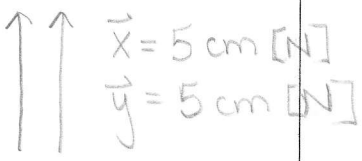
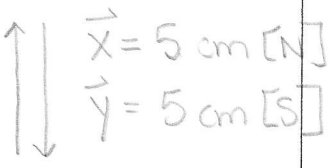
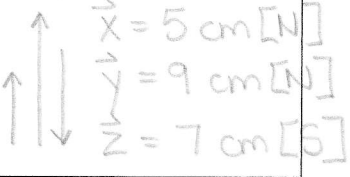
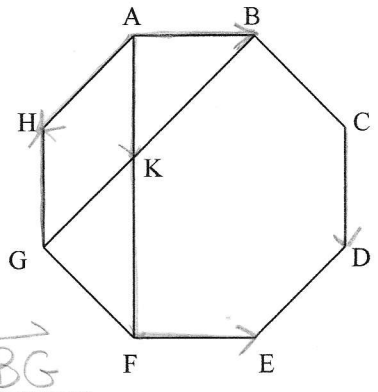
Introduction To Vectors

Term	Definition	Notation	Examples
Magnitude	The amount/size of a quantity	$\lvert\vec{x}\rvert$	5 1056 10 0.3
Direction	Which way a vector is pointing	[direction]	[Left] [Down] [S] [N 10° E]
Scalar	A measurement/quantity with no direction	$X = 5$ cm	Temperature Distance Age Mass
Vector	A measurement/quantity with a direction	$\vec{X} = 5$ cm [N]	Displacement Velocity Force Weight
Unit Vector	A vector that is 1 unit long	$\hat{X} = 1$ cm [N]	
Equal Vectors *(Equivalent)*	Vectors with the same magnitude and direction	$\vec{x} = \vec{y}$	
Opposite Vectors	Vectors with the same magnitude and opposite directions	$\vec{x} = -\vec{y}$	
Parallel Vectors	Vectors with the same or opposite directions	$\vec{x} \parallel \vec{y}$	
Coincident Vectors	Vectors with the same direction that lie on top of each other		

Classifying Vectors

For the following diagram, state:



a) Two vectors that are equal. $\vec{AB}$ $\vec{FE}$

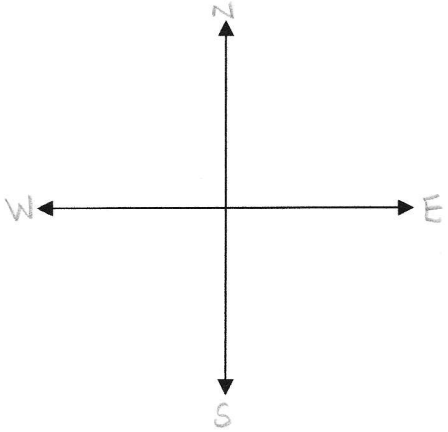
b) Two vectors that are opposite. $\vec{GH}$ $\vec{CD}$

c) Two vectors that are coincident. $\vec{AK}$ $\vec{AF}$

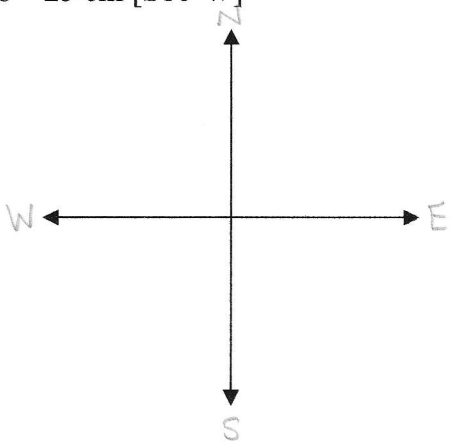
d) Two vectors that parallel, but have different magnitudes. $\vec{AH}$ $\vec{BG}$ or $\vec{GB}$

Drawing Vectors

$\vec{a}$ = 10 cm [N25°E]

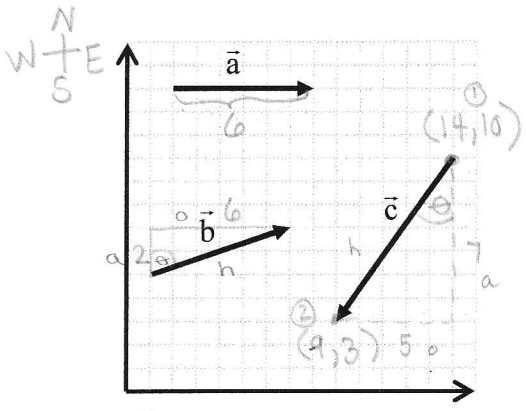


$\vec{b}$ = 25 cm [S60°W]



Calculating Vectors

Calculate the magnitude and direction of the following vectors:



$\vec{a}$ = 6 units [E] or [right]

Pythagorean thm.

