec{N} = (-4, 16, -1)$

A B C

$Ax + By + Cz + D = 0$

$-4x + 16y - 1z + D = 0$

$-4(2) + 16(-1) - 1(7) + D = 0$

$-8 - 16 - 7 + D = 0$

$-31 + D = 0$

$D = +31$

$\therefore -4x + 16y - z + 31 = 0$

or

$4x - 16y + z - 31 = 0$ ← Proper standard Form

Find the scalar equation of the plane that passes through the points A(5, 14, -3), B(2, 1, -2), C$\underline{(0, 4, -1)}$.

$\vec{m_1} = \vec{AB}$

$= (2-5, 1-14, -2-(-3))$

$= (-3, -13, 1)$

$= (3, 13, -1)$

$m_2 = \vec{AC}$

$= (0-5, 4-14, -1-(-3))$

$= (-5, -10, 2)$

$= (5, 10, -2)$

$Ax + By + Cz + D = 0$

$-16x + 1y - 35z + D = 0$

$-16(0) + 1(4) - 35(-1) + D = 0$

$0 + 4 + 35 + D = 0$

$39 + D = 0$

$D = -39$

$$\begin{array}{c} 13 \times 10 \\ -1 \times -2 \\ 3 \times 5 \\ 13 \times 10 \end{array}$$

$\vec{N} = \vec{m_1} \times \vec{m_2}$

$= (-26-(-10), -5-(-6), 30-65)$

$= (-16, 1, -35)$

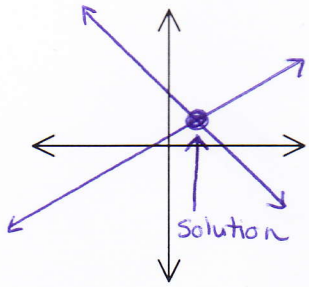
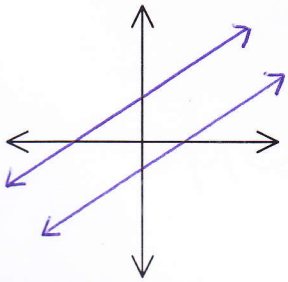
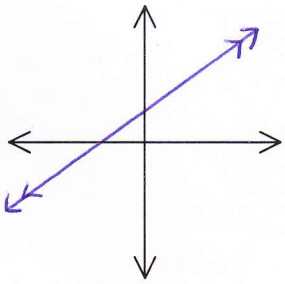
$\therefore -16x + y - 35z - 39 = 0$

or

$16x - y + 35z + 39 = 0$

Intersection of 2 Lines (2D)

Types of Intersection/Solutions

Type	One Solution	No Solution	Infinite Solutions
Graph			
Description	Lines intersect at a point.	Lines are parallel and do not intersect.	Lines are the same and intersect EVERYWHERE.
Scalar Equations	(different slopes) $y = 2x + 3$ $y = 5x - 1$	(same slope) $y = 2x + 3$ $y = 2x - 1$	(Equations are multiples) $2x + 3y + 6 = 0$ $4x + 6y + 12 = 0$
Vector Equations	(different directions) $(x,y) = (2,4) + t(1,3)$ $(x,y) = (1,5) + s(-2,7)$	(directions are same or multiples) $(x,y) = (2,4) + t(1,3)$ $(x,y) = (3,1) + t(1,3)$	(same direction, share a point) $(x,y) = (2,4) + t(1,3)$ $(x,y) = (2,4) + t(1,3)$
Solving by Sub or Elim	$x = \#$ or $y = \#$	$0 = \#$	$0 = 0$
Solving by Matrices	$\left[\begin{array}{cc\|c} 1 & 0 & \# \\ 0 & 1 & \# \end{array}\right]$	$\left[\begin{array}{cc\|c} \# & \# & \# \\ 0 & 0 & \# \end{array}\right]$	$\left[\begin{array}{cc\|c} \# & \# & \# \\ 0 & 0 & 0 \end{array}\right]$
Solving by Vectors	$s = \#$ or $t = \#$	$0 = \#$	$0 = 0$

Scalar Equations

- To determine the intersection point of two lines, you can solve by **substitution, elimination** or **matrices.**

Solve: $3x + 2y + 5 = 0$
$x - y - 10 = 0$

Method 1: Substitution

① Rearrange one of the equations to isolate x or y

$x = y + 10$

② Substitute into other equation and solve.

$3(y+10) + 2y + 5 = 0$
$3y + 30 + 2y + 5 = 0$
$5y + 35 = 0$
$5y = -35$
$\boxed{y = -7}$

③ Sub answer into equation from ①

$x = y + 10$
$x = -7 + 10$
$\boxed{x = 3}$

∴ They intersect at $(3, -7)$

Method 2: Elimination

① Write equations in the same order and write one on top of the other.

$3x + 2y + 5 = 0$
$x - y - 10 = 0 \quad \ldots \times 3$

② Choose to eliminate x or y and multiply to get same coefficient.

$3x + 2y + 5 = 0$
$3x - 3y - 30 = 0$

③ Subtract and solve.

$5y + 35 = 0$
$5y = -35$
$\boxed{y = -7}$

④ Sub back into one of the equations + solve.

$x - (-7) - 10 = 0$
$x + 7 - 10 = 0$
$x - 3 = 0$
$\boxed{x = 3}$

∴ They intersect at $(3, -7)$

Method 3: Matrices

① Write equations in the form $ax + by = -c$

$3x + 2y = -5$
$x - y = 10$

② Write in a matrix without x, y and =. Put a line where the = goes.

$$\begin{array}{cccc} x & y & = & \# \end{array}$$
$$\left[\begin{array}{cc|c} 3 & 2 & -5 \\ 1 & -1 & 10 \end{array}\right]$$

③ Perform operations to turn first two columns into $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & 2 & -5 \\ 1 & -1 & 10 \end{array}\right] R2 \times 3$$
$$= \left[\begin{array}{cc|c} 3 & 2 & -5 \\ 3 & -3 & 30 \end{array}\right] R1 - R2$$
$$= \left[\begin{array}{cc|c} 3 & 2 & -5 \\ 0 & 5 & -35 \end{array}\right] \begin{array}{l} R1 \times 5 \\ R2 \times 2 \end{array}$$
$$= \left[\begin{array}{cc|c} 15 & 10 & -25 \\ 0 & 10 & -70 \end{array}\right] R1 - R2$$
$$= \left[\begin{array}{cc|c} 15 & 0 & 45 \\ 0 & 10 & -70 \end{array}\right] \begin{array}{l} R1 \div 15 \\ R2 \div 10 \end{array}$$
$$= \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -7 \end{array}\right] \begin{array}{l} \leftarrow 1x + 0y = 3 \\ \leftarrow 0x + 1y = -7 \end{array}$$
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & \# \end{array}$$

$\boxed{\begin{array}{l} x = 3 \\ y = -7 \end{array}}$

∴ They intersect at $(3, -7)$

Vector Equations

Solve: $\vec{r} = (2, 3) + t(-1, 4)$

$\vec{r} = (-10, 7) + s(9, 8)$

① Write in parametric form.

