Algebraic Vectors

Geometric Vectors

- Geometric vectors are vectors with no fixed location
- Geometric vectors are written as a magnitude and a direction

i.e. $\vec{a}$ = 16 km [N 25° W]

Algebraic Vectors

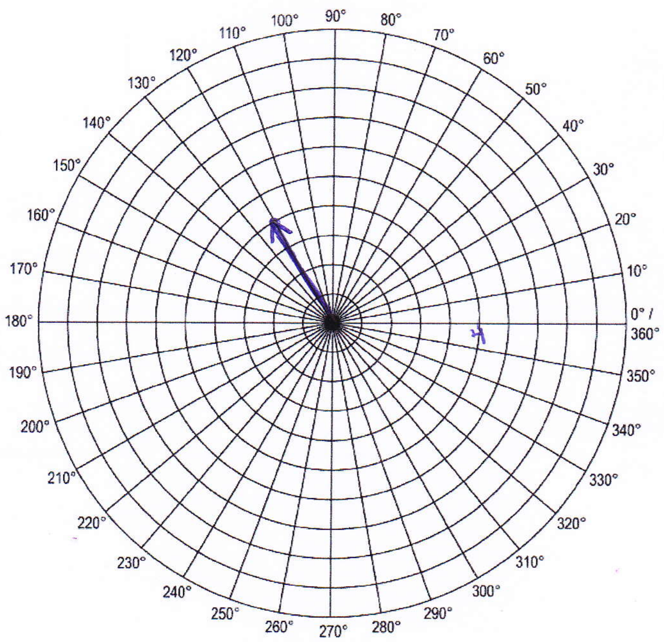
- Algebraic vectors are vectors that are drawn on a coordinate plane with the tail at (0, 0).

Polar Coordinates

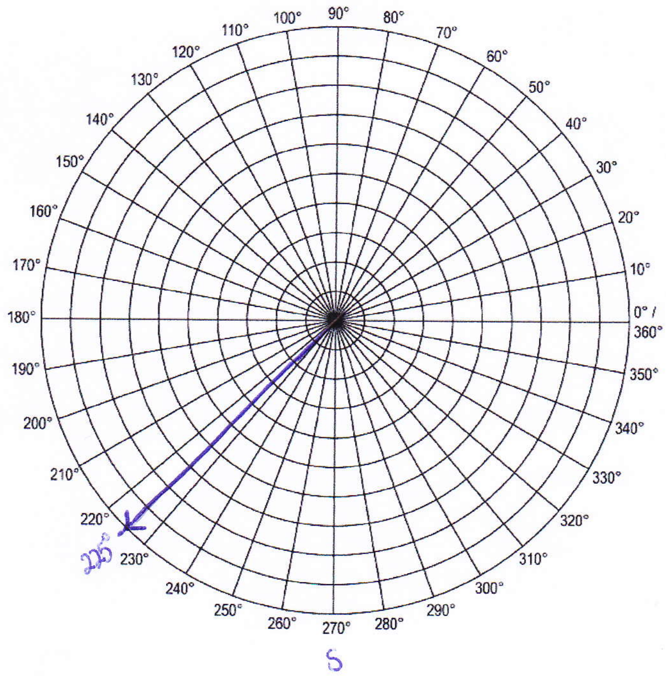
- Algebraic vectors can be written using polar coordinates in the form (Magnitude, Direction) where the angle is measured from the terminal arm (positive x-axis).

i.e. (5, 10°) or (3.79, 187°)

1. Plot the vector $\vec{a} = (4, 120°)$



2. Plot the vector $\vec{a}$ = 9 units [S45°W]



$\vec{a} = (9, 225°)$

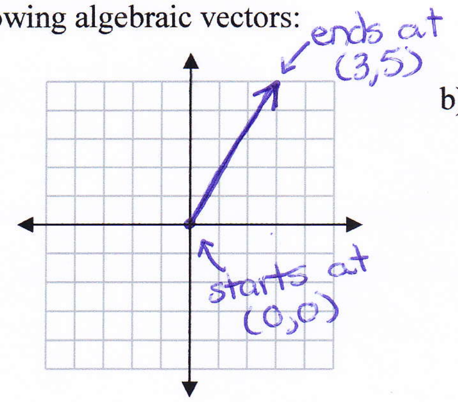
Cartesian Coordinates

- Algebraic vectors can be written using Cartesian coordinates in cartesian (coordinate) or unit vector form.

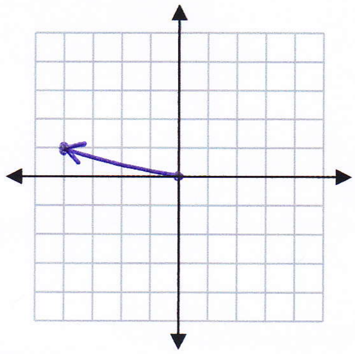
i.e. $\vec{a} = (5, 7)$ or $\vec{a} = 5\hat{i} + 7\hat{j}$

1. Draw the following algebraic vectors:

a) $\vec{a} = (3, 5)$



b) $\vec{a} = (-4, 1)$



2. Write the vector $\overrightarrow{AB}$ in component form if:

Head - Tail

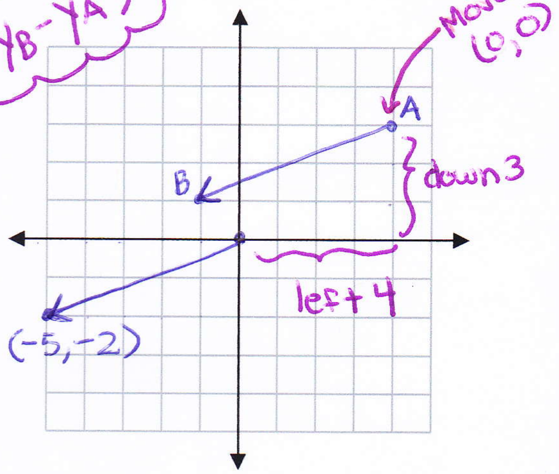
$\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$

a) A = (4, 3) and B = (-1, 1)

$\overrightarrow{AB} = (-5, -2)$

-1-4

1-3



b) A = (25, 70) and B = (15, 100)

$\overrightarrow{AB} = (15-25, 100-70)$

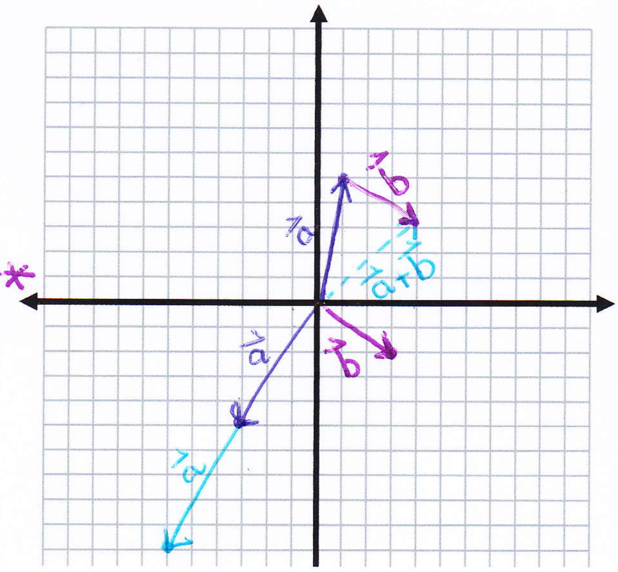
$\overrightarrow{AB} = (-10, 30)$

3. Determine $\vec{a} + \vec{b}$ if $\vec{a} = (1, 5)$ and $\vec{b} = (3, -2)$

$\vec{a} + \vec{b} = (4, 3)$

1+3   5+(-2)