$a^2+b^2=c^2$

$2^2+6^2=c^2$

$4+36=c^2$

$40=c^2$

$\sqrt{40}=c$

$\sqrt{4}\sqrt{10}=c$

$2\sqrt{10}=c$

$\vec{b}$ = $2\sqrt{10}$ units [N 71.57° E]

$\text{Tan}\theta=\frac{o}{a}$

$\text{Tan}\theta=\frac{6}{2}$

$\theta=\text{Tan}^{-1}[3]$

$\theta \doteq 71.57°$

Length Formula

$|\vec{c}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(9-14)^2+(10-3)^2}$

$= \sqrt{(-5)^2+(7)^2}$

$= \sqrt{25+49}$

$= \sqrt{74}$

$\text{Tan}\theta = \frac{o}{a}$

$\text{Tan}\theta = \frac{5}{7}$

$\theta=\text{Tan}^{-1}\left(\frac{5}{7}\right)$

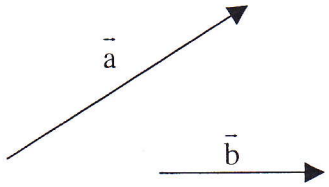
$\theta \doteq 35.54°$

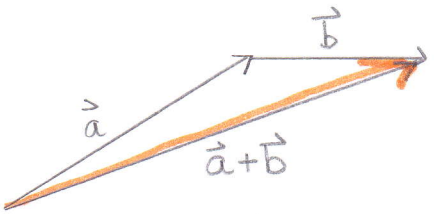
$\vec{c}$ = $\sqrt{74}$ units [S 35.54° W]

Vector Addition & Subtraction

Vector Addition

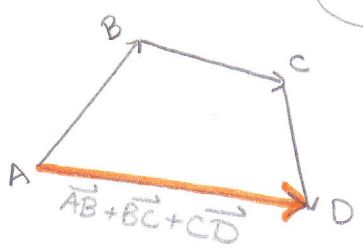
- To add vectors, line them up from head to tail.
- The sum of the vectors is drawn from tail of first to head of last.
- The sum of the vectors is also called the resultant vector.





State the resultant vector for each of the following:

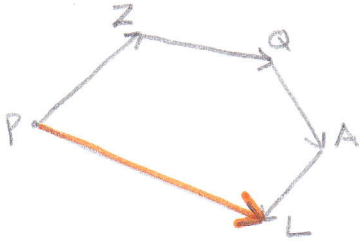
a) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$ $= \overrightarrow{AD}$



b) $\overrightarrow{AL}+\overrightarrow{PZ}+\overrightarrow{QA}+\overrightarrow{ZQ}$

start with unique letter

$\vec{PZ}+\vec{ZQ}+\vec{QA}+\vec{AL}$ $= \vec{PL}$

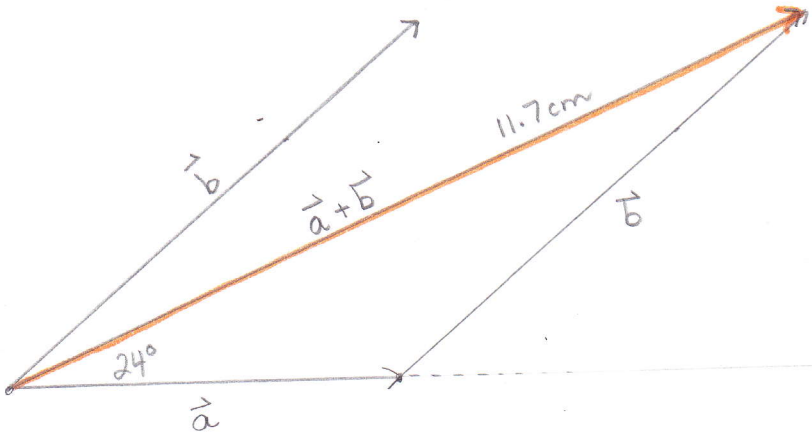


Determine $\vec{a}+\vec{b}$ if $|\vec{a}|=5.13$ cm, $|\vec{b}|=7.2$ cm, and the angle between the two vectors is 40.00°.