$x = 2 - 1t$ $\quad$ $x = -10 + 9s$

$y = 3 + 4t$ $\quad$ $y = 7 + 8s$

② Set x=x and y=y

$2 - 1t = -10 + 9s$ $\quad$ $3 + 4t = 7 + 8s$

$12 = 9s + t$ $\quad$ $-4 = 8s - 4t$

③ Solve by substitution, elimination or matrices.

Substitution

$12 = 9s + t$ $\quad$ $-4 = 8s - 4t$

$12 - 9s = t \rightarrow -4 = 8s - 4(12 - 9s)$

$-4 = 8s - 48 + 36s$

$-4 = 44s - 48$

$44 = 44s$

$\boxed{1 = s}$

$12 - 9(1) = t$

$12 - 9 = t$

$\boxed{3 = t}$

Elimination

$12 = 9s + t \ldots \times 4$

$-4 = 8s - 4t$

$48 = 36s + 4t$

$+ \quad -4 = 8s - 4t$

$44 = 44s$

$\boxed{1 = s}$

$-4 = 8(1) - 4t$

$-4 = 8 - 4t$

$-12 = -4t$

$\boxed{3 = t}$

Matrices

$$\begin{array}{c} \begin{array}{cc|c} s & t & \# \end{array} \\ \left[\begin{array}{cc|c} 9 & 1 & 12 \\ 8 & -4 & -4 \end{array}\right] \end{array} \begin{array}{l} R1 \times 8 \\ R2 \times 9 \end{array}$$

$$= \left[\begin{array}{cc|c} 72 & 8 & 96 \\ 72 & -36 & -36 \end{array}\right] R1 - R2$$

$$= \left[\begin{array}{cc|c} 72 & 8 & 96 \\ 0 & 44 & 132 \end{array}\right] \begin{array}{l} R1 \div 8 \\ R2 \div 44 \end{array}$$ To simplify Numbers to make next step easier

$$= \left[\begin{array}{cc|c} 9 & 1 & 12 \\ 0 & 1 & 3 \end{array}\right] R1 - R2$$

$$= \left[\begin{array}{cc|c} 9 & 0 & 9 \\ 0 & 1 & 3 \end{array}\right] R1 \div 9$$

$$= \begin{array}{c} \begin{array}{cc|c} s & t & \# \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right] \end{array}$$

$\therefore \boxed{\begin{array}{l} s = 1 \\ t = 3 \end{array}}$

④ Sub s and t values back into parametric equations

$x = 2 - 1t$ $\quad$ $y = 3 + 4t$

$x = 2 - 1(3)$ $\quad$ $y = 3 + 4(3)$

$x = 2 - 3$ $\quad$ $y = 3 + 12$

$\boxed{x = -1}$ $\quad$ $\boxed{y = 15}$

Check with 2nd Equation

$x = -10 + 9s$ $\quad$ $y = 7 + 8s$

$x = -10 + 9(1)$ $\quad$ $y = 7 + 8(1)$

$x = -10 + 9$ $\quad$ $y = 7 + 8$

$\boxed{x = -1}$ ✓ $\quad$ $\boxed{y = 15}$ ✓

$\therefore (-1, 15)$

Intersection of a Line and a Plane

Determine the intersection of the following lines with the plane $4x + 3y - 2z - 5 = 0$.

a) $\vec{r} = (5, 1, -3) + t(2, -4, 1)$

$x = 5 + 2t$
$y = 1 - 4t$
$z = -3 + 1t$

$4x + 3y - 2z - 5 = 0$
$4(5+2t) + 3(1-4t) - 2(-3+t) - 5 = 0$
$20 + \underline{8t} + 3 - \underline{12t} + 6 - \underline{2t} - 5 = 0$
$-6t + 24 = 0$
$-6t = -24$
$\boxed{t = 4}$

$x = 5 + 2(4)$ $y = 1 - 4(4)$ $z = -3 + 1(4)$
$x = 13$ $y = -15$ $z = 1$

$\therefore$ They intersect at the point $(13, -15, 1)$

b) $\vec{r} = (-2, 3, 1) + t(-3, 4, 0)$

$x = -2 - 3t$
$y = 3 + 4t$
$z = 1 + 0t$

$4x + 3y - 2z - 5 = 0$
$4(-2-3t) + 3(3+4t) - 2(1) - 5 = 0$
$-8 - \underline{12t} + 9 + \underline{12t} - 2 - 5 = 0$
$-6 = 0$ ← Not Possible

$\therefore$ No Solution (parallel)

c) $\vec{r} = (1, 3, 4) + t(-1, -2, -5)$

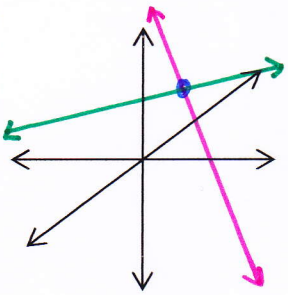
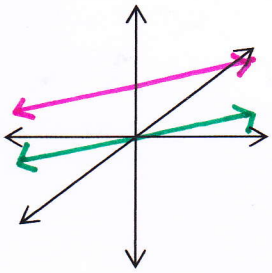
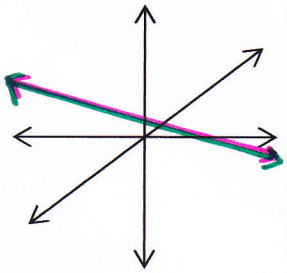
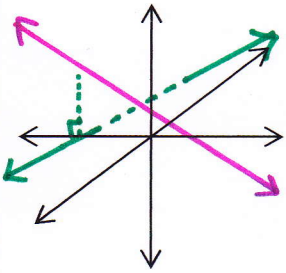
$x = 1 - 1t$
$y = 3 - 2t$
$z = 4 - 5t$

$4x + 3y - 2z - 5 = 0$
$4(1-1t) + 3(3-2t) - 2(4-5t) - 5 = 0$
$4 \underline{-4t} + 9 \underline{-6t} - 8 \underline{+10t} - 5 = 0$
$0 = 0$ ← true for any value of t

$\therefore$ Infinite Solutions (coincident)

Intersection of 2 Lines (3D)

