Maximum and Minimum Problems

Numbers

1. The sum of two positive numbers is 20. Find the two numbers such that
   a) the sum of the square is minimum,
   b) the product of one and the square of the other is maximum.
   c) the product of the square of one and the cube of the other is maximum.

2. Product of two positive numbers is 16. Find the two numbers such that
   a) the sum is minimum
   b) the sum of one and the square of the other is minimum.

3. If the sum of two positive numbers is k. show that the sum of the square of the two numbers will be greater or equal to $\frac{1}{2}k^2$.

4. The sum of two positive numbers is k. Show that the sum of one and the square of the other is at least $k-\frac{1}{4}$.

5. The sum of two positive numbers is 8. Find the minimum value of the sum of the square of one and the cube of the other.

6 Find the minimum value of the sum of a positive number and its reciprocal.

7 The sum of two positive numbers is 12. Find the two numbers such that the product of one and the cube of the other is maximum.

8. Find a positive number which exceeds its square by a maximum amount.

Area

9 A rectangle is inscribed in a circle $x^2+y^2=9$. Find the dimensions of the largest rectangle. What is its area?

10 A rectangle is inscribed in a ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. Find the area of the largest rectangle.

11 A rectangle is inside a parabola $y=12-x^2$ with its base lying on the x-axis and the other two vertices lying on the parabola. Find the dimension of the largest rectangle.

12. An isosceles trapezoid ABCD, with $AB \parallel DC$, is inscribed in semi circle of radius 2. If AB is the diameter of the semicircle, find the length of ~~D~~ CD at which the area of the trapezoid is maximum.

13. Find the area of the largest isosceles triangle ABC, AB = AC, that can be inscribed in an ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, with vertex A =(0,2).

14. The volume of an open top cylinder is V. Find the ratio of height to radius that gives the minimum surface area.

15 Show that of all the rectangles with a given perimeter P, the one with the greatest area is a square.

16 Three sides of a trapezoid are 10cm each. Find the length of the fourth side such that the area of the trapezoid is largest.

when you unwrap the cylinder to minimize area, use squares for top and bottom instead of circles, since that is the material you use +waste

17. In constructing a cylindrical can of given volume, nothing is wasted in making the side of the can. But the top and bottom are cut from square sheets and the remainder is wasted. Find the ratio of h to r so that the material used is a minimum.

18. A rectangle is inscribed in an isosceles right angle triangle of two sides each equal to 10 cm. One side of the rectangle rests on the hypotenuse and the other two vertices on the shorter sides. Find the dimension of the largest rectangle.

Poster

19. A rectangular poster is of area 6912 cm^2. Side margin is 8 cm each and top and bottom margin are each 6 cm. Find the dimensions of the poster that give the maximum printing area.

20. A poster contains 200 cm^2 of printing area. The top and bottom margins are both 8 cm and the side margins are both 4 cm. What are the dimensions of the poster of minimum area?

Wire

21. A wire of length 8m is divided into two parts. One forms a circle and the other forms a square. How should the wire be cut so that the total area is:

a) minimum?
b) maximum?

22. A wire of length 8m is divided into two parts. One forms a rectangle with length equals to twice the width and the other part forms an equilateral triangle. How should the wire be cut so that the total area is:

a) maximum?
b) minimum?

Fence

23. A man has 600m of fence. He wants to enclose a rectangular field with one side along a wall. Find the dimensions of the rectangular field of maximum area.

24. A man wants to fence an rectangular field of area 3750 m^2 and divide it in half with a fence parallel to one of the sides of the rectangular field. What are the dimensions of the field that can minimize the cost of the fence?

25. A rectangular field is to have 100 m^2 in area. It was enclosed by fence. The north-south sides costs $20/m, east-west sides cost $5/ m. What are the dimensions so that the cost is a minimum?

Window

26. A window of fixed perimeter 8 m consist of a rectangle surmounted by a semicircle. What is the ratio of height to radius that will admit most sunlight?

27. A window consists of a rectangle surmounted by a semicircle. The perimeter is 8m. Suppose the rectangular part admits twice as much sunlight as the circular part. Find the width of the window which admits most sunlight. twice as much maybe different glass filters more light...

28. Suppose there is a semicircle below the rectangle as well as above it. Find the dimensions of the window which admit most sunlight for a given perimeter 8m.