* Add xs together, add ys together *



4. Determine $2\vec{a}$ if $\vec{a} = (-3, -5)$

$2\vec{a} = (-6, -10)$

2(-3)   2(-5)

* Multiply x and y by coefficient *

5. Simplify $10\vec{a} - 3\vec{b}$ if $\vec{a} = (-2, 7)$ and $\vec{b} = (3, 1)$

$$10\vec{a} - 3\vec{b} = 10(-2,7) - 3(3,1)$$
$$= (-20,70) - (9,3)$$
$$= (-29, 67)$$

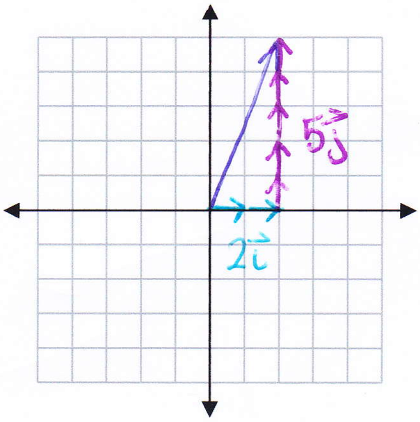
Unit Vector Form

- Algebraic vectors can be written using unit vectors.

$\vec{i}$ = unit vector on x-axis

$\vec{j}$ = unit vector on y-axis

1. Write each of the following vectors in unit vector form:



a) $\vec{a} = (2, 5)$

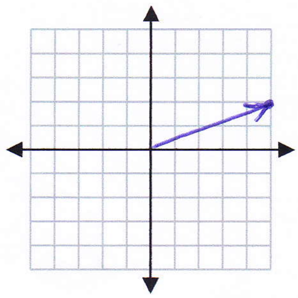
$\vec{a} = 2\vec{i} + 5\vec{j}$

b) $\vec{b} = (-3, 10)$

$\vec{b} = -3\vec{i} + 10\vec{j}$

Magnitude of an Algebraic Vector

Calculate the magnitude of the following algebraic vectors:



a) $\vec{a} = (5, 2)$

$D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$D = \sqrt{(5-0)^2 + (2-0)^2}$

$D = \sqrt{(5)^2 + (2)^2}$

$D = \sqrt{25+4}$

$D = \sqrt{29}$

$|\vec{a}| = \sqrt{29}$

b) $\vec{a} = (-7, 3)$

$|\vec{a}| = \sqrt{(-7)^2 + (3)^2}$

$= \sqrt{49+9}$

$= \sqrt{58}$

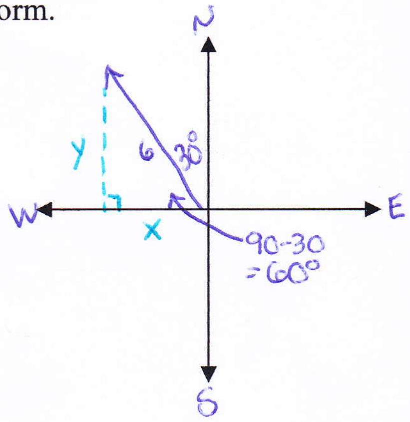
$$|\vec{a}| = \sqrt{x^2 + y^2}$$

Converting Between Forms

1. Write $\vec{a} = -4\vec{i} + 3\vec{j}$ in component form:

$\vec{a} = (-4, 3)$

2. Write the vector $\vec{a} = 6$ m [N30°W] as an algebraic vector in component form.



$\sin\theta = \frac{o}{h}$

$\sin 60 = \frac{y}{6}$

$6(\sin 60) = y$

$\boxed{5.20 \doteq y}$

$\cos\theta = \frac{a}{h}$

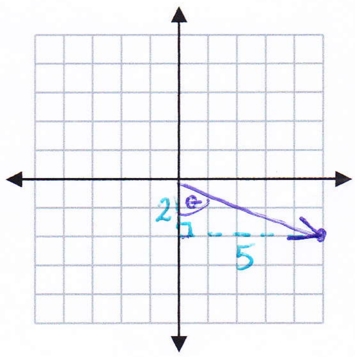
$\cos 60 = \frac{x}{6}$

$6(\cos 60) = x$

$\boxed{3 = x}$

$\therefore \vec{a} = (-3, 5.20)$

3. Write the vector $\vec{a} = (5, -2)$ as a geometric vector.



$|\vec{a}| = \sqrt{x^2 + y^2}$

$= \sqrt{(5)^2 + (-2)^2}$

$= \sqrt{25 + 4}$

$= \sqrt{29}$

$\tan\theta = \frac{o}{a}$

$\tan\theta = \frac{5}{2}$

$\theta = \tan^{-1}\left(\frac{5}{2}\right)$

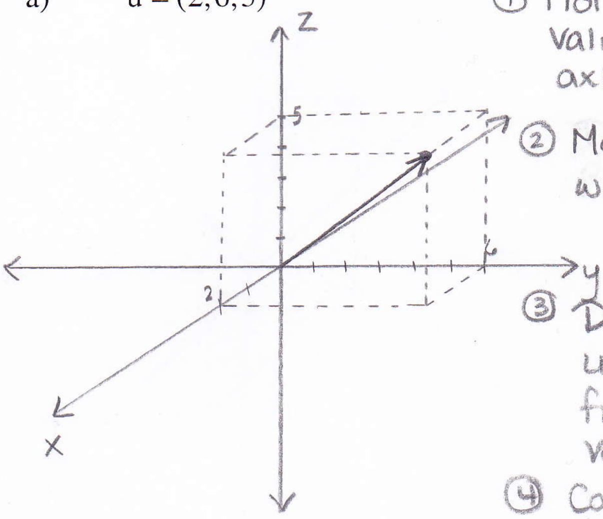
$\theta \doteq 68.20$

$\therefore \vec{a} = \sqrt{29}$ units [S68.20°E]

Drawing Vectors in Three Dimensions

Draw each of the following vectors:

a) $\vec{u} = (2,6,5)$



① Plot x, y, z values on axis

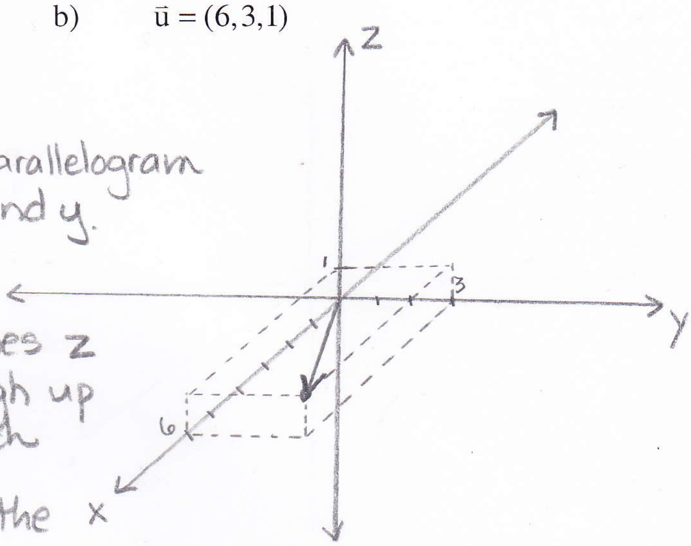
② Make a parallelogram with x and y.