5.2 cm

Method 1 - Measure / Draw

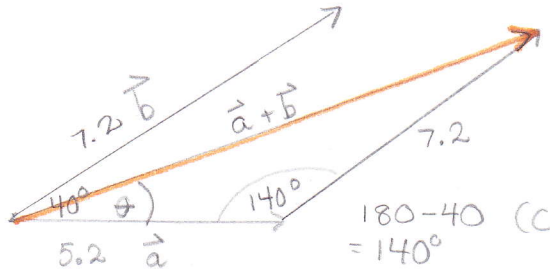
$\vec{a}+\vec{b} = 11.7\text{ cm}\ [24^\circ \text{ to } \vec{a}]$



PROS:
- easy

CONS:
- not very accurate
- must use ruler + protract

Method 2: Cosine Law



180 − 40 (C Pattern) = 140°

PROS:
- easier to draw
- more accurate

CONS:
- calculations!

CAUTION: Sine Law Does Not Find Obtuse Angles!

$a^2 = b^2 + c^2 - 2bc\cos A$

$\sqrt{|\vec{a}+\vec{b}|^2} = \sqrt{5.2^2 + 7.2^2 - 2(5.2)(7.2)\cos 140}$

$|\vec{a}+\vec{b}| \doteq 11.67$ cm

$a^2 = b^2 + c^2 - 2bc\cos A$

$7.2^2 = 5.2^2 + 11.67^2 - 2(5.2)(11.67)\cos\theta$

$7.2^2 - 5.2^2 - 11.67^2 = -2(5.2)(11.67)\cos\theta$

$-111.3889 = -121.368\cos\theta$

$\cos^{-1}\left(\frac{-111.3889}{-121.368}\right) = \theta$

$23.40° = \theta$

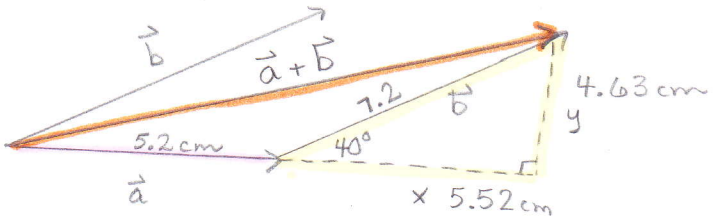
(23.36° if exact)

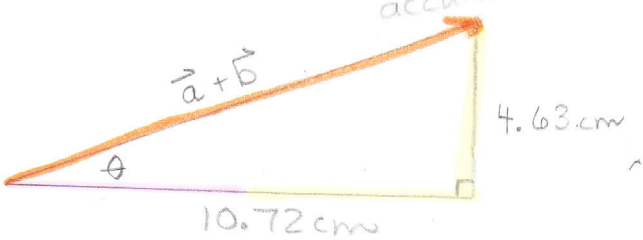
$\therefore \vec{a}+\vec{b} = 11.67$ cm [23.40° to $\vec{a}$]

Method 3: All SohCahToa All The Time

PROS
- easy draw
- more accurate

CONS
- calcs





$\sin\theta = \frac{o}{h}$ $\quad \cos\theta = \frac{a}{h}$

$\sin 40 = \frac{y}{7.2}$ $\quad \cos 40 = \frac{x}{7.2}$

$7.2\sin 40 = y$ $\quad 7.2\cos 40 = x$

$4.63 \doteq y$ $\quad 5.52 \doteq x$

$5.2 + 5.52 = 10.72$

$a^2 + b^2 = c^2$

$10.72^2 + 4.63^2 = |\vec{a}+\vec{b}|^2$

$\sqrt{136.3553} = |\vec{a}+\vec{b}|$

$11.68 \doteq |\vec{a}+\vec{b}|$

$\tan\theta = \frac{o}{a}$

$\tan\theta = \frac{4.63}{10.72}$

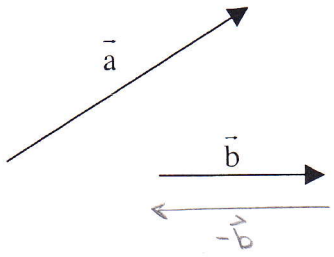
$\theta = \tan^{-1}\left(\frac{4.63}{10.7}\right)$