Types of Intersection/Solutions

Type	Intersecting	Parallel	Coincident	Skew
Graph				
Description	Lines intersect at one point	Lines do not meet (evenly spaced)	The same line (intersect everywhere)	Lines do not cross
Number of Solutions	1	0	∞	0
Vector Equations	~~different $\vec{m}$ (must use calculations to check if skew)~~	same $\vec{m}$ different points (points not on both lines)	same $\vec{m}$ Same point or points exist on both lines	different $\vec{m}$ different points (must use calcs to tell if intersecting or not)
Solving	$t = \#$ (same answers for x, y, z)	$0 = \#$	$0 = 0$	$t = \#$ (x, y, z values don't match up)

Solve: $\vec{r} = (2, 4, -1) + t(2, 1, -1)$

$\vec{r} = (4, 5, 7) + s\boxed{(-2, -1, 1)} \Rightarrow -1(2,1,-1)$ ∴ Parallel $\vec{m}$

~~$x_1 = 2 + 2t$~~

check point

$\vec{r} = (2,4,-1) + t(2,1,-1)$ pt $= (4,5,7)$

$x = 2 + 2t$	$y = 4 + t$	$z = -1 - t$
$4 = 2 + 2t$	$5 = 4 + t$	$7 = -1 - t$
$2 = 2t$	$1 = t$	$8 = -t$
$1 = t$		$-8 = t$ ← doesn't match

∴ Lines are different

∴ The lines are parallel (No Solution)

Solve: $\vec{r} = (10, -3, 1) + t(1, 1, -1)$

$\vec{r} = (7, -6, 4) + s\boxed{(2, 2, -2)} \Rightarrow 2(1,1,-1)$ ∴ Parallel $\vec{m}$

$x = 10 + t$ $\quad y = -3 + t$ $\quad z = 1 - t$

$7 = 10 + t$ $\quad -6 = -3 + t$ $\quad 4 = 1 - t$

$-3 = t$ $\quad -3 = t$ $\quad 3 = -t$

$-3 = t$

∴ Point is on both lines

∴ The lines are Coincident (Infinite Solutions)

Solve: $\vec{r} = (4, 2, -1) + t(3, 0, 1)$

$\vec{r} = (-3, 3, -7) + s(1, -1, 4)$] Not parallel

$x_1 = 4 + 3t$ $\quad 4 + 3t = -3 + 1s$ $\quad 2 + 0t = 3 - 1s$ $\quad -1 + 1t = -7 + 4s$

$y_1 = 2 + 0t$ $\quad 7 = s + 3t$ $\quad -1 = -s$ $\quad 6 = 4s - t$

$z_1 = -1 + 1t$ $\quad \boxed{1 = s}$

$7 = 1 + 3t$

$x_2 = -3 + 1s$ $\quad 6 = -3t$

$y_2 = 3 - 1s$ $\quad \boxed{-2 = t}$

$z_2 = -7 + 4s$

$x_1 = 4 + 3(-2)$
$= \cancel{10} -2$

$y_1 = 2$

$z_1 = -1 + 1(-2)$
$= \cancel{3} -3$

$x_2 = -3 + 1(1)$
$= -2$

$y_2 = 3 - 1(1)$
$= 2$

$z_1 = -7 + 4(1)$
$= -3$

~~∴ No Solution (skew)~~

∴ The solution is $(-2, 2, -3)$

Solve: $\dfrac{x-1}{4} = \dfrac{y-3}{3} = \dfrac{z-3}{1}$ $\quad \dfrac{x-2}{3} = \dfrac{y-2}{3} = \dfrac{z-10}{2}$

Not Parallel

$x_1 = 1 + 4t$ $\quad 1 + 4t = 2 + 3s$ $\quad 3 + 3t = 2 + 3s$ $\quad 3 + t = 10 + 2s$

$y_1 = 3 + 3t$ $\quad -1 = 3s - 4t$ $\quad 1 = 3s - 3t$ $\quad -7 = 2s - t$

$z_1 = 3 + t$

$-1 = 3s - 4t$
$-\ \ 1 = 3s - 3t$

$-2 = -1t$

$\boxed{2 = t}$

$x_2 = 2 + 3s$

$y_2 = 2 + 3s$

$z_2 = 10 + 2s$

$-1 = 3s - 4(2)$
$-1 = 3s - 8$
$7 = 3s$
$\boxed{7/3 = s}$

$x_1 = 1 + 4(2)$
$= 9$

$y_1 = 3 + 3(2)$
$= 9$

$z_1 = 3 + 2$
$= 5$

$x_2 = 2 + 3(7/3)$
$= 2 + 7$
$= 9$

$y_2 = 2 + 3(7/3)$
$= 9$

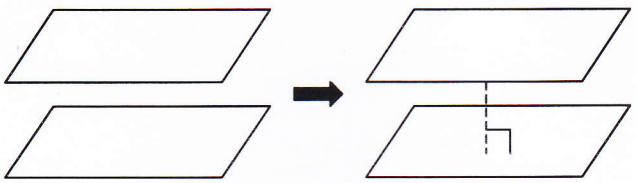
$z_2 = 10 + 2(7/3)$
$= 10 + 14/3$
$= 44/3$

∴ No Solution (skew)

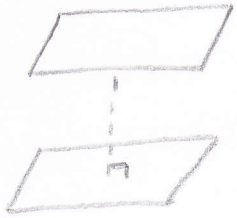
How to Draw Planes – 2 Planes

Type	Solution	Description
Parallel	No Solution	Do Not Intersect

How To Draw



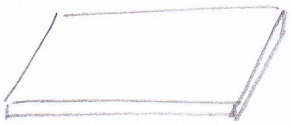
You Try



Type	Solution	Description
Coincident	Infinite	Intersect in a plane

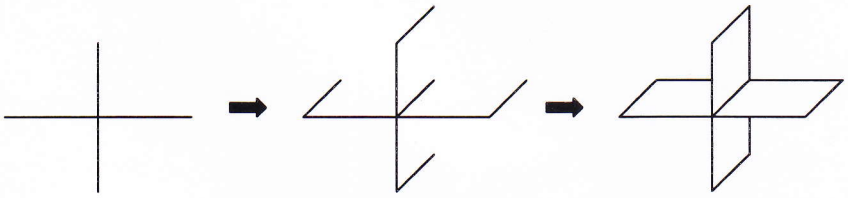
How To Draw

You Try

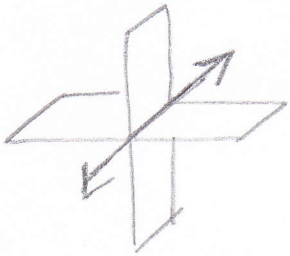


Type	Solution	Description
Intersecting	Infinite Solutions	Intersect in a Line

How To Draw

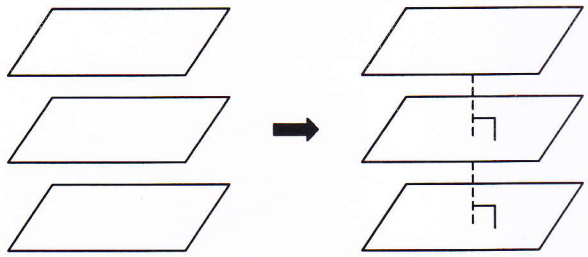
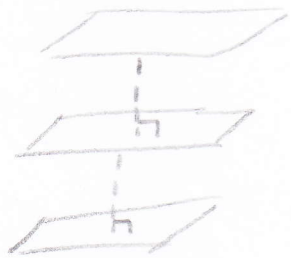