29. A window of perimeter 8 m consists of a rectangle surmounted by an equilateral triangle. Find the width of the window that admit most sunlight.

Volume

30. Squares of equal length are cut from the 4 comers of a square sheet of side 12cm. The four sides are then folded to form an open top box. Find the length of the side of the squares cut that give a maximum volume.

31. A closed container is made with a hemisphere on top of a cylinder. The height and radius of the cylinder are h and r respectively. Find h:r so that the volume is maximum if the total surface area is a constant A.

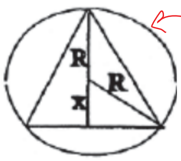
32. The total surface area of a squared base open top rectangular box is 12 square units. Find the dimensions of the box such that the volume is maximum.

33 Of all the circular cone of slant height 8 cm, find the dimensions of the cone of largest volume.

34. A cylinder is inscribed in a fixed cone. Find the ratio of the volume of the largest cylinder to the volume of the cone.

35. For a circular cylinder of given volume, find the ratio of height to radius so that the total surface is minimum.

36. Find the volume of the largest cylinder that can be inscribed in a sphere of radius R.



37. Find the volume of the cone of maximum value that can be inscribed in a sphere of radius R. hint: use pythag th on small triangle with x,R-radius of sphere,r-radius of cone

38. A trough is to be made by bending along rectangular piece of tin 3 units wide. The cross section of the trough is a isosceles trapezoid with sides making angles of 120 degree with the base. Find the length of one of the sides that is bent which give a maximum capacity.

Distance and shortest time

39. A man in ~~an island~~ a boat is 4 km from the shore. He wants to go to a pub which is 8 km down the shore He can row at 3 km/h and walk at 5 km/h. Where should he land if he want to reach the pub as soon as possible

40. The cost of laying power line underwater is 3 times that of underground. An island is 4 km from the shore and a power station is at distance 8 km from the point on the shore which is closest to the island. How should the power line be laid so that the cost is minimum

41. Two vertical posts 7m apart are of lengths 3 and 4 m. A wire is to run from the top of a post, reaches the ground and then goes to the top of another post. Find the minimum length of the wire.

42. A and B are points on the opposite side of a straight line. P and Q are points on line such that AP and BO are perpendicular to the line. AP= 3, BQ= 4, PQ=7.R is a point on the line. At what point should R be located so that the total distance form A to B through R is minimum?

Closest Distance

43. Find a point on the line 3x + 4y - 25 = 0 which is closest to the origin.

44. Find the minimum distance from the point (4,2) to y2= 8x.

Revenue, profit and cost

45. A shop can sell 30 radio at $20 each per week. If the price is increased, for each dollar increase there will be a lost of one sale per week. The cost of radio is $10 each. What is the price that will give the maximum profit?

46. In a certain type of soil, if 20 apple trees are planted, each will yield 100 apples. But if more trees are planted for each addition tree planted, the yield will be reduced by 2 apples. How many trees should be planted so that the total yield is maximum?

47. A open top rectangular box is of volume 250 cm^3. The width of the box is 5cm. The cost is \$2/cm^2 for the base and \$ 1/cm^2 for the side. What is minimum cost for making the box?

Miscellaneous

48. A rectangular beam is made from a right circular cylindrical rod of radius 10 cm. If the strength of the beam is proportional to the product of the width and the square of the depth, find the dimensions of the strongest beam.

49. A ship A leaves a dock at noon and travels due south at 20 km/h. Another ship B has been heading due east at 30 km/h and reach the same dock at 3:00 p.m . At what time will the two ship be closest?

50. The area of a rectangle is 4 cm2. Find the dimension of the rectangle so that the distance from one vertex to the midpoint of a non-adjacent side is minimum.

51 A line passes (1, l) cuts the x-and y-axis at A and B respectively. Find the minimum value of the length of AB.

52. A line passes (3,4) in the first quadrant cut the x-axis at A and y-axis at B where 0 is the origin. Find the minimum value of the area of the triangle OAB.

53. An isosceles triangle has one of its vertexes at origin. Its base is parallel to and above the x-axis. The other two vertices lies on the curve $y = 12-x^2$. Find the largest possible area of the triangle.