③ Draw lines z units high up from each vertex

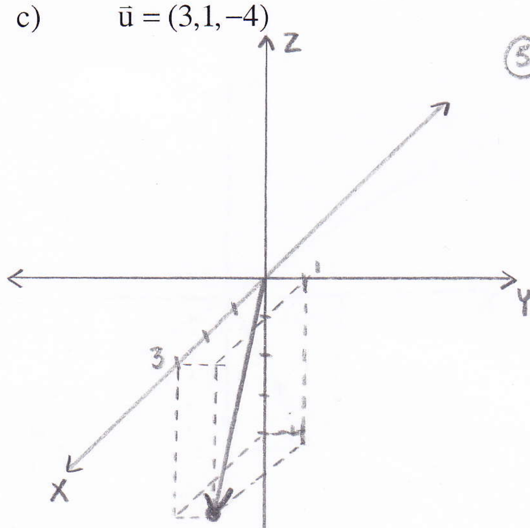
④ Complete the box

⑤ Point is in opposite corner from vertex.

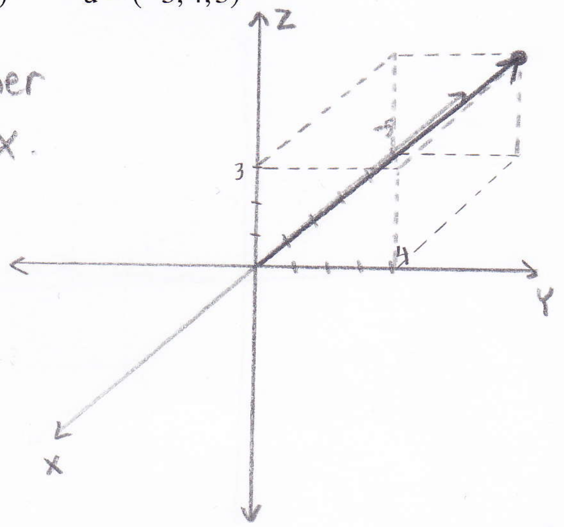
b) $\vec{u} = (6,3,1)$



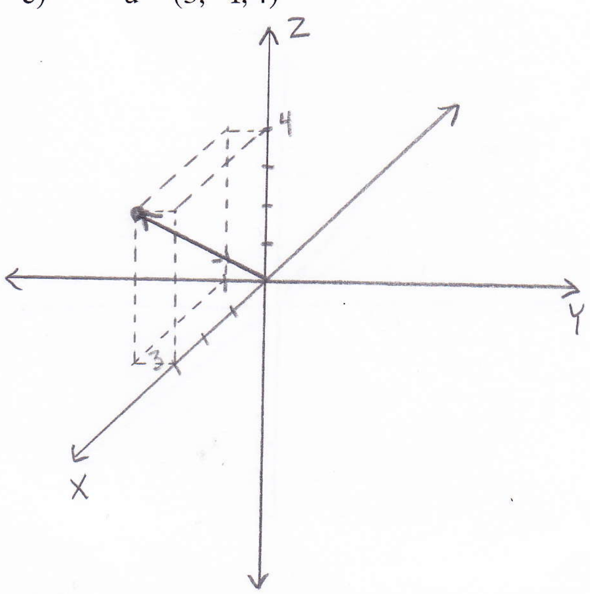
c) $\vec{u} = (3,1,-4)$



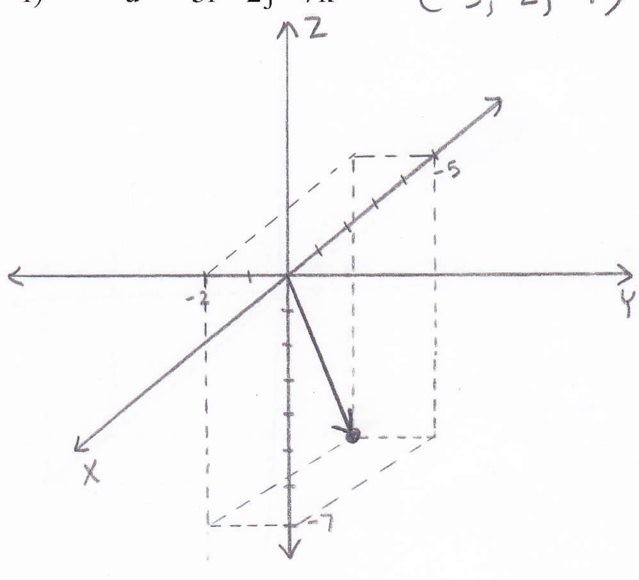
d) $\vec{u} = (-5,4,3)$



e) $\vec{u} = (3,-1,4)$

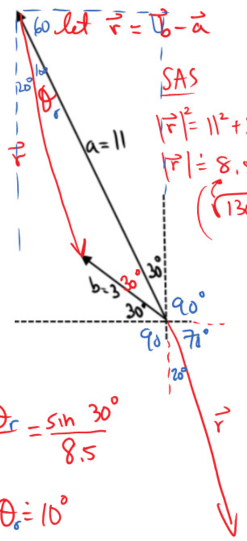


f) $\vec{u} = -5\hat{i} - 2\hat{j} - 7\hat{k}$ $= (-5,-2,-7)$



Velocity Again ANS

Find $\vec{b}-\vec{a}$ using Geometric vectors with Sine and Cosine laws	Find $\vec{b}-\vec{a}$ using Algebraic Vectors then check if the answers are the same.



let $\vec{r}=\vec{b}-\vec{a}$

SAS

$|\vec{r}|^2 = 11^2+3^2-2(3)(11)\cos 30°$

$|\vec{r}| \doteq 8.5$

$\left(\sqrt{130-66\frac{\sqrt{3}}{2}} \text{ or } \sqrt{130-33\sqrt{3}}\right)$

$\frac{\sin\theta_r}{3} = \frac{\sin 30°}{8.5}$

$\theta_r \doteq 10°$

Note: Ambiguous Case happens when you use $\sin^{-1}(\quad)$ on an obtuse angle.

$\therefore$ geo. version

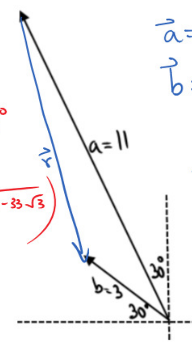
$\vec{r} = (8.5, 290°)$
  r θ

convert $8.5(\cos 290°, \sin 290°)$?

yes :)

OR $|\vec{r}| = 8.5$, $\angle(\vec{r},\vec{a}) = 170°, 190°$

$\angle(\vec{r},\vec{b}) = 140°, 220°$



$\vec{a} = (11, 120°)$
  r θ

$\vec{b} = (3, 150°)$
  r θ

geo → convert to alg.