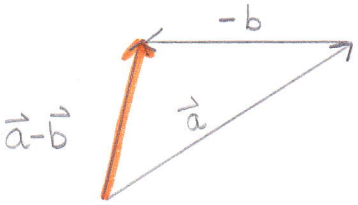
$\theta \doteq 23.36$

$\therefore \vec{a}+\vec{b} = 11.68$ cm [23.36° to $\vec{a}$]

Vector Subtraction $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

- To subtract vectors, add the opposite of the vector being subtracted. (flip)





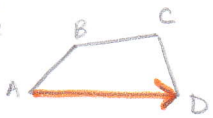
State the resultant vector for each of the following:

a) $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

$= \overrightarrow{AB} + (-\overrightarrow{CB}) + (-\overrightarrow{DC})$

$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{AD}$



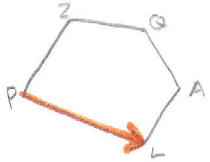
b) $\overrightarrow{PZ} - \overrightarrow{LA} - \overrightarrow{AQ} - \overrightarrow{QZ}$

$= \overrightarrow{PZ} + (-\overrightarrow{LA}) + (-\overrightarrow{AQ}) + (-\overrightarrow{QZ})$

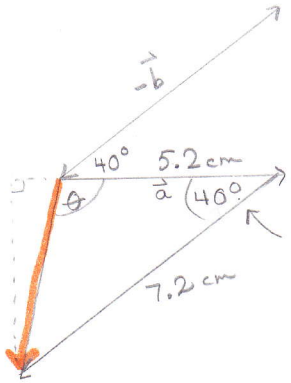
$= \overrightarrow{PZ} + \overrightarrow{AL} + \overrightarrow{QA} + \overrightarrow{ZQ}$

$= \overrightarrow{PZ} + \overrightarrow{ZQ} + \overrightarrow{QA} + \overrightarrow{AL}$

$= \overrightarrow{PL}$



Determine $\vec{a} - \vec{b}$ if $|\vec{a}| = 5.2$ cm, $|\vec{b}| = 7.2$ cm, and the angle between the two vectors is 40.00°.



$a^2 = b^2 + c^2 - 2bc\cos A$

$\sqrt{|\vec{a} - \vec{b}|^2} = \sqrt{5.2^2 + 7.2^2 - 2(5.2)(7.2)\cos 40}$

$|\vec{a} - \vec{b}|^2 \doteq 4.64$ cm

Sine Law Lies!

$\therefore \vec{a} - \vec{b} = 4.64$ cm [93.89° to $\vec{a}$

$a^2 = b^2 + c^2 - 2bc\cos A$

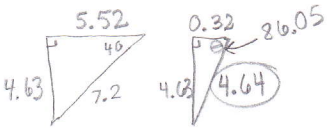
$7.2^2 = 5.2^2 + 4.64^2 - 2(5.2)(4.64)\cos\theta$

$7.2^2 - 5.2^2 - 4.64^2 = -2(5.2)(4.64)\cos\theta$

$3.2704 = -48.256\cos\theta$

$\cos^{-1}\left(\frac{3.2704}{-48.256}\right) = \theta$

$93.89^\circ \doteq \theta$



Zero Vector

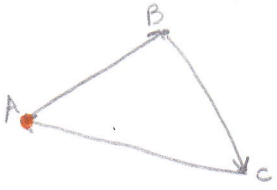
- A vector with a magnitude of 0 and no direction.

i.e. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC}$

$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

$= \overrightarrow{AA}$

$= \vec{0}$



Vector Operations

Properties of Vectors Activity

Property #1

$\vec{a}+\vec{0}=\vec{a}$

True / False

Property #2

$\vec{a}-\vec{a}=\vec{0}$

True / False

Property #3

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$

True / False

Property #4

$\vec{a}-\vec{b}=-\vec{b}+\vec{a}$

True / False

Property #5

$\vec{a}-\vec{b}=\vec{b}-\vec{a}$

True / False

Property #6

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

True / False

Property #7

$(\vec{a}-\vec{b})+\vec{c}=\vec{a}-(\vec{b}+\vec{c})$

True / False

Property #8

$\vec{a}+\vec{a}=2\vec{a}$

True / False

Property #9

$\vec{a}+\vec{b}+\vec{a}+\vec{b}+\vec{a}=3\vec{a}+2\vec{b}$

True / False

Property #10

$3(2\vec{a})=2(3\vec{a})$

True / False

Property #11

$2(\vec{a}+\vec{b})=2\vec{a}+2\vec{b}$

True / False

Property #12

$2(\vec{a}-\vec{b})=2\vec{a}-2\vec{b}$

True / False

Vector Operations

Simplify each of the following:

a) $3(5\vec{a}+\vec{b})-(2\vec{b}-4\vec{a})$

$=15\vec{a}+3\vec{b}-2\vec{b}+4\vec{a}$

$=13\vec{a}+7\vec{b}$

b) $6(3\vec{a}-2\vec{b}+5\vec{c})+\frac{1}{2}(4\vec{a}+4\vec{b})-10(3\vec{c}-\vec{b}+2\vec{a})$

$=18\vec{a}-12\vec{b}+30\vec{c}+2\vec{a}+2\vec{b}-30\vec{c}+10\vec{b}-20\vec{a}$

$=0\vec{a}+0\vec{b}+0\vec{c}$

$=\vec{0}$

If $\vec{a} = 2\vec{x} + 3\vec{y} - 4\vec{z}$ and $\vec{b} = \vec{x} + 5\vec{z}$, express $10\vec{b} - 2\vec{a}$ in terms of $\vec{x}$, $\vec{y}$ and $\vec{z}$.

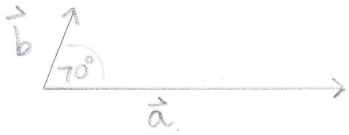
$10\vec{b} - 2\vec{a}$

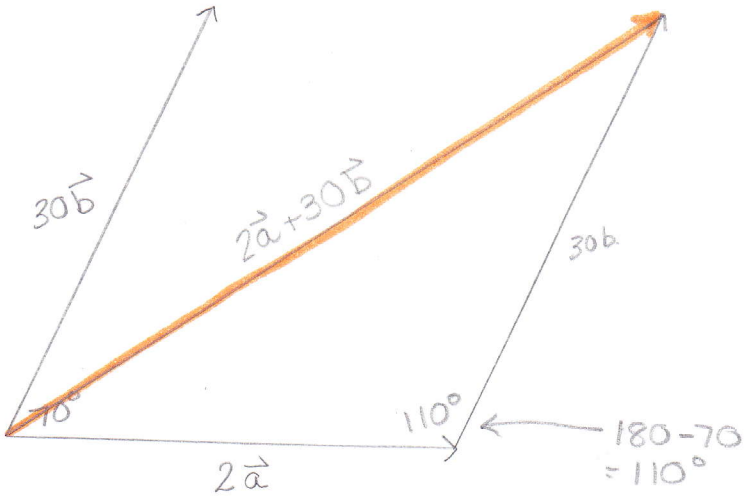
$= 10(\vec{x} + 5\vec{z}) - 2(2\vec{x} + 3\vec{y} - 4\vec{z})$

$= 10\vec{x} + 50\vec{z} - 4\vec{x} - 6\vec{y} + 8\vec{z}$

$= 6\vec{x} - 6\vec{y} + 58z$

Determine the values of $|2\vec{a} + 30\vec{b}|$ if $|\vec{a}| = 10$ cm, $\vec{b}$ is a unit vector, and the angle between the two vectors is 70°.





$|2\vec{a}| = 2(10)$

$= 20$ cm

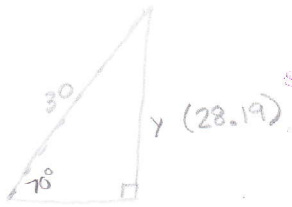
$|30\vec{b}| = 30(1)$

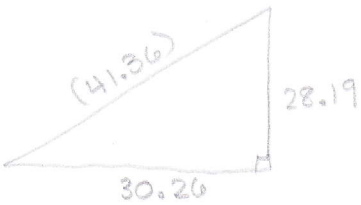
$= 30$ cm

$a^2 = b^2 + c^2 - 2bc\cos A$

$\sqrt{|2\vec{a} + 30\vec{b}|^2} = \sqrt{20^2 + 30^2 - 2(20)(30)\cos 110}$

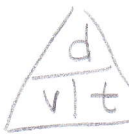
$|2\vec{a} + 30\vec{b}| \doteq 41.36$ cm