You Try

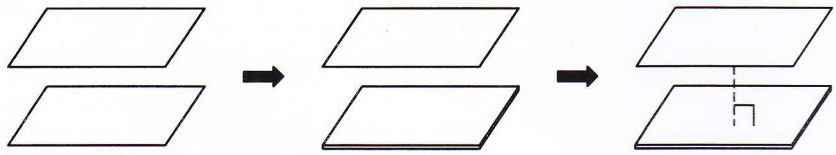
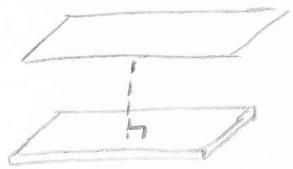


How to Draw Planes – 3 Planes

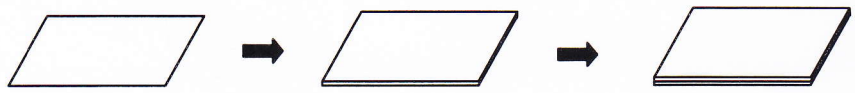
Type	Solution	Description
3 Parallel	No Solution	Do Not Intersect

How To Draw	You Try
	

Type	Solution	Description
2 Parallel 1 Coincident	No Solution	Do Not Intersect (in same place)

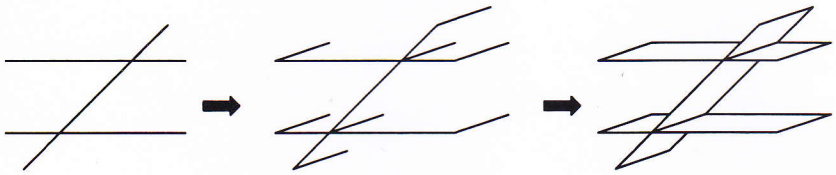
How To Draw	You Try
	

Type	Solution	Description
3 Coincident	Infinite	Intersect in a Plane

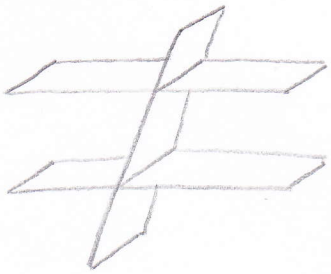
How To Draw	You Try
	

Type	Solution	Description
2 Parallel 1 Intersecting	No Solution	Do Not Intersect (all in same place)

How To Draw

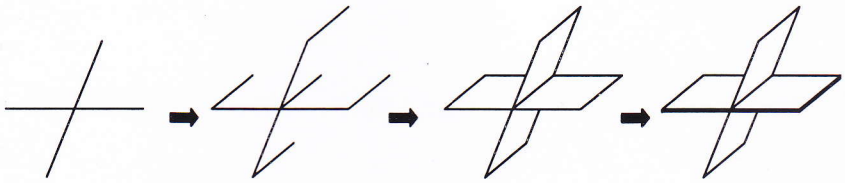


You Try



Type	Solution	Description
2 Coincident 1 Intersecting	Infinite	Intersect in a Line

How To Draw



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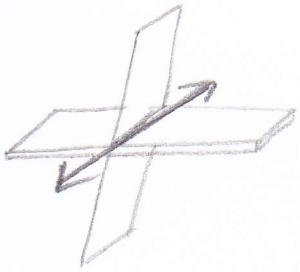
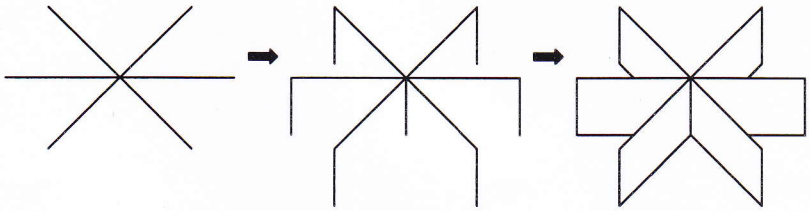
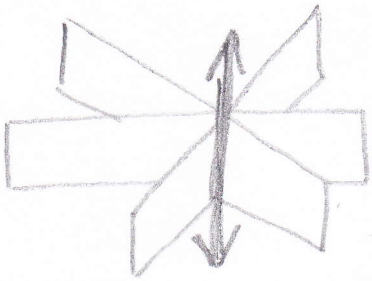


Type	Solution	Description
3 Intersecting	Infinite	Intersect in a Line

How To Draw

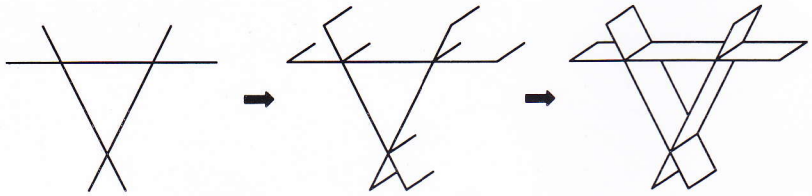


You Try

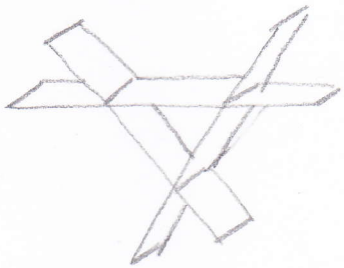


Type	Solution	Description
3 Intersecting	No Solution	Do Not All Intersect In Same Place

How To Draw

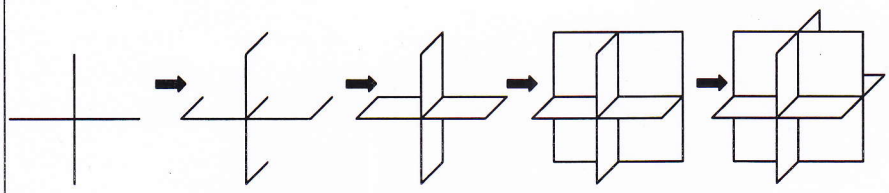


You Try

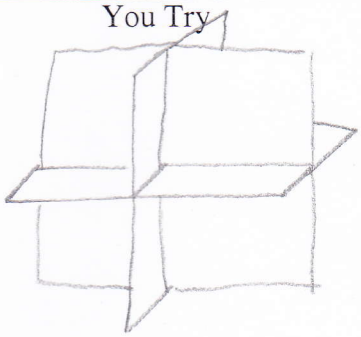


Type	Solution	Description
3 Intersecting	One Solution	Intersect at a point

How To Draw



You Try



Configurations of Planes

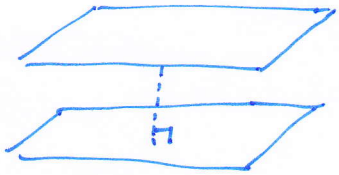
To determine whether or not planes intersect, they need to be written in cartesian (standard) form.