$\vec{a} = 11(\cos 120°, \sin 120°)$

$\vec{a} \doteq (-5.5, 9.5)$
 left up

$\vec{b} = 3(\cos 150°, \sin 150°)$

$\vec{b} \doteq (-2.6, 1.5)$

$\therefore \vec{r} = \vec{b} - \vec{a}$

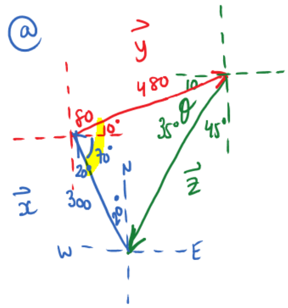
$= (-2.6, 1.5) - (-5.5, 9.5)$

$\vec{r} = (2.9, -8.0)$

A search and rescue aircraft, travelling at a speed of 240km/h, starts out at a heading of $N20^o W$. After travelling for one hour and fifteen minutes, it turns to a heading of $N80^o E$ and continues for another 2 hours before returning to base.

a) Determine the displacement vector for each leg of the trip. – use a method of your choice
b) Find the total distance the aircraft travelled and how long it took.

(a)



$\therefore \vec{x} = 300\text{km}\,[N20°W]$

$\vec{y} = 480\text{ km}\,[N80°E]$

$\vec{z} = 520\text{km}\,[S45°W]$

$D = VT$

$= \left(240\frac{\text{km}}{\text{h}}\right)(1.25\text{ h})$

$= 300\text{ km}$

$D = (240)(2)$

$= 480\text{ km}$

$|\vec{z}|^2 = 300^2 + 480^2 - 2(300)(480)\cos 80°$

$|\vec{z}| \doteq 520\text{ km}$

$\frac{\sin\theta}{300} = \frac{\sin 80}{520}$

$\theta = \sin^{-1}\left(\frac{300\sin 80}{520}\right)$

$\theta = 35°$

(b) Total distance = 1300 km
$\therefore$ (scalar)

Total Time = 1 hr 15_{min} + 2hr + 2hr 10min

= 5 hrs 25 min

$T = \frac{D}{V}$

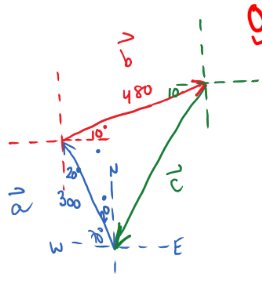
$= \frac{520}{240}$

$= 2.1\bar{6}$

$0.1\bar{6} \times 60\text{min}$

$\sim 10\text{min}$

OR Component Method



$\vec{a} = 300(\cos 110°, \sin 110°)$

$\vec{b} = 480(\cos 10°, \sin 10°)$

$\vec{c} = \vec{a} + \vec{b} \doteq (370.102, 365.259)$ * algebraic

$\therefore |\vec{c}| = \sqrt{365.259^2 + 370.102^2}$

$\doteq 520\text{ km}$

$\theta = \tan^{-1}\left(\frac{370.102}{365.259}\right)$

$\theta = 45°$

$\therefore \vec{c} \doteq 520\text{km}\,[S45°W]$ * geometric

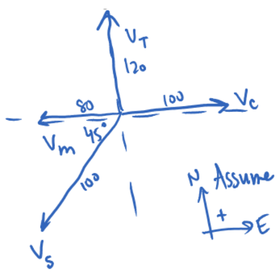
The **relative velocity** of the object B traveling at $\vec{v}_B$ relative to the object A traveling at $\vec{v}_A$ is given by: $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$

For **boat in water** (similarly for **plane with wind**) questions: $\vec{v}_{BG} = \vec{v}_{BW} + \vec{v}_{WG}$ where $\vec{v}_{BG}$ is velocity of boat relative to ground, $\vec{v}_{BW}$ is velocity of boat in still water, and $\vec{v}_{WG}$ is velocity of water relative to ground.

A car is traveling at $\vec{v}_c = 100km/h[E]$, a motorcycle is traveling at $\vec{v}_m = 80km/h[W]$, a truck is traveling at $\vec{v}_t = 120km/h[N]$ and an SUV is traveling at $\vec{v}_s = 100km/h[SW]$. Find the relative velocity of the car relative to:

a) motorcycle
b) truck
c) SUV

Give answers in both algebraic and geometric forms



ⓐ $\vec{V}_{cm} = \vec{V}_c - \vec{V}_m$

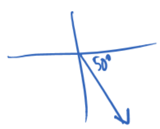
$= (100,0) - (-80,0)$

alg $= (180,0)$

geo $= 180 km/h$ [E]

ⓑ $\vec{V}_{CT} = \vec{V}_c - \vec{V}_T$

$= (100,0) - (0,120)$



alg $= (100,-120)$

geo $\doteq 156.2$ [S 40° E]

ⓒ $\vec{V}_{cs} = \vec{V}_c - \vec{V}_s$

$= (100,0) - \left(100\frac{\sqrt{2}}{2}, -100\frac{\sqrt{2}}{2}\right)$

$= (100 + 50\sqrt{2},\ 50\sqrt{2})$

alg $\doteq (170.71, 70.71)$

geo $\doteq 184.8$ [N 67° E]

Vectors in Three Dimensions

Algebraic Form

Component Form	Unit Vector Form
$\vec{u} = (x, y, z)$	$\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$

Write $\vec{u} = (2, -3, 1)$ in unit vector form

$\vec{u} = 2\vec{i} - 3\vec{j} + \vec{k}$

Write $\vec{u} = -7\vec{i} + 5\vec{j} + 9\vec{k}$ in component form

$\vec{u} = (-7, 5, 9)$

Geometric Form

Magnitude	Direction (Direction Cosines)
$\lvert\vec{u}\rvert = \sqrt{x^2 + y^2 + z^2}$	$\text{Cos}\alpha = \frac{x}{\lvert\vec{u}\rvert}$ $\text{Cos}\beta = \frac{y}{\lvert\vec{u}\rvert}$ $\text{Cos}\gamma = \frac{z}{\lvert\vec{u}\rvert}$

Write the vector $\vec{u} = (3, 1, -4)$ as a geometric vector.

$$\lvert\vec{u}\rvert = \sqrt{x^2+y^2+z^2}$$
$$= \sqrt{(3)^2+(1)^2+(-4)^2}$$
$$= \sqrt{9+1+16}$$
$$\boxed{\lvert\vec{u}\rvert = \sqrt{26}}$$

$$\text{Cos}\,\alpha = \frac{x}{\lvert\vec{u}\rvert}$$
$$\text{Cos}\,\alpha = \frac{3}{\sqrt{26}}$$
$$\alpha = \text{Cos}^{-1}\left(\frac{3}{\sqrt{26}}\right)$$
$$\boxed{\alpha \doteq 53.96^\circ}$$

$$\text{Cos}\,\beta = \frac{y}{\lvert\vec{u}\rvert}$$
$$\text{Cos}\,\beta = \frac{1}{\sqrt{26}}$$
$$\beta = \text{Cos}^{-1}\left(\frac{1}{\sqrt{26}}\right)$$
$$\boxed{\beta \doteq 78.69^\circ}$$

$$\text{Cos}\,\gamma = \frac{z}{\lvert\vec{u}\rvert}$$
$$\text{Cos}\,\gamma = \frac{-4}{\sqrt{26}}$$
$$\gamma = \text{Cos}^{-1}\left(\frac{-4}{\sqrt{26}}\right)$$
$$\boxed{\gamma \doteq 141.67^\circ}$$