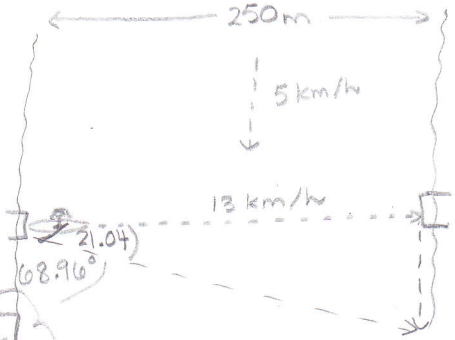
Applications of Vectors – Velocity

- Velocity is a vector because it has both magnitude and direction.
- Air speed (water speed) is the speed of a plane (boat) relative to a person on board.
- Ground speed is the speed of a plane (boat) relative to a person on the ground and includes the effect of wind (current).



Using vectors to represent velocities

Connie the canoeist wants to cross a river (that is) 250 m wide. The current flows at 5 km/h and Connie can paddle at 13 km/h in still water. If she directs her boat towards a dock directly across the river determine:



a) her actual velocity as she crosses the river.

$a^2 + b^2 = c^2$

$13^2 + 5^2 = |\vec{v}|^2$

$194 = |\vec{v}|^2$

$\sqrt{194} = |\vec{v}|$

$13.93 \doteq |\vec{v}|$

$\text{Tan}\theta = \frac{o}{a}$

$\text{Tan}\theta = \frac{5}{13}$

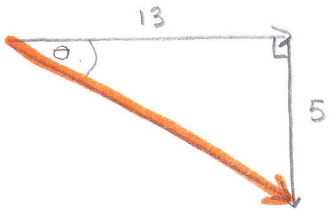
$\theta = \text{Tan}^{-1}\left(\frac{5}{13}\right)$

$\theta \doteq 21.04°$

$90 - 21.04 = 68.96°$

∴ 13.93 km/h [68.93° to bank]

b) how far downstream Connie will end up.



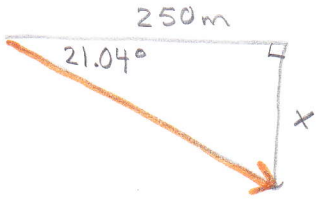
$\text{Tan}\theta = \frac{o}{a}$

$\text{Tan}\,21.04 = \frac{x}{250}$

$250\,\text{Tan}\,21.04 = x$

$96.17\text{ m} \doteq x$

∴ She will end up 96.17m downstream.



c) how long it takes her to cross the river.

$t = \frac{d}{v}$

$t = \frac{0.26786}{13.93}$

$t = 0.02\text{ h}$

$\text{Cos}\theta = \frac{a}{h}$

$\text{Cos}\,21.04 = \frac{250}{h}$

$h = \frac{250}{\text{Cos}\,21.04}$

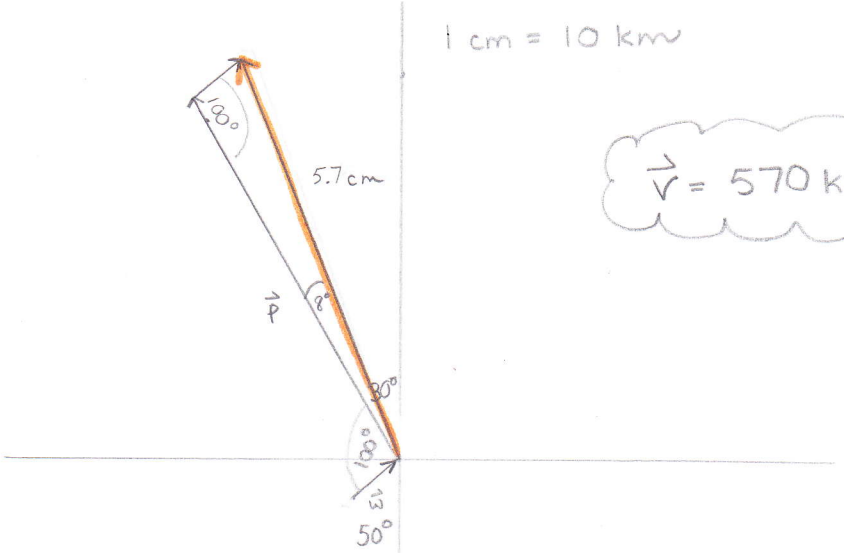
$h \doteq 267.86\text{ m}$

$h = 0.26786\text{ km}$

∴ it will take 0.02h (or 1.15 min) (or 69.22 sec) to cross the river.