54. Given a triangle of sides 5, 12, 13 units. A rectangle is inscribed in the triangle such that two sides of the rectangle lie along the two shorter sides of the triangle. The remaining vertex lies on the hypotenuse. Find the area of the largest rectangle.

55. Find the minimum vertical distance between the two curves $y = x^2$ and $y = -(x - 2)^2$.

56. A, B are two light source 100 m apart. A is twice as strong as B. P is a point between A, B. Intensity of illumination varies inversely as the square of the distance from the light source. Find the length of AP so that the point P is darkest.

57. A wall 27 m high is parallel to a tall building which is 8 m from the wall. Find the length of the shortest ladder that will reach from the ground across the top of the wall to reach the building.

58. A metal rod is being carried down a corridor 4 m wide. At the end of the corridor there is a right angle turn to another corridor of 3 m wide. What is the longest rod that can be carried around the corner horizontally?

Ex 8.3 p.25

(Note: Some variables are defined in the diagrams. In most solutions, the testing of the critical points for maxima and minima is omitted.

For cones, cylinders and spheres, diagram represents the cross section through the axis of the solids)

1a). Let $S = x^2 + y^2$ $\quad x + y = 20 \Rightarrow y = 20 - x$ $\quad$ so, $S = x^2 + (20-x)^2$

$S' = 2x + 2(20-x)(-1) = 4x - 40 = 0 \Rightarrow x = 10$

$S'' = 4 > 0$, for $x = 10$ $\quad \therefore S$ attains a min value when $x = 10$ $\quad$ when $x = 10$, $y = 10$.

Note: In doing a max/ min problem, we have to test the critical values for max and min.

But from now on this step will be skipped.

Students are advised to do this step unless otherwise instructed by the teacher.

1b) let $x = 20 - y$ $\quad P = xy^2 = (20-y)y^2 = 20y^2 - y^3$

$\frac{dP}{dy} = 40y - 3y^2$, $\quad P' = 0 \Rightarrow y(40-3y) = 0 \Rightarrow y = \frac{40}{3}$ or $y = 0$ (rejected) $\quad$ when $y = \frac{40}{3}$, $x = 20 - \frac{40}{3} = \frac{20}{3}$

1c) $P = x^2y^3 = (20-y)^2 y^3$ $\quad P' = (20-y)^2(3y^2) + y^3(2)(20-y)(-1) = 0 \Rightarrow y = 12$ $\quad$ when $y = 12$, $x = 8$

2a) $xy = 16 \Rightarrow y = \frac{16}{x}$ $\quad S = x + y = x + \frac{16}{x}$ $\quad S' = 1 - \frac{16}{x^2}$ $\quad S' = 0 \Rightarrow x = 4$ $\quad$ when $x = 4$, $y = 4$

b) $S = x + y^2 = \frac{16}{y} + y^2$ $\quad S' = \frac{-16}{y^2} + 2y$ $\quad S' = 0 \Rightarrow y = 2$ $\quad$ when $y = 2$, $x = 8$

p.68

3. $x+y=k \Rightarrow y=k-x \qquad S=x^2+y^2=x^2+(k-x)^2=2x^2-2xk+k^2$

$S'=4x-2k=0 \Rightarrow x=\frac{k}{2} \qquad when\ x=\frac{k}{2},\quad y=k-\frac{k}{2}=\frac{k}{2} \qquad so\ \min\ of\ S=\left(\frac{k}{2}\right)^2+\left(\frac{k}{2}\right)^2=\frac{k^2}{2}$

4. $x+y=k \Rightarrow x=k-y \qquad S=x+y^2=(k-y)+y^2 \qquad S'=-1+2y=0 \Rightarrow y=\frac{1}{2} \qquad S=k-\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2=k-\frac{1}{4}$

5. $x+y=8 \Rightarrow x=8-y \qquad S=x^2+y^3=(8-y)^2+y^3=y^3+y^2-16y+64$

$S'=3y^2+2y-16=0 \qquad S'=0 \Rightarrow (3y+8)(y-2)=0 \Rightarrow y=2 \qquad when\ y=2,\ \ x=6 \qquad S=6^2+2^3=44.$

6. $S=x+\frac{1}{x} \qquad S'=1-\frac{1}{x^2}=0 \Rightarrow x=1\ or\ -1\ (reject) \qquad \min\ of\ S=1+1=2$