Scalar Multiplication

a) $3(4\vec{i}+2\vec{j}-\vec{k})$

$= 12\vec{i}+6\vec{j}-3\vec{k}$

b) $10(3,7,1)$

$= (30,70,10)$

Vector Addition

a) $(5,7,-3)+(6,2,4)$

$= (11,9,1)$

b) $(2\hat{i}+15\hat{j}+3\hat{k})-(6\hat{i}-4\hat{j}+2\hat{k})$

$= -4\vec{i}+19\vec{j}+\vec{k}$

Parallel Vectors

- Parallel vectors have the same or opposite direction.
- Two vectors are parallel if they are scalar multiples.

a) For the points A(3, 2, 7), B(4, 5, 1), C(-4, 7, 1), D(-6, 1, 13), determine whether $\overline{AB}$ is parallel to $\overline{CD}$.

$\vec{AB} = (4-3, 5-2, 1-7)$
$= (1,3,-6)$

$\vec{CD} = (-6-(-4), 1-7, 13-1)$
$= (-2,-6,12)$

FACTOR

$= -2(1,3,-6)$

$\therefore$ Parallel ✓

Collinear Points

- Three points (A, B, C) are collinear if $\vec{AB}$ and $\vec{AC}$ are Parallel.

a) Determine whether the points A (4, -2, 3), B(3, 2, 7), C(1, 10, 15) are collinear.

$\vec{AB} = (3-4, 2-(-2), 7-3)$
$= (-1,4,4)$

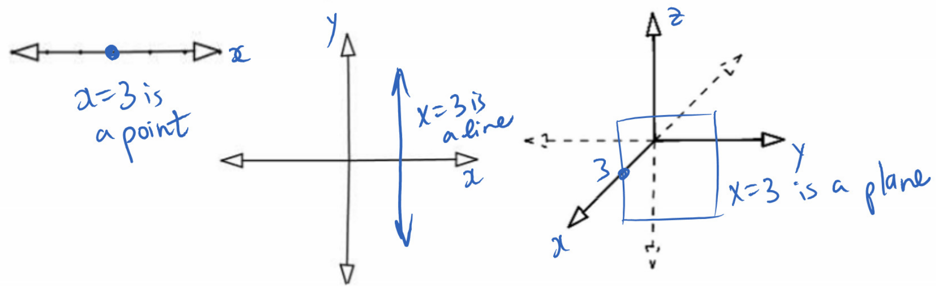
$\vec{AC} = (1-4, 10-(-2), 15-3)$
$= (-3,12,12)$
$= 3(-1,4,4)$

$\therefore$ Collinear ✓

Linear Combinations ANS

Describe what $x = 3$ would mean in each dimension.

R – line | **R^2 – plane** | **R^3 – space**

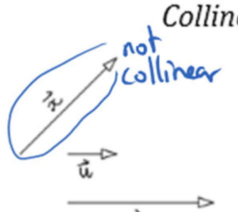


$x=3$ is a point

$x=3$ is a line

$x=3$ is a plane

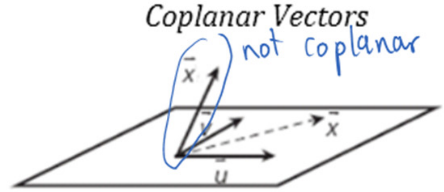
Collinear Vectors



not collinear

$\vec{x}$ and $\vec{u}$ are collinear $\stackrel{\text{iff}}{\Longleftrightarrow}$ there is a scalar k such that $\vec{x} = k\vec{u}$

Coplanar Vectors



not coplanar

$\vec{u}, \vec{v}$ and $\vec{x}$ are coplanar $\stackrel{\text{iff}}{\Longleftrightarrow}$ there are scalars a and b such that $\vec{x} = a\vec{u} + b\vec{v}$

Parallel/Collinear Vectors can be written as <u>scalar multiples</u> of each other

Ex. if $\exists$ k such that $\vec{a} = k\vec{b}$ then $\vec{a} \parallel \vec{b}$

symbol for "there exist" ($\exists$); symbol for "parallel" ($\parallel$)

Coplanar Vectors can be written as <u>linear combination</u> of each other.
Vectors can also be said to be **Linearly Dependent**

Ex. if $\exists$ k and m such that $\vec{c} = k\vec{a} + m\vec{b}$ then $\vec{c}, \vec{a}$ and $\vec{b}$ on same plane.

Spanning Set for 1D – LINE in R^2 or R^3

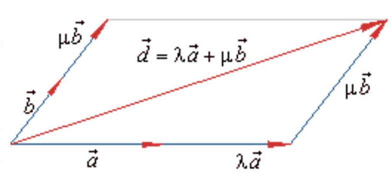
2 points
OR one vector

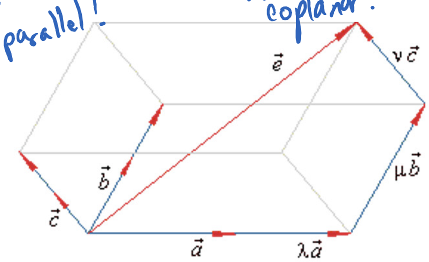
Spanning Set for 2D – PLANE in R^3

3 points
OR 2 vectors but must be non parallel!

Spanning Set for 3D – SPACE=R^3

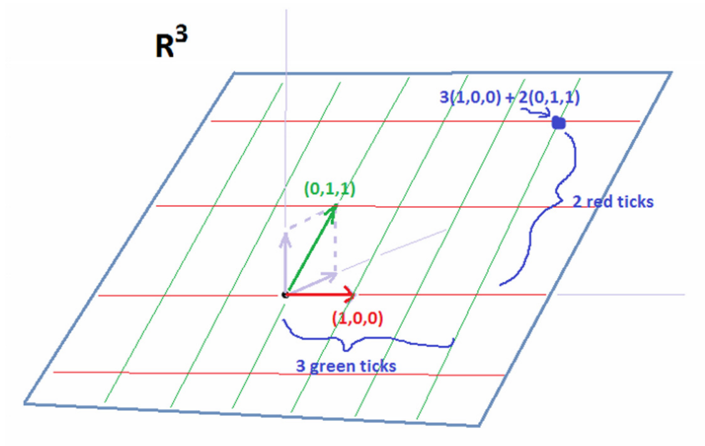
3 vectors – but must be non coplanar!





What is the set of all linear combinations of (1, 0, 0) and (0, 1, 1)?