Helpful Information		
Parallel Planes	**Coincident Planes**	**Intersecting Planes**
The Normals are the same or scalar multiples	The ~~Normals~~ Equations are the same or scalar multiples	The Normals are different.
$2x + 5y - 3z + 10 = 0$ $2x+5y-3z+1=0$ $4x+10y-6z-9=0$	$2x + 5y - 3z + 10 = 0$ $2x+5y-3z=-10$ $4x+10y-6z+20=0$	$2x + 5y - 3z + 10 = 0$ $3x-2y+z+5=0$

Determine the configuration of each of the following sets of planes. Draw a geometric representation of each.

a) $4x - 2y + z - 3 = 0$
$8x - 4y + 2z - 3 = 0 \rightarrow 2(4x-2y+z-\frac{3}{2})=0$

$\therefore$ Parallel and distinct

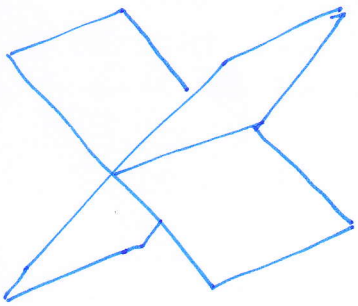


b) $10x - 6y + 4z - 8 = 0 \rightarrow 2[5x-3y+2z-4]=0$
$15x - 9y + 6z - 12 = 0 \rightarrow 3[5x-3y+2z-4]=0$

$\therefore$ Coincident Planes

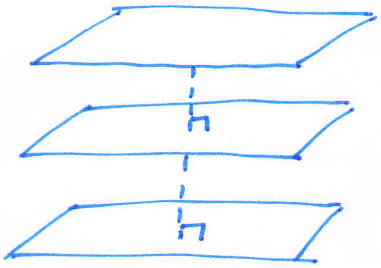
c) $3x + 5y - 2z + 1 = 0$
$2x + 5y + z - 6 = 0$

$\therefore$ Intersecting Planes



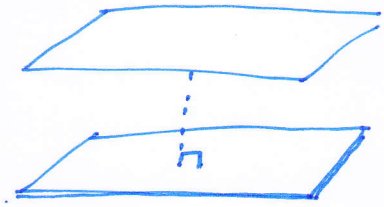
d) $2x + 3y + 5z - 4 = 0$
$4x + 6y + 10z + 3 = 0 \rightarrow 2\left[2x + 3y + 5z + \frac{3}{2}\right] = 0$
$10x + 15y + 25z - 7 = 0 \rightarrow 5\left[2x + 3y + 5z - \frac{7}{5}\right] = 0$

$\therefore$ 3 Parallel + Distinct Planes



e) $5x + 3y - 4z + 2 = 0$
$5x + 3y - 4z + 10 = 0$
$35x + 21y - 28z + 14 = 0 \rightarrow 7\left[5x + 3y - 4z + 2\right] = 0$

$\therefore$ 2 coincident, 1 parallel

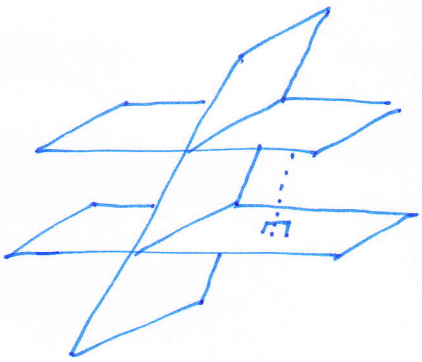


f) $x - y + 7z - 3 = 0$
$2x - 2y + 14z - 6 = 0 \rightarrow 2(x - y + 7z - 3) = 0$
$10x - 10y + 70z - 30 = 0 \rightarrow 10(x - y + 7z - 3) = 0$

$\therefore$ 3 coincident planes

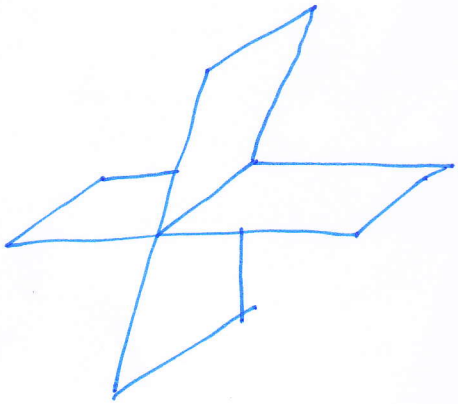
g) $9x - 3y + 12z + 6 = 0 \rightarrow 3(3x - y + 4z + 2) = 0$
$x - y + 3z + 2 = 0$
$15x - 5y + 20z - 10 = 0 \rightarrow 5(3x - y + 4z - 2) = 0$

$\therefore$ 2 parallel, 1 intersecting



h) $x - 7y - 10z + 9 = 0$
$3x + 4y - z + 4 = 0$
$18x + 24y - 6z + 24 = 0 \quad 6(3x + 4y - z + 4) = 0$

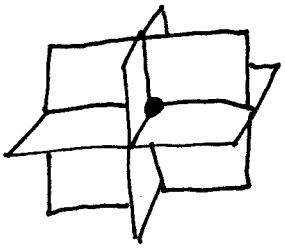
$\therefore$ 2 Coincident, 1 Intersecting



Solving For The Intersection of 2 & 3 Planes

Solving means finding the point, line, or plane where two or more planes intersect.

Determine the point of intersection of the following:



$$\boxed{\begin{aligned}3x &+ y + 4z + 3 = 0\\ 2x &- 5y + 3z + 13 = 0\\ 5x &+ 3y - 2z + 11 = 0\end{aligned}}$$

$3x+y+4z+3=0 \ldots \times 2$
$2x-5y+3z+13=0 \ldots \times 3$

$$\begin{array}{rl} & 6x+2y+8z+6=0\\ - & 6x-15y+9z+39=0\\ \hline & 17y-1z-33=0\end{array}$$

$2x-5y+3z+13=0 \ldots \times 5$
$5x+3y-2z+11=0 \ldots \times 2$

$$\begin{array}{rl} & 10x-25y+15z+65=0\\ - & 10x+6y-4z+22=0\\ \hline & -31y+19z+43=0\end{array}$$

$17y-z-33=0 \ldots \times 19$
$-31y+19z+43=0$

$$\begin{array}{rl} & 323y-19z-627=0\\ + & -31y+19z+43=0\\ \hline & 292y-584=0\end{array}$$