7. $P=xy^3=(12-y)y^3=12y^3-y^4 \qquad P'=36y^2-4y^3=-4y^2(y-9)$

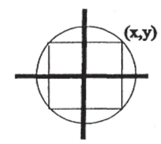
$P'=0 \Rightarrow y=9\ or\ y=0\ (reject) \qquad when\ y=9,\ \ x=12-9=3$

8. $S=x-x^2 \qquad S'=1-2x \qquad S'=0 \Rightarrow x=\frac{1}{2}$

9. Let $(x,\ y)$ be a vertex on the circle in the first quadrant

$A=(2x)(2y)=4xy$. Instead of finding the maximum of A we find the maximum of A^2

$A^2=16x^2y^2=16x^2(9-x^2)=144x^2-16x^4 \qquad \frac{d(A^2)}{dx}=\overset{288}{\cancel{299}}x-64x^3=0 \Rightarrow x=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$

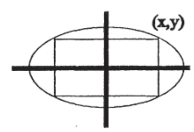


When $x=\frac{3\sqrt{2}}{2},\quad y=\frac{3\sqrt{2}}{2}$ the dim is $3\sqrt{2}$ by $3\sqrt{2}$ the area is $3\sqrt{2}\times 3\sqrt{2}=18$

10. *Let* (x,y) *be the vertex on the ellipse in first quadrant* $\qquad A=4xy \qquad A^2=16x^2y^2$

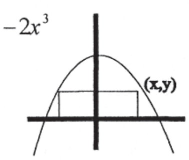
$y^2=\frac{9}{16}(16-x^2) \Rightarrow A^2=9x^2(16-x^2) \qquad \frac{d(A^2)}{dx}=288x-36x^3$

$\frac{d(A^2)}{dx}=0 \Rightarrow x=2\sqrt{2} \qquad when\ x=2\sqrt{2},\quad y=\frac{3\sqrt{2}}{2} \qquad A=4(2\sqrt{2})\left(\frac{3\sqrt{2}}{2}\right)=24$



11. *Let* (x,y) *be a pt on the parabola in the first quadrant* $\qquad A=2xy=2x(12-x^2)=24x-2x^3$

$A'=24-6x^2 \qquad A'=0 \Rightarrow x=2 \qquad when\ x=2,\quad y=12-(x)^2=8 \qquad$ *the dim is* 4 *by* 8



12. Let (x,y) be point C on the semicircle. $x^2+y^2=4,\quad A=\frac{(2x+4)y}{2}=(x+2)y$

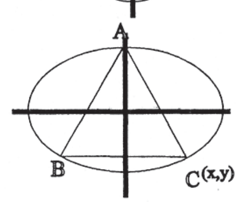
$A^2=(x+2)^2y^2=(x+2)^2(4-x^2)$

$(A^2)'=(x+2)^2(-2x)+(4-x^2)2(x+2)=-4(x+2)^2(x-1)=0, x=1, \qquad CD=2x=2(1)=2$

13. Let (x,y) be vertex C in the fourth quadrant.

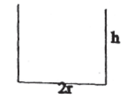
$A=\frac{(2-y)(2x)}{2}=(2-y)(x),\quad A^2=(2-y)^2(x^2)=(2-y)^2\left(\frac{9}{4}\right)(4-y^2)$

$(A^2)'=\frac{9}{4}\left((2-y)^2(-2y)+2(2-y)(-1)(4-y^2)\right)=0,\quad y=-1 \quad x=\frac{3\sqrt{3}}{2} \qquad A=\frac{9\sqrt{3}}{2}$

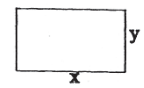


14. $V=\pi r^2 h \qquad h=\dfrac{V}{\pi r^2} \qquad A=\pi r^2+2\pi rh=\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)=\pi r^2+\dfrac{2V}{r}$

$A'=2\pi r-\dfrac{2V}{r^2} \qquad A'=0 \Rightarrow 2\pi r-\dfrac{2V}{r^2}=0 \Rightarrow r=\left(\dfrac{V}{\pi}\right)^{\frac{1}{3}} \qquad h=\dfrac{V}{\pi\left(\dfrac{V}{\pi}\right)^{\frac{2}{3}}}=\dfrac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}=r \qquad so\ \ h\!:\!r=1\!:\!1$