- given 2 vectors they can either span a line if they are parallel or a plane if they are not
- Since they are not multiples of each other they span a plane. This is seen by taking a random linear combination as shown:



if we stretch (1,0,0) by factor 3 and (0,1,1) by factor 2 we get another vector:

$3(1,0,0) + 2(0,1,1)$

$= (3,2,2)$ which is still on the same plane as the original vectors

Prove that the vectors $\vec{a} = (-1,2,-3)$, $\vec{b} = (2,0,-1)$, and $\vec{c} = (-7,6,-7)$ are linear dependant.

$\vec{c} = k\vec{a} + m\vec{b}$

$(-7,6,-7) = k(-1,2,-3) + m(2,0,-1)$

x parts: $-7 = -k + 2m$ sub in $-7 = -3 + 2m$

y parts: $6 = 2k$ $\therefore k = 3$ $\quad -4 = 2m$

z parts: $-7 = -3k - m$ $\quad -2 = m$

sub both to check if contradiction → not coplanar

if ok → coplanar

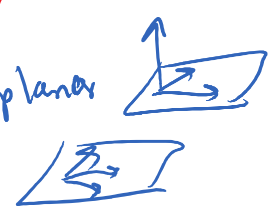
$-7 \overset{?}{=} -3(3) - (-2)$

$-9 + 2$

-7

✓

✓✓



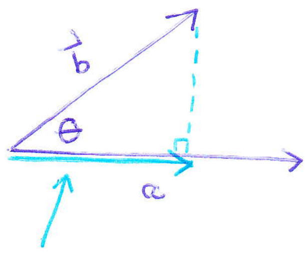
$\therefore$ Vectors are coplanar or linearly dependent

$\therefore$ span PLANE ✓

Dot Product

Dot product is the <u>Scalar multiplication</u> of one vector with the scalar projection of the other vector.

(Answer is a number)



$\cos\theta = \frac{a}{h}$

$\cos\theta = \frac{|x|}{|\vec{b}|}$

$|\vec{b}|\cos\theta = x$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Component of $\vec{b}$ in same direction as $\vec{a}$

Determine $\vec{a}\bullet\vec{b}$, if $|\vec{a}| = 20$, $|\vec{b}| = 45$ and $\theta = 55°$.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= |\vec{a}||\vec{b}|\cos\theta \\ &= 20(45)\cos 55 \\ &\doteq 516.22\end{aligned}$$

<u>Perpendicular Vectors</u>

If two vectors are perpendicular, their dot product <u>Zero!</u>

Determine $\vec{a}\bullet\vec{b}$, if $|\vec{a}| = 5$, $|\vec{b}| = 10$ and $\theta = 90°$.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= |\vec{a}||\vec{b}|\cos\theta \\ &= 5(10)\underline{\cos 90} \leftarrow \text{Always equals Zero!} \\ &= 5(10)(0) \\ &= 0\end{aligned}$$

Algebraic Vectors

$$\boxed{\vec{a}\cdot\vec{b} = x_a x_b + y_a y_b + \underbrace{z_a z_b}_{\text{If 3D}}}$$

1. If $\vec{u} = (2, -3, 1)$ and $\vec{v} = (-5, 2, 4)$, calculate $\vec{u}\cdot\vec{v}$.

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (2,-3,1)\cdot(-5,2,4) \\
&= 2(-5) + (-3)(2) + 1(4) \\
&= -10 - 6 + 4 \\
&= -12
\end{aligned}$$

2. If $\vec{u} = 3\vec{i} + 2\vec{j} + 7\vec{k}$ and $\vec{v} = 5\vec{i} - 9\vec{k}$, find $\vec{u}\cdot\vec{v}$.

no $\vec{j}$ term

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (3,2,7)\cdot(5,0,-9) \\
&= 3(5) + 2(0) + 7(-9) \\
&= 15 + 0 - 63 \\
&= 48
\end{aligned}$$

3. Find the angle between the vectors $\vec{u} = (-2, 3, 4)$ and $\vec{v} = (1, 5, 2)$.

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (-2,3,4)\cdot(1,5,2) \\
&= -2(1) + 3(5) + 4(2) \\
&= -2 + 15 + 8 \\
&= 21
\end{aligned}$$

$$\begin{aligned}
|\vec{u}| &= \sqrt{x^2+y^2+z^2} \\
&= \sqrt{(-2)^2+(3)^2+(4)^2} \\
&= \sqrt{29}
\end{aligned}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{(1)^2+(5)^2+(2)^2} \\
&= \sqrt{30}
\end{aligned}$$

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta \\
21 &= \sqrt{29}\sqrt{30}\cos\theta \\
\frac{21}{\sqrt{870}} &= \cos\theta
\end{aligned}$$

$$\cos^{-1}\left(\frac{21}{\sqrt{870}}\right) = \theta$$

$$\boxed{44.60^\circ \doteq \theta}$$

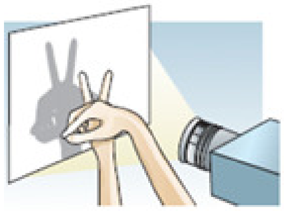
Properties of Dot Product

$k(\vec{u}\cdot\vec{v}) = k\vec{u}\cdot\vec{v}$ or $\vec{u}\cdot k\vec{v}$

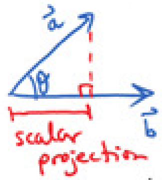
$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

$\vec{u}\cdot\vec{u} = |\vec{u}|^2$

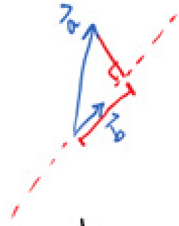
Projections ANS



$\cos\theta = \dfrac{\text{shadow}}{|\vec{a}|}$



$\longrightarrow$ vector projection



Scalar Projection

of $\vec{a}$ on $\vec{b}$ $= \left|\text{proj } \vec{a} \text{ on } \vec{b}\right|$

$= \left|\vec{a} \downarrow \vec{b}\right|$

$= \left|\,|\vec{a}|\cos\theta\,\right|$

$= \left|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right|$

Vector Projection of $\vec{a}$ on $\vec{b}$

$\text{proj}(\vec{a} \text{ on } \vec{b})$

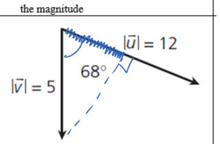
$= \vec{a} \downarrow \vec{b}$

$= \underbrace{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}_{\text{mag.}} \underbrace{\hat{b}}_{\text{dir.}}$

using unit vector $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|}$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$

Explain and show how to draw the scalar projection $|\vec{v} \downarrow \vec{u}|$ on the diagram. Find the magnitude



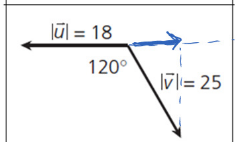
- think of $\vec{u}$ as "floor"
- drop perpendicular from $\vec{v}$'s head to the floor
- draw the "shadow"

$|\vec{v} \downarrow \vec{u}| = |\vec{v}|\cos 68°$

$= 5\cos 68°$

$\doteq 1.87$

Explain and show how to draw the vector projection $\vec{v} \downarrow \vec{u}$. Find the vector.