$292y=584$

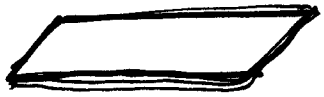
$\boxed{y=2}$

$17y-1z-33=0$
$17(2)-z-33=0$
$34-z-33=0$
$1-z=0$
$\boxed{1=z}$

$3x+y+4z+3=0$
$3x+2+4(1)+3=0$
$3x+9=0$
$3x=-9$
$\boxed{x=-3}$

$\therefore$ They intersect at the point $(-3,2,1)$

Determine the equation of the plane of intersection of the following:



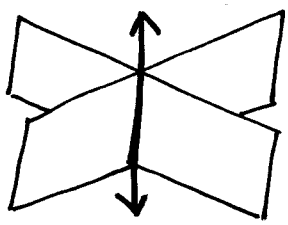
$2x-7y+4z-3=0$
$4x-14y+8z-6=0 \rightarrow 2(2x-7y+4z-3)=0$
$10x-35y+20z-15=0 \quad 5(2x-7y+4z-3)=0$

$\therefore$ Intersect in the plane $2x-7y+4z-3=0$

3. Determine the equation of the **line** of intersection of the following planes. Include a geometric representation.

$$\begin{array}{r} 2x + 10y - 3z + 1 = 0 \\ -\quad 2x - 8y + 3z - 5 = 0 \\ \hline 18y - 6z + 6 = 0 \quad \div 6 \end{array}$$

$3y - z + 1 = 0$ (reduced)



line of intersection

$$\boxed{\vec{r} = \underbrace{(1, 0, 1)}_{\text{point on line}} + t\underbrace{(-\tfrac{1}{2}, 1, 3)}_{\text{direction vector}}}$$

① Find a point on the line by choosing any value for either x/y/z and solve for the other two. (zero is an easy # to use)

② Find the direction vector by choosing a # for the same letter as ① and solve for the other values (one is a good #)

$\therefore y = 0 + 1t$
(parametric)

$3y - z + 1 = 0$
$3(0 + 1t) - z + 1 = 0$
$0 + 3t - z + 1 = 0$
$-z = -1 - 3t$
$\boxed{z = 1 + 3t}$
(parametric)

$2x + 10y - 3z + 1 = 0$
$2x + 10(0 + 1t) - 3(1 + 3t) + 1 = 0$
$2x + 10t - 3 - 9t + 1 = 0$
$2x + t - 2 = 0$
$2x = 2 - t$
$\boxed{x = 1 - \tfrac{1}{2}t}$
(parametric)

$\therefore$ The intersect along the line
$\vec{r} = (1, 0, 1) + t(-\tfrac{1}{2}, 1, 3)$

Aside
Reducing $\vec{m}$

$\vec{m} = (2, 4, 6)$
$\div 2$
$\vec{m} = (1, 2, 3)$
$\div 4$
$\vec{m} = (\tfrac{1}{2}, 1, \tfrac{3}{2})$
$\div 6$
$\vec{m} = (\tfrac{1}{3}, \tfrac{2}{3}, 1)$

4. Determine the equation of the **line** of intersection of the following planes. Include a geometric representation.

$$\begin{aligned} x + 4y + 3z - 6 &= 0 \\ 3x + 2y - z + 2 &= 0 \\ 3x + 14y + 11z - 22 &= 0 \end{aligned}$$

$x + 4y + 3z - 6 = 0 \ldots \times 3$
$3x + 2y - z + 2 = 0$

$$\begin{array}{r} 3x + 12y + 9z - 18 = 0 \\ -\ 3x + 2y - z + 2 = 0 \\ \hline 10y + 10z - 20 = 0 \div 10 \end{array}$$

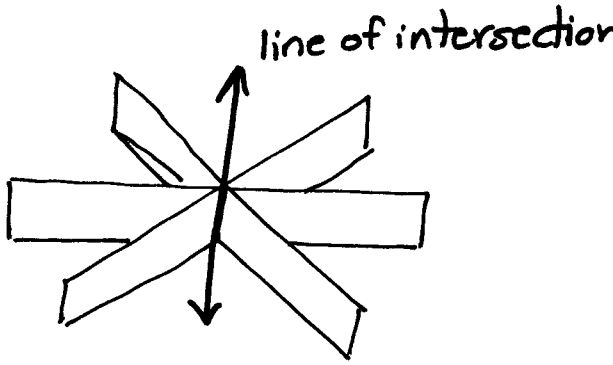
$y + z - 2 = 0$ reduced

$$\begin{array}{r} 3x + 2y - z + 2 = 0 \\ -\ 3x + 14y + 11z - 22 = 0 \\ \hline -12y - 12z + 24 = 0 \div (-12) \end{array}$$

$y + z - 2 = 0$ reduced

$$\begin{array}{r} y + z - 2 = 0 \\ -\ y + z - 2 = 0 \\ \hline 0 = 0 \end{array}$$

$\therefore$ Infinite Solutions
(Intersect in a line)



Line Of Intersection

$$\boxed{\vec{r} = (-2, 2, 0) + t(1, -1, 1)}$$

$\therefore\ z = 0 + 1t$

$y + z - 2 = 0$
$y + (0 + 1t) - 2 = 0$
$y = 2 - t$

$x + 4y + 3z - 6 = 0$
$x + 4(2 - t) + 3(0 + 1t) - 6 = 0$
$x + 8 - 4t + 3t - 6 = 0$
$x - t + 2 = 0$
$x = -2 + t$

Solve:

$$10x - 3y \boxed{-7z} + 18 = 0$$
$$12x - 11y \boxed{-z} + 66 = 0$$
$$22x - 14y \boxed{-8z} + 158 = 0$$

$$10x - 3y - 7z + 18 = 0$$
$$12x - 11y - z + 66 = 0 \ldots \times 7$$

$$\begin{array}{rl} & 10x - 3y - 7z + 18 = 0 \\ - & 84x - 77y - 7z + 462 = 0 \\ \hline & -74x + 74y - 444 = 0 \div (-74) \end{array}$$

$$x - y + 6 = 0 \text{ (reduced)}$$

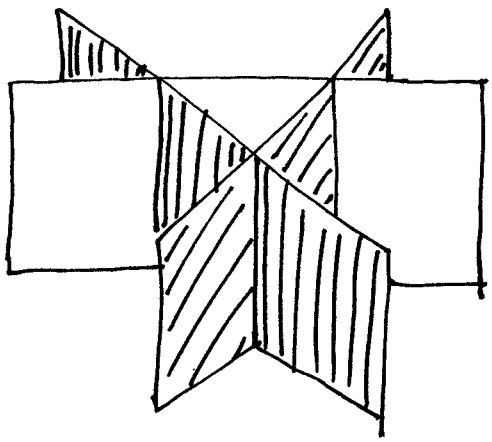
$$12x - 11y - z + 66 = 0 \ldots \times 8$$
$$22x - 14y - 8z + 158 = 0$$

$$\begin{array}{rl} & 96x - 88y - 8z + 528 = 0 \\ - & 22x - 14y - 8z + 158 = 0 \\ \hline & 74x - 74y + 370 = 0 \div 74 \end{array}$$

$$x - y + 5 = 0 \text{ (reduced)}$$

$$\begin{array}{rl} & x - y + 6 = 0 \\ - & x - y + 5 = 0 \\ \hline & 1 \neq 0 \end{array}$$