15. Let x and y be the length of the two adjacent sides of the rectangle

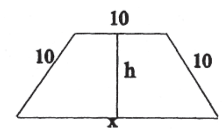


$2x+2y=P \qquad A=xy=x\left(\dfrac{P-2x}{2}\right) \qquad A'=\dfrac{P}{2}-2x=0,\ \ x=\dfrac{P}{4} \qquad y=\dfrac{P-2\left(\dfrac{P}{4}\right)}{2}=\dfrac{P}{4}=x,$ it is a square

16. Let the length of the fourth side be x and the height be h cm. $h^2+\left(\dfrac{x-10}{2}\right)^2=10^2,$

$A=\dfrac{1}{2}(x+10)h \qquad A^2=\dfrac{1}{4}(x+10)^2h^2=\dfrac{1}{4}(x+10)^2\left(100-\dfrac{1}{4}(x-10)^2\right)$

$(A^2)'=\dfrac{1}{4}(x+10)^2\left(-\dfrac{1}{4}(2)(x-10)\right)+\dfrac{1}{4}(2)(x+10)\left(100-\dfrac{1}{4}(x-10)^2\right)=0 \quad x=20\,\text{cm}$



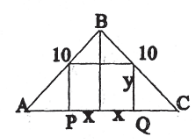
17. $V=\pi r^2h \Rightarrow h=\dfrac{V}{\pi r^2} \qquad A=2\pi rh+2(2r)^2==2\pi r\left(\dfrac{V}{\pi r^2}\right)+8r^2=\dfrac{2V}{r}+8r^2$

$\dfrac{dA}{dr}=-\dfrac{2V}{r^2}+16r=0 \Rightarrow r=\dfrac{V^{\frac{1}{3}}}{2} \qquad h=\dfrac{V}{\pi\left(\dfrac{V^{\frac{1}{3}}}{2}\right)^2}=\dfrac{4V^{\frac{1}{3}}}{\pi} \qquad so\ \ h\!:\!r=\left(\dfrac{4V^{\frac{1}{3}}}{\pi}\right)\!:\!\left(\dfrac{V^{\frac{1}{3}}}{2}\right)=8\!:\!\pi$

18. Let 2x be the length of the side of rectangle that lies on the hypotenuse and y be the length of the other side

$AC=10\sqrt{2} \qquad y=QC=5\sqrt{2}-x \qquad A=2xy=2x\left(5\sqrt{2}-x\right)=10\sqrt{2}x-2x^2$

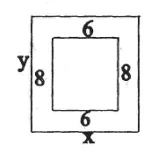
$A'=10\sqrt{2}-4x=0 \Rightarrow x=\dfrac{5\sqrt{2}}{2}, \qquad y=5\sqrt{2}-x=5\sqrt{2}-\dfrac{5\sqrt{2}}{2}=\dfrac{5\sqrt{2}}{2}, \qquad \text{dim } is\ 5\sqrt{2}\ by\ \dfrac{5\sqrt{2}}{2}$



19. $xy=6912 \qquad A=(x-16)(y-12)=xy-16y-12x+192=6912-\dfrac{16(6912)}{x}-12x+192$

$A'=-12+\dfrac{(16)(6912)}{x^2} \qquad A'=0 \Rightarrow x=96$

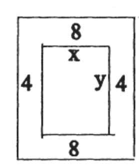
$when\ x=96, \quad y=\dfrac{6912}{96}=72 \qquad \text{dim } is\ 96cm\ by\ 72cm$



20. $xy=200, \qquad y=\dfrac{200}{x}, \qquad A=(x+8)(y+16)=200+16x+\dfrac{(8)(200)}{x}+128$

$A'=16-\dfrac{(8)(200)}{x^2} \qquad A'=0 \Rightarrow x=10 \qquad when\ x=10, \quad y=\dfrac{200}{10}=20$

$the\ \text{dim } is\ (10+8)\ by\ (20+16), \quad i.e.\ 18\ cm\ by\ 36\ cm$



21. Let x be the length of the part for circle and $(8-x)$ be that for square. Let r be the radius of the circle, $r=\dfrac{x}{2\pi}$

$$A=\pi\left(\frac{x}{2\pi}\right)^2+\left(\frac{8-x}{4}\right)^2=\left(\frac{1}{4\pi}\right)x^2+\frac{(8-x)^2}{16} \qquad A'=\frac{1}{4\pi}(2x)+\frac{2(8-x)}{16}(-1)=0\Rightarrow x=\frac{8\pi}{4+\pi} \qquad 0\le x\le 8$$