- think of $\vec{u}$ as "floor"
- drop perpendicular from $\vec{v}$'s head to the floor
- draw the "shadow"
- add arrow in direction of $\vec{v}$'s pull

$\vec{v} \downarrow \vec{u} = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|}\hat{u}$ or $\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\vec{u}$

$= \dfrac{\cancel{|\vec{u}|}|\vec{v}|\cos\theta}{\cancel{|\vec{u}|}}\hat{u}$

$= 25\cos 120°\,\hat{u}$

$= -12.5\,\hat{u}$

$\vec{a} = [5, 4, -1]$ and $\vec{b} = [1, -2, 3]$.

a) Find $|\vec{a} \downarrow \vec{b}|$

b) Find $\vec{b} \downarrow \vec{a}$

a) $|\vec{a} \downarrow \vec{b}| = \left|\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right|$

$= \left|\dfrac{(5)(1) + (4)(-2) + (-1)(3)}{\sqrt{1^2 + 2^2 + 3^2}}\right|$

$= \left|\dfrac{-6}{\sqrt{14}}\right|$ or $\dfrac{3\sqrt{14}}{7}$

b) $\vec{b} \downarrow \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\hat{a}$ or $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a}$

$= \dfrac{(5)(1) + (4)(-2) + (-1)(3)}{5^2 + 4^2 + 1^2}(5, 4, -1)$

$= \dfrac{-6}{42}(5, 4, -1)$

$= -\dfrac{1}{7}(5, 4, -1)$

Under what circumstances is

a. $\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \text{Proj}(\vec{v} \text{ onto } \vec{u})$?

b. $|\text{Proj}(\vec{u} \text{ onto } \vec{v})| = |\text{Proj}(\vec{v} \text{ onto } \vec{u})|$?

(a) Case 1 any vector size $\vec{u}, \vec{v}$ at $90°$:

$LS \stackrel{?}{=} RS$	
$\vec{u}\downarrow\vec{v}$	$\vec{v}\downarrow\vec{u}$
0	0 since $\vec{u} \perp \vec{v}$ $\cos 90° = 0$

Case 2 any vector size $\vec{u}, \vec{v}$ other angles:

$$\vec{u}\downarrow\vec{v} \stackrel{?}{=} \vec{v}\downarrow\vec{u}$$

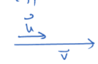
$$\frac{\cancel{\vec{u}\cdot\vec{v}}}{|\vec{v}|^2}\vec{v} \stackrel{?}{=} \frac{\cancel{\vec{u}\cdot\vec{v}}}{|\vec{u}|^2}\vec{u}$$

$$\frac{\vec{v}}{|\vec{v}|^2} \stackrel{?}{=} \frac{\vec{u}}{|\vec{u}|^2}$$

$$\vec{v} \stackrel{?}{=} \frac{|\vec{v}|}{|\vec{u}|}\vec{u}$$

looks like scalar mult.

$\vec{v} = k\vec{u} \implies \vec{u} \parallel \vec{v}$ same direction



But we need to get equal "shadows" one to the other

$\therefore$ we see that $\vec{v} = \vec{u}$

$\therefore$ for $\vec{u}\downarrow\vec{v} = \vec{v}\downarrow\vec{u}$ either vectors are at $90°$, any size OR vectors are same.

(b) $$|\vec{u}\downarrow\vec{v}| \stackrel{?}{=} |\vec{v}\downarrow\vec{u}|$$

$$\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|} \stackrel{?}{=} \frac{|\vec{v}\cdot\vec{u}|}{|\vec{u}|}$$

$$\frac{|\vec{u}|\cancel{|\vec{v}|}\cos\theta}{\cancel{|\vec{v}|}} = \frac{|\vec{v}|\cancel{|\vec{u}|}\cos\theta}{\cancel{|\vec{u}|}}$$

$\therefore |\vec{u}| = |\vec{v}|$

$\therefore |\vec{u}\downarrow\vec{v}| = |\vec{v}\downarrow\vec{u}|$ if $|\vec{u}| = |\vec{v}|$ at any angle.

Applications of Dot Product

Work

- Work occurs when a force is applied on an object, resulting in the displacement of the object.
- Measured in Joules

$$W = \vec{F}\vec{d}$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\cos\theta = \frac{|\vec{F}|}{|\vec{F}_A|}$$

$$|\vec{F}_A|\cos\theta = |\vec{F}|$$

$$W = (\vec{F})(\vec{d})$$

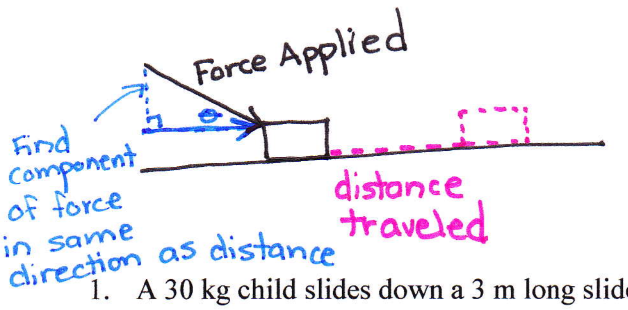
$$W = \left[|\vec{F}_A|\cos\theta\right]|\vec{d}|$$

$$W = |\vec{F}_A||\vec{d}|\cos\theta$$

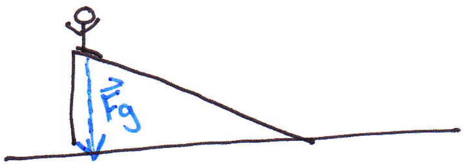
cos

or

$$\boxed{W = \vec{F}\cdot\vec{d}}$$



1. A 30 kg child slides down a 3 m long slide that makes an angle of 35° with the ground. Determine the work done by the force of gravity.



$$W = \vec{F}\cdot\vec{d}$$

$$W = |\vec{F}||\vec{d}|\cos\theta$$

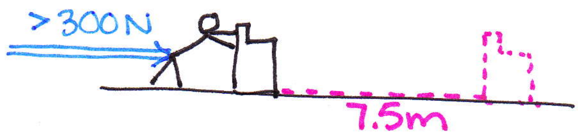
$$W = 294.3(3)\cos 35^\circ$$

$$\boxed{W \doteq 723.23\ J}$$

$$|\vec{F_g}| = 9.81(30) = 294.3\ N$$

∴ The work done by gravity is 723.23 J.

2. A mover pushes a washing machine 7.5 m across a basement floor. Calculate the work done by the mover, if he is pushing against a frictional force of 300 N.



$$W = \vec{F}\cdot\vec{d}$$

$$W = |\vec{F}||\vec{d}|\cos\theta$$

$$W > 300(7.5)\cos 0$$

$$W > 2250$$

∴ He must do <u>more than</u> 2250 J of work.

Cross Product

* answer will be a vector *

The cross product is the Vector multiplication of two vectors, which results in a vector that is perpendicular to the original two vectors.