A plane is steering at N30°W at an air speed (speed in still air) of 550 km/h. If the wind is from S50°W at 80 km/h, find the ground speed and the course of the plane.

METHOD 1:

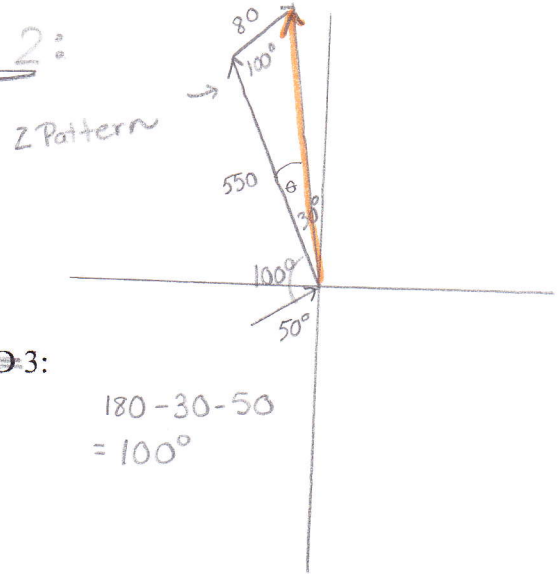


$30 - 8 = 22°$

$\vec{v} = 570$ km/h [N 22° W]

~~METHOD 2:~~

Method 2:



$180 - 30 - 50 = 100°$

$a^2 = b^2 + c^2 - 2bc\,\text{Cos}A$

$\sqrt{|\vec{a}+\vec{b}|^2} = \sqrt{80^2 + 550^2 - 2(80)(550)\,\text{Cos}\,100}$

$|\vec{a}+\vec{b}| \doteq 569.37$ km/h

$a^2 = b^2 + c^2 - 2bc\,\text{Cos}A$

$80^2 = 550^2 + 569.37^2 - 2(550)(569.37)\,\text{Cos}$

$-620282.1969 = -626307\,\text{Cos}A$

$\text{Cos}^{-1}\left(\frac{-620282.1969}{-626307}\right) = A$

$7.95° \doteq A$

~~METHOD 3:~~

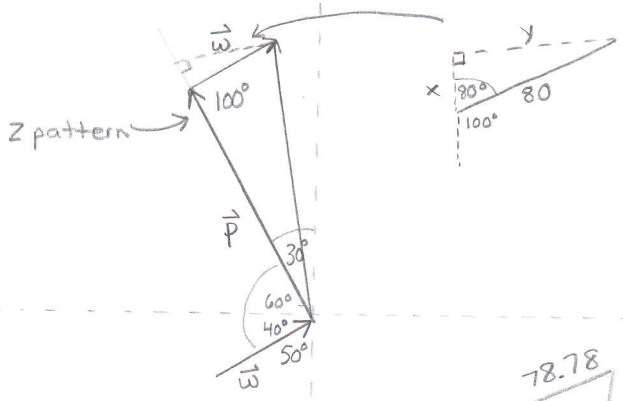
$30 - 7.95 = 22.05°$

$\therefore \vec{v} = 569.37$ km/h [N22.05°W]

A plane is steering at N30°W at an air speed (speed in still air) of 550 km/h. The wind is from S50°W at 80 km/h.

METHOD 3

a) Find the ground speed and the course of the plane.



$\sin\theta = \frac{o}{h}$

$\sin 80 = \frac{y}{80}$

$80 \sin 80 = y$

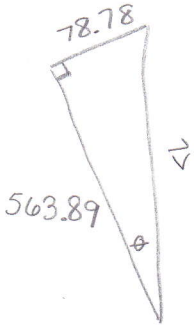
$78.78 \text{ km/h} \doteq y$

$\cos\theta = \frac{a}{h}$

$\cos 80 = \frac{x}{80}$

$80 \cos 80 = x$

$13.89 \text{ km/h} \doteq x$



$a^2 + b^2 = c^2$

$563.89^2 + 78.78^2 = |\vec{v}|^2$

$\sqrt{324178.2205} = |\vec{v}|$

$569.37 \text{ km/h} = |\vec{v}|$

$\tan\theta = \frac{o}{a}$

$\tan\theta = \frac{78.78}{563.89}$

$\theta = \tan^{-1}\left(\frac{78.78}{563.89}\right)$

$\theta = 7.95°$

$30 - 7.95$

$= 22.05°$

$\therefore \vec{v} = 569.39 \text{ km/h}$ [N 22.05° W]

b) How far will the plane have travelled in 2 h?