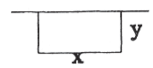
$$A(0)=4,\quad A\left(\frac{8}{4+\pi}\right)=2.24,\quad A(8)=5.09,\quad x=8 \text{ gives max}, \quad \text{and}\quad x=\frac{8\pi}{4+\pi} \text{ give the min total area}$$

22. Let x be the length of the part for the rectangle and (L-x) be the length of the part for the triangle.

$$A=\left(\frac{x}{3}\right)\left(\frac{x}{6}\right)+\left(\frac{1}{2}\right)\left(\frac{8-x}{3}\right)^2\sin\frac{\pi}{3}=\frac{x^2}{18}+\frac{\sqrt{3}}{36}(8-x)^2 \qquad A'=\frac{x}{9}+\frac{\sqrt{3}}{36}(2)(8-x)(-1)=0\Rightarrow x=\left(2\sqrt{3}-3\right)8$$

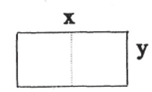
$$0\le x\le 8,\quad A(0)=3.07,\quad A\left(\left(2\sqrt{3}-3\right)8\right)=1.62,\ \text{min} \qquad A(8)=3.56,\ \text{max}$$

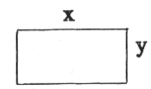
23. $A=xy \qquad 2y+x=600 \qquad A=(600-2y)y=600y-2y^2$



$A'=600-4y \qquad A'=0\Rightarrow y=150 \qquad \text{when } y=150,\ x=600-2(150)=300 \qquad$ dim *is* 300*m by* 150*m*

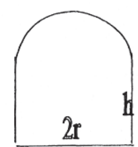
24. $xy=3750,\quad L=2x+3y=2x+3\left(\dfrac{3750}{x}\right),\quad \dfrac{dL}{dx}=2-\dfrac{11250}{x^2}=0\quad x=75m\quad y=\dfrac{3750}{75}=50m$





25. Let y be the length of $N-S$ side and x be the length of $E-W$ side. $\quad A=xy=100$

Cost $=C=5(2x)+20(2y)=10x+\dfrac{4000}{x} \qquad C'=10-\dfrac{4000}{x^2}=0\Rightarrow x=20,\quad y=5 \qquad$ dim *is* 20*m by* 5*m*



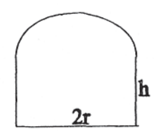
26. Let h be the height of the rectangular part and r be the radius of the semicircle.

$$8=2r+2h+\pi r \qquad A=2rh+\frac{\pi r^2}{2}=r(8-2r-\pi r)+\frac{\pi r^2}{2} \qquad A'=0\Rightarrow 8-4r-\pi r=0$$

$$\text{so}\quad r=\frac{8}{4+\pi} \qquad \text{hence } h=\frac{8}{4+\pi}=r \qquad \text{the height of the window: the radius is } (h+r):r=2:1$$

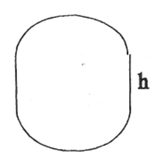
27. Let h ne the height of the rectangular part and r be the radius of the semicircle, so width $=2r$

$$2r+2h+\frac{2\pi r}{2}=8\Rightarrow 2h=8-2r-\pi r \qquad A=2(2r)(h)+\frac{\pi r^2}{2}=2r(8-2r-\pi r)+\frac{\pi r^2}{2}$$



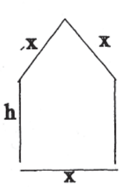
$$A'=16-8r-3\pi r=0\Rightarrow r=\frac{16}{8+3\pi} \qquad width=2r=\frac{32}{8+3\pi}$$

28. $8=2h+2\pi r \qquad A=2rh+\pi r^2\Rightarrow A=r(8-2\pi r)+\pi r^2 \qquad A'=8-4\pi r+2\pi r$



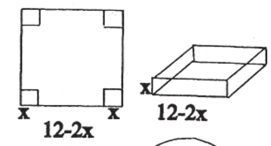
$$A'=0\Rightarrow 8-2\pi r=0\Rightarrow r=\frac{8}{2\pi} \qquad \text{when } r=\frac{8}{2\pi}\Rightarrow h=0 \qquad \text{the window is a circle}$$