$\therefore$ No Solution



Plane Practice

Solve for the intersection of the following planes. Draw a geometric representation of each.

a)
$$5x + 3y - 2z + 1 = 0$$
$$2x + 4y + 2z - 8 = 0$$
$$4x + 3y - z + 5 = 0 \quad \times 2$$

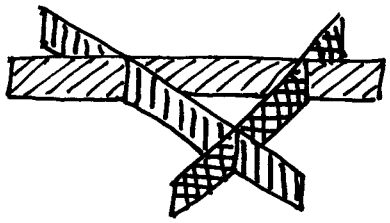
$$\begin{array}{r} + \quad 5x+3y-2z+1=0 \\ 2x+4y+2z-8=0 \\ \hline 7x+7y-7=0 \quad \div 7 \\ x+y-1=0 \end{array}$$

$$\begin{array}{r} + \quad 2x+4y+2z-8=0 \\ 8x+6y-2z+10=0 \\ \hline 10x+10y+2=0 \quad \div 2 \\ 5x+5y+1=0 \end{array}$$

$$x+y-1=0 \ldots \times 5$$
$$5x+5y+1=0$$

$$\begin{array}{r} 5x+5y-5=0 \\ - \quad 5x+5y+1=0 \\ \hline -6=0 \end{array}$$

$\therefore$ No Solution!



b)
$$\pi_1: \quad x + 7y - 6z + 17 = 0$$
$$\pi_2: \quad \vec{p} = (7,-2,3) + s(1,1,2) + t(9,-5,-3)$$
$$\pi_3: \quad \begin{cases} x = 2 + 4s + t \\ y = 4 + s + 9t \\ z = 4 - 2s + 3t \end{cases}$$

$$\pi_2 = (7,-2,3) + s(1,1,2) + t(9,-5,-3)$$

$$\begin{array}{rr} 1 & -5 \\ 2 & -3 \\ 1 & 9 \\ 1 & -5 \end{array}$$

$$\vec{N} = (1,1,2) \times (9,-5,-3)$$
$$\vec{N} = (-3+10,\ 18+3,\ -5-9)$$
$$\vec{N} = (7, 21, -14) \text{ reduce}$$
$$\vec{N} = (1, 3, -2)$$

$$1(7) + 3(-2) - 2(3) + D = 0$$
$$7 - 6 - 6 + D = 0$$
$$-5 + D = 0$$
$$D = 5$$

$$Ax + By + Cz + D = 0$$
$$1x + 3y - 2z + D = 0$$

$$\boxed{\pi_2: x + 3y - 2z + 5 = 0}$$

π_3 : $x = 2 + 4s + t$
$y = 4 + s + 9t$
$z = 4 - 2s + 3t$

$\vec{r} = (2,4,4) + s(4,1,-2) + t(1,9,3)$

$$\begin{matrix} 1 & 9 \\ -2 & 3 \\ 4 & 1 \\ 1 & 9 \end{matrix}$$

$\vec{N} = (4,1,-2) \times (1,9,3)$
$\vec{N} = (3+18, -2-12, 36-1)$
$\vec{N} = (21, -14, 35)$ reduce
$\vec{N} = (3, -2, 5)$

$Ax + By + Cz + D = 0$
$3x - 2y + 5z + D = 0$
$3(2) - 2(4) + 5(4) + D = 0$
$6 - 8 + 20 + D = 0$
$18 + D = 0$
$D = -18$

$$\boxed{\pi_3 : \ 3x - 2y + 5z - 18 = 0}$$

$x + 7y - 6z + 17 = 0$
$x + 3y - 2z + 5 = 0 \ldots \times 3$
$3x - 2y + 5z - 18 = 0$

$$\begin{array}{rl} - & x + 7y - 6z + 17 = 0 \\ & x + 3y - 2z + 5 = 0 \\ \hline & 4y - 4z + 12 = 0 \ \div 4 \\ & y - z + 3 = 0 \end{array}$$

$$\begin{array}{rl} - & 3x + 9y - 6z + 15 = 0 \\ & 3x - 2y + 5z - 18 = 0 \\ \hline & 11y - 11z + 33 = 0 \ \div 11 \\ & y - z + 3 = 0 \end{array}$$

same ~~inner~~ equation

$\therefore$ Intersect in a line (Infinite Solutions)

$$\boxed{\vec{r} = (4, -3, 0) + t(-13, 1, 1)}$$

$z = 0 + 1t$

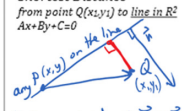
$y - z + 3 = 0$
$y - (0 + 1t) + 3 = 0$
$y - 1t + 3 = 0$
$y = -3 + 1t$

$x + 7y - 6z + 17 = 0$
$x + 7(-3 + 1t) - 6(0 + 1t) + 17 = 0$
$x - 21 + 7t - 0 + 6t + 17 = 0$
$x + 13t - 4 = 0$
$x = 4 - 13t$

Distances ANS

METHOD 1

Shortest Distance
from point $Q(x_1,y_1)$ to line in R^2
$Ax+By+C=0$



Distance = shadow of $\vec{PQ}$ on $\vec{n}$

$= \left|\vec{PQ} \downarrow \vec{n}\right| = \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right|$

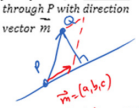
$= \left|\frac{(x_1-x,\ y_1-y)\cdot(A,B)}{\sqrt{A^2+B^2}}\right|$

$= \frac{|Ax_1+By_1-Ax-By|}{\sqrt{A^2+B^2}}$

$= \left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$

pt. not on the line.

Shortest Distance
from point Q to line in R^3
through P with direction vector $\vec{m}$



$\vec{m}=(a,b,c)$

infinitely many $\vec{n}$
can't do shadow of $\vec{PQ}$ on $\vec{n}$ here!

$A_{\square} = |\vec{m}\times\vec{PQ}|$

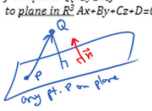
$\therefore$ distance = height of $\square$

$= \frac{A_{\square}}{\text{base}}$

$= \frac{|\vec{m}\times\vec{PQ}|}{|\vec{m}|}$

don't reduce here!