$\vec{d} = vt$

$= 569.39 \times 2$

$= 1138.78 \text{ km}$

$\therefore$ The plane will have travelled 1138.78 km [N 22.05° W]

Applications of Vectors – Force

- Force is something that either pushes or pulls an object.
- Force is measured using the unit Newtons [N].
- A mass of 1kg exerts a force of 9.81 N/kg [down].

Determine the downward force exerted by a 10 kg box of textbooks.

$10\text{ kg} \times 9.81\ \frac{\text{N}}{\text{kg}}$

$= 98.1\text{ N}$

Components of Force

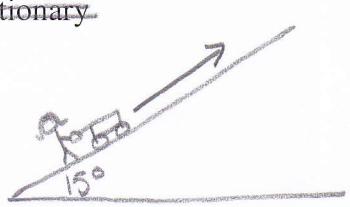
A force applied to an object can broken down into two components:

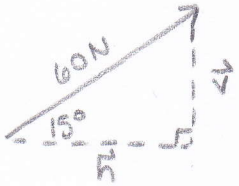
horizontal – moves object forward/backward

vertical – moves object up/down

Marsha pushes shopping cart up a 15° incline using a force of 60 N. Calculate the horizontal and vertical forces being exerted on the cart.

~~Stationary~~





$\sin\theta = \frac{o}{h}$

$\sin 15 = \frac{|\vec{v}|}{60}$

$60 \sin 15 = |\vec{v}|$

$15.53\text{ N} \doteq |\vec{v}|$

$\cos\theta = \frac{a}{h}$

$\cos 15° = \frac{|\vec{h}|}{60}$

$60 \cos 15 = |\vec{h}|$

$57.96\text{ N} \doteq |\vec{h}|$

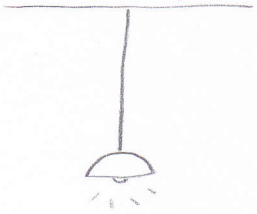
∴ Horizontal: 57.96 N [forward]
Vertical: 15.53 N [up]

Forces in Equilibrium

- When an object is stationary, an opposite equal force is used to counteract the applied forces.

~~Determine the force that must be applied to prevent a 8 kg shopping cart from rolling down a 10° ramp.~~

Determine the tension in a cable that is supporting a suspended 0.75 kg pendant light fixture.

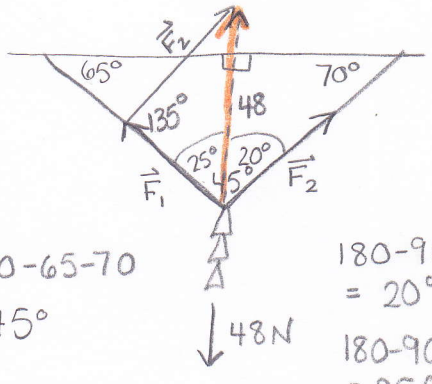


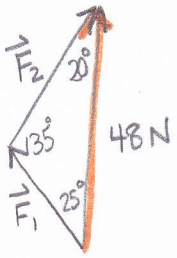
$|\vec{F}_g| = 9.81\ N/kg \times 0.75\ kg$

$\vec{F}_g \doteq 7.3575\ N\ [down]$

∴ The tension in the cable is 7.3575 N [up]

A piece of mobile art is suspended from the ceiling using two cables that make angles of 65° and 70° with the ceiling. If the mobile exerts a downward force of 48N, what is the tension in each cable?





180-65-70
= 45°

180-45
= 135°

180-90-70
= 20°

180-90-65
= 25°

180-135-25°
= 20°

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin 135}{48} = \frac{\sin 20}{|\vec{F}_1|}$

$|\vec{F}_1| \sin 135 = 48 \sin 20$

$|\vec{F}_1| = \frac{48 \sin 20}{\sin 135}$

$|\vec{F}_1| \doteq 23.22\ N$

$\frac{\sin 135}{48} = \frac{\sin 25}{|\vec{F}_2|}$

$|\vec{F}_2| \sin 135 = 48 \sin 25$

$|\vec{F}_2| = \frac{48 \sin 25}{\sin 135}$

$|\vec{F}_2| \doteq 28.69\ N$

∴ The tensions in the cables are:
23.22 N [65° to ceiling]
28.69 N [70° to ceiling]