29. *Let x be the width of the window* $\qquad 2h+3x=8 \qquad h=\dfrac{8-3x}{2}$



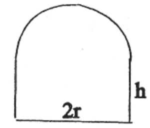
$$A=xh+\frac{1}{2}x^2\sin\frac{\pi}{3}=xh+\frac{\sqrt{3}}{4}x^2=x\left(\frac{8-3x}{2}\right)+\frac{\sqrt{3}}{4}x^2 \qquad A'=4-3x-\frac{\sqrt{3}}{2}x=0\Rightarrow x=\frac{8(6+\sqrt{3})}{33}m$$

30. $V=(12-2x)^2x \qquad V'=(12-2x)(12-6x) \qquad V'=0\Rightarrow x=\frac{12}{6}=2\,cm$



31. $A=\pi r^2+2\pi rh+\frac{4\pi r^2}{2} \qquad V=\frac{\frac{4}{3}\pi r^3}{2}+\pi r^2h=\frac{2\pi r^3}{3}+\pi r^2\left(\frac{A-3\pi r^2}{2\pi r}\right)$

$V'=0\Rightarrow r=\sqrt{\frac{A}{5\pi}} \qquad when\ r=\sqrt{\frac{A}{5\pi}},\ \ h=\sqrt{\frac{A}{5\pi}} \qquad h:r=1:1$



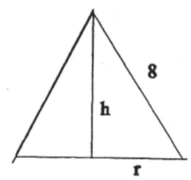
32. Let x be the length of one side of the square base and y be the height

$x^2+4(xy)=12\Rightarrow y=\frac{12-x^2}{4x}, \qquad V=x^2y=x^2\left(\frac{12-x^2}{4x}\right)=3x-\frac{x^3}{4} \qquad V'=3-\frac{3x^2}{4}=0\Rightarrow x=2,$

$when\ x=2, \quad y=\frac{12-4}{4(2)}=1 \qquad \dim\ is\ 2\ by\ 2\ by\ 1$

33. Let h be the height and r be the radius. $r^2+h^2=8^2=64 \qquad V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi(64-h^2)h$

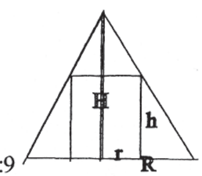
$V'=\frac{1}{3}\pi(64-3h^2)=0, \quad h^2=\frac{64}{3} \qquad h=\frac{8\sqrt{3}}{3}\,cm, \quad r^2=64-\frac{64}{3}=\frac{64\times 2}{3}, \quad r=\frac{8\sqrt{6}}{3}\,cm$



34. Let r, h be the radius of the cylinder and R, H be the radius and height of the fixed cone

$\frac{R-r}{h}=\frac{R}{H}\Rightarrow h=\frac{H(R-r)}{R}, \qquad V=\pi r^2h=\pi r^2\left(\frac{H(R-r)}{R}\right)=\pi Hr^2-\frac{\pi H}{R}r^3$

$V'=\pi H(2)r-\frac{\pi H}{R}(3)r^2\Rightarrow r=\frac{2R}{3} \quad so\ \ h=\frac{H}{3} \qquad V_{cylinder}:V_{cone}=\pi\left(\frac{2R}{3}\right)^2\left(\frac{H}{3}\right):\frac{1}{3}\pi R^2H=4:9$

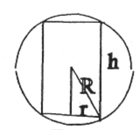


35. $V=\pi r^2h\Rightarrow h=\frac{V}{\pi r^2} \qquad A=2\pi r^2+2\pi rh=2\pi r^2+\frac{2V}{r}$

$A'=4\pi r-\frac{2V}{r^2}=0\Rightarrow r^3=\frac{V}{2\pi} \qquad \frac{h}{r}=\frac{\frac{V}{\pi r^2}}{r}=\frac{V}{\pi r^3}=\frac{V}{\pi\left(\frac{V}{2\pi}\right)}=2:1$

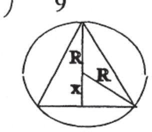
36. *Let r and h be the radius and height of the cylinder* $r^2+\left(\frac{h}{2}\right)^2=R^2\Rightarrow r^2=R^2-\frac{h^2}{4}$