Shortest Distance
from point $Q(x_1,y_1,z_1)$
to plane in R^3 $Ax+By+Cz+D=0$



distance = shadow of $\vec{PQ}$ on $\vec{n}$

$= \left|\vec{PQ} \downarrow \vec{n}\right|$

$= \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right|$ Like before

$= \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$

METHOD 2:

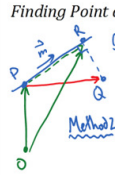
Shortest Distance
from point Q to line in R^2 or R^3 through R with direction vector $\vec{m}$

- convert line into parametric + vector forms (to see $\vec{m}$)
- general form of pt. R using parameter
- $\vec{m}\cdot\vec{QR}=0$ solve for parameter
- dist $=|\vec{QR}|$

Shortest Distance
from point $Q(x_1,y_1,z_1)$
to plane in R^3 $Ax+By+Cz+D=0$

- convert plane into parametric + vector forms to see $\vec{m}_1, \vec{m}_2$
- general form of pt. R
- $\vec{m}_1\cdot\vec{QR}=0$, $\vec{m}_2\cdot\vec{QR}=0$ } solve this system for 2 parameters
- dist $=|\vec{QR}|$

Finding Point at Minimum Distance



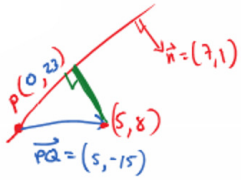
Method 1 $\vec{OR} = \vec{OP} + \text{Proj}(\vec{PQ} \text{ on } \vec{m})$

$= \vec{OP} + \frac{\vec{PQ}\cdot\vec{m}}{|\vec{m}|^2}\vec{m}$

Method 2
- convert line to parametric and vector forms to see $\vec{m}$
- state general pt. R using parameter
- $\vec{m}\cdot\vec{QR}=0$ solve for parameter

random pt. on line
sub $x=0$ or anything
$y=23$

Find the distance from the point $Q(5, 8)$ to the line $7x + y - 23 = 0$.



$D = \left|\text{proj}(\vec{PQ} \text{ on } \vec{n})\right|$

$= \left|\frac{(5,-15)\cdot(7,1)}{\sqrt{7^2+1^2}}\right|$

$= \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$(0,23)+t\underbrace{(1,-7)}_{\vec{m}}$

$x=t$
$y=-7t+23$

OR

pt $R=(t,-7t+23)$

$\vec{QR}=(t-5,\ -7t+23-8)$

$=(t-5,\ -7t+15) = \left(-\frac{14}{5}, \frac{-2}{5}\right)$

$\vec{m}\cdot\vec{QR}=0$

$t-5-7(-7t+15)=0$

$t-5+49t-105=0$

$50t=110$

$t=\frac{11}{5}$

$|\vec{QR}| = \sqrt{\frac{196}{25}+\frac{4}{25}}$

$\sqrt{8}$
$=2\sqrt{2}$

c. Calculate the distance between the two parallel planes $2x - y + 2z + 4 = 0$ and $2x - y + 2z + 16 = 0$

Q - find a random pt. Q on π_1

let $x=0$, $z=0$, then $y=4$

$$\text{dist.} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

from the different plane than where Q is

$$= \frac{|2(0) + -1(4) + 2(0) + 16|}{\sqrt{2^2+1^2+2^2}}$$

$$= 4$$

$x = t$
$y = 2t + 2r + 16$
$z = r$

$\vec{m}_1 = (1,2,0)$
$\vec{m}_2 = (0,2,1)$

pt. R $(t, 2t+2r+16, r)$

$\vec{QR} = (t-0, 2t+2r+16-4, r-0)$
$= (t, 2t+2r+12, r)$

$\vec{m}_1 \cdot \vec{QR} = 0$
$t + 4t + 4r + 24 = 0$
$5t + 4r = -24$
$\times 4$ $20t + 16r = -96$

$\vec{m}_2 \cdot \vec{QR} = 0$
$4t + 4r + 24 + r = 0$
$4t + 5r = -24$
$\times 5$ $20t + 25r = -120$

subtract
$9r = -24$
$r = -\frac{8}{3}$

$t = -\frac{8}{3}$

$\vec{QR} = \left(-\frac{8}{3}, \frac{4}{3}, -\frac{8}{3}\right)$

$\sqrt{\frac{64+16+64}{9}}$

$\sqrt{16}$

4

d. Determine the distance between point Q(-2,1,0) and line $\vec{r} = (0,1,0) + t(1,1,2), t \in R$

$$\text{dist} = \frac{|\vec{PQ} \times \vec{m}|}{|\vec{m}|}$$

P Q

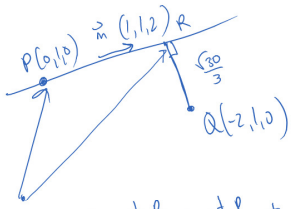
$\vec{PQ} = (0+2, 1-1, 0-0)$
$= (2,0,0)$

$\vec{PQ}$: 2/ 0 0 2 0 0
$\vec{m}$: 1 2 1 1 2

$(0,-4,2)$

$$= \frac{\sqrt{0^2+4^2+2^2}}{\sqrt{1^2+1^2+2^2}} = \frac{\sqrt{20}}{\sqrt{6}} = \frac{\sqrt{10}}{\sqrt{3}} = \frac{\sqrt{30}}{3}$$

e. For question d. determine point R on the line at which minimum distance occurs (use METHOD 1 &2 then check if your answer in d. is correct).



$\vec{PQ} = (-2-0, 1-1, 0-0)$
$= (-2,0,0)$

Method 1

pt. R = pt. P + proj($\vec{PQ}$ on $\vec{m}$)

$$\vec{OR} = \vec{OP} + \frac{\vec{PQ} \cdot \vec{m}}{|\vec{m}|^2}\vec{m}$$

$$= (0,1,0) + \frac{-2(1)}{1^2+1^2+2^2}(1,1,2)$$

$$= (0,1,0) - \frac{2}{6}(1,1,2)$$

$$= (0,1,0) + \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

pt. R $= \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$

Method 2

pt. R $(t, 1+t, 2t)$

$\vec{QR}$ $(t+2, t, 2t)$

$\vec{m} \cdot \vec{QR} = 0$

$(1,1,2) \cdot (t+2, t, 2t) = 0$

$t + 2 + t + 4t = 0$
$6t = -2$
$t = -\frac{1}{3}$

$\therefore$ pt. R $\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$

check

$\therefore \vec{RQ} = \left(-2+\frac{1}{3}, 1-\frac{2}{3}, 0+\frac{2}{3}\right)$
$= \left(-\frac{5}{3}, \frac{1}{3}, \frac{2}{3}\right)$

dist $= \sqrt{\frac{25}{9} + \frac{1}{9} + \frac{4}{9}}$

$\sqrt{\frac{30}{9}} = \frac{\sqrt{30}}{3}$