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$$

$$\vec{a} \times \vec{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)$$

1. If $|\vec{u}| = 10$, $|\vec{v}| = 13$, and the angle between them is 25°, find $\left|\vec{u} \times \vec{v}\right|$.

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$
$$= 10(13)\sin 25$$
$$\boxed{|\vec{u} \times \vec{v}| \doteq 54.94}$$

write vector vertically starting with y

2. If $\vec{a} = (2, 3, 5)$ and $\vec{b} = (4, 7, 6)$, find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = [3(6) - 5(7), 5(4) - 2(6), 2(7) - 3(4)]$$
$$= (18 - 35, 20 - 12, 14 - 12)$$
$$\boxed{= (-17, 8, 2)}$$

$$\begin{matrix} 3 & \times & 7 \\ 5 & \times & 6 \\ 2 & \times & 4 \\ 3 & \times & 7 \end{matrix}$$

Multiply L to R

Check: $(2, 3, 5) \cdot (-17, 8, 2)$
$= -34 + 24 + 10 = 0$ ✓

Since answer is perp to $\vec{a}$ and $\vec{b}$ you can check if dot product = 0

3. Find a vector perpendicular to $\vec{u} = 10\vec{i} + 3\vec{j} + 7\vec{k}$ and $\vec{v} = -2\vec{i} - 1\vec{j}$.

cross product

No k term

$$\vec{u} \times \vec{v} = (10, 3, 7) \times (-2, -1, 0)$$
$$= [3(0) - 7(-1), 7(-2) - 10(0), 10(-1) - 3(-2)]$$
$$= (0 + 7, -14 - 0, -10 + 6)$$
$$= (7, -14, -4)$$

$$\begin{matrix} 3 & \times & -1 \\ 7 & & 0 \\ 10 & \times & -2 \\ 3 & \times & -1 \end{matrix}$$

check on calculator with dot product ✓

$$\begin{matrix} 2 & -5 \\ 1 & 7 \\ 3 & 4 \\ 2 & -5 \end{matrix}$$

4. Calculate the magnitude of $\vec{a} \times \vec{b}$ when $\vec{a} = (3, 2, 1)$ and $\vec{b} = (4, -5, 7)$

$$\vec{a} \times \vec{b} = (3,2,1) \times (4,-5,7)$$
$$= [2(7)-1(-5),\ 1(4)-3(7),\ 3(-5)-4(2)]$$
$$= (14+5,\ 4-21,\ -15-8)$$
$$= (19,\ -17,\ -23)$$

$$|\vec{a} \times \vec{b}| = \sqrt{x^2+y^2+z^2}$$
$$= \sqrt{(19)^2+(-17)^2+(-23)^2}$$
$$= \sqrt{480}$$

Properties of Cross Product

$\vec{a} \times \vec{a} = 0$

$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$k(\vec{a} \times \vec{b}) = k\vec{a} \times \vec{b}$ or $\vec{a} \times k\vec{b}$

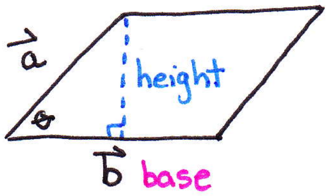
1. Simplify $(\vec{a} + \vec{b}) \times (\vec{a} + \vec{b})$.

$$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{b} \times \vec{a} + \vec{b} \times \vec{b}$$
$$= 0 + \vec{a} \times \vec{b} - (\vec{a} \times \vec{b}) + 0$$
$$= 0$$

Applications of ~~Dot and~~ Cross Product

Area of a Parallelogram

Find the equation for the area of a parallelogram with sides $\vec{a}$ and $\vec{b}$.



$A = bh$

$$\boxed{A = |\vec{b}||\vec{a}|\sin\theta}$$

or

$$\boxed{A = |\vec{a} \times \vec{b}|}$$

$\sin\theta = \frac{o}{h}$

$\sin\theta = \frac{\text{height}}{|\vec{a}|}$

$|\vec{a}|\sin\theta = \text{height}$

1. Calculate the area of the parallelogram with sides $\vec{u} = (10, 4, -1)$ and $\vec{v} = (-2, 5, 3)$.

$A = |\vec{a} \times \vec{b}|$

$= \sqrt{x^2+y^2+z^2}$

$= \sqrt{(17)^2+(-28)^2+(58)^2}$

$= \sqrt{4437}$

$= \sqrt{9}\sqrt{493} \Rightarrow 3\sqrt{493}$

$\vec{a} \times \vec{b} = (10,4,-1) \times (-2,5,3)$

$= [12-(-5),\ 2-30,\ 50-(-8)]$

$= (17, -28, 58)$

$$\begin{array}{ccc} 4 & \times & 5 \\ -1 & \times & 3 \\ 10 & \times & -2 \\ 4 & \times & 5 \end{array}$$

$\therefore$ Area $= 3\sqrt{493}$ units^2

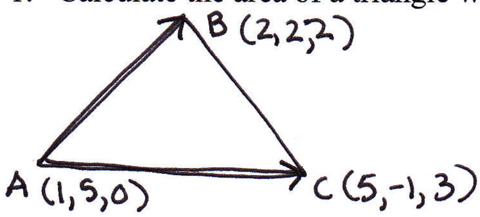
Area of a Triangle

Find the equation for the area of a triangle

$A = \frac{bh}{2} \Rightarrow \boxed{A = \frac{|\vec{a} \times \vec{b}|}{2}}$

$$\begin{array}{ccc} -3 & \times & -6 \\ 2 & \times & 3 \\ 1 & \times & 4 \\ -3 & \times & -6 \end{array}$$

1. Calculate the area of a triangle with vertices A(1, 5, 0), B(2, 2, 2), and C(5, -1, 3).



$\vec{AB} = (2-1, 2-5, 2-0)$

$= (1, -3, 2)$

$\vec{AC} = (5-1, -1-5, 3-0)$

$= (4, -6, 3)$

$A = \frac{|\vec{a} \times \vec{b}|}{2}$

$A = \frac{\sqrt{3^2+5^2+6^2}}{2}$

$A = \frac{\sqrt{9+25+36}}{2}$

$A = \frac{\sqrt{70}}{2}$

$\vec{a} \times \vec{b}$

$= (1,-3,2) \times (4,-6,3)$

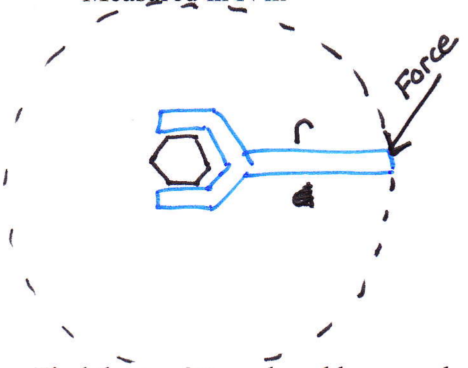
$= (-9-(-12),\ 8-3,\ -6-(-12))$

$= (3, 5, 6)$

$\therefore$ Area $= \frac{\sqrt{70}}{2}$ units^2

Torque

- Torque occurs when a force is applied on an object, resulting in rotational motion.
- Measured in N m



$$\boxed{\tau = |\vec{r} \times \vec{F}|}$$

1. Find the torque produced by a mechanic exerting a force of 95 N on the end of a 20 cm long wrench at an angle of 30°.

* Convert $|\vec{r}|$ into metres

20 cm = 0.20 m

$$\begin{aligned}\tau &= |\vec{r} \times \vec{F}| \\ &= |\vec{r}||\vec{F}|\sin\theta \\ &= (.20)(95)\sin 30° \\ &= 9.5 \text{ Nm}\end{aligned}$$

$\therefore$ The torque is 9.5 Nm