$V=\pi r^2h=\pi\left(R^2-\frac{h^2}{4}\right)h \qquad V'=\pi R^2-\frac{3}{4}\pi h^2=0\Rightarrow h=\frac{2}{\sqrt{3}}R, \quad r^2=\frac{2}{3}R^2, \quad V=\pi\left(\frac{2}{3}R^2\right)\left(\frac{2}{\sqrt{3}}R\right)=\frac{4\sqrt{3}}{9}\pi R^3$



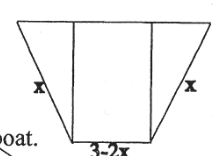
37. Let x be the distance from the centre of the sphere to the base of the cone. So the height of the cone is $x+R$, the radius of the base of the cone is $\sqrt{R^2-x^2}$

$V=\frac{1}{3}\pi\left(R^2-x^2\right)(x+R) \qquad V'=\frac{1}{3}\pi(R-3x)(x+R)=0\Rightarrow x=\frac{R}{3} \qquad V=\frac{1}{3}\pi\left(R^2-\frac{R^2}{9}\right)\left(\frac{R}{3}+R\right)=\frac{32}{81}\pi R^3$



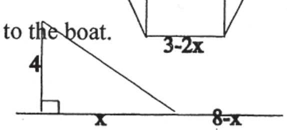
38. *Let x be length of one side* $A=\frac{1}{2}\left(3-2x+(3-2x+\frac{x}{2}+\frac{x}{2})\right)\left(\frac{\sqrt{3}}{2}x\right)=\frac{3\sqrt{3}}{2}x-\frac{3\sqrt{3}x^2}{4}$

$A'=3\sqrt{3}-3\sqrt{3}x=0\Rightarrow x=1\,unit$



39. Let x be the distance between the pt he land to the pt on the shore which is closest to the boat.

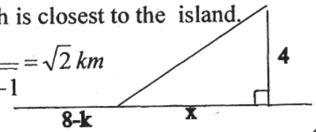
$Time=T=\frac{\sqrt{x^2+16}}{3}+\frac{8-x}{5}, \quad T'=\frac{2x}{6\sqrt{x^2+16}}-\frac{1}{5}=0, \quad x=3\,km$



p.72

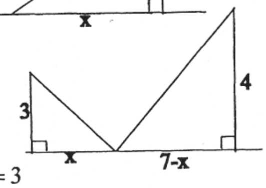
K cancels

40 Let x be the distance between the "turning" pt and the pt on the shore which is closest to the island.



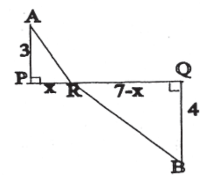
$C=(3k)\sqrt{4^2+x^2}+(k)(8-x)$ $\qquad C'=\dfrac{3kx}{\sqrt{4^2+x^2}}-k$ $\qquad C'=0\Rightarrow x=\dfrac{4}{\sqrt{3^2-1}}=\sqrt{2}\,km$

41. *Let one part be x that is close to the post of length 3*



$L=\sqrt{3^2+x^2}+\sqrt{4^2+(7-x)^2}$ $\qquad L'=\dfrac{x}{\sqrt{3^2+x^2}}-\dfrac{7-x}{\sqrt{4^2+(7-x)^2}}$

$L'=0\Rightarrow \dfrac{x^2}{3^2+x^2}=\dfrac{(7-x)^2}{4^2+(7-x^2)}$ *simplify, we get* $\dfrac{3^2}{x^2}=\dfrac{4^2}{(7-x)^2}$ $\quad \dfrac{3}{x}=\dfrac{4}{7-x}\Rightarrow x=3$



$7-3=4 \qquad L=\sqrt{3^2+3^2}+\sqrt{4^2+4^2}=7\sqrt{2}$

42 $D=\sqrt{3^2+x^2}+\sqrt{4^2+(7-x)^2}\ 2$

Similar to # 41, $x=3$ *units*

43. *Let* (x,y) *be a pt on* $3x+4y-25=0$

$D=\sqrt{x^2+y^2}\Rightarrow D^2=x^2+y^2=x^2+\left(\dfrac{25-3x}{4}\right)^2$ $\qquad (D^2)'=2x+2\left(\dfrac{25-3x}{4}\right)\left(-\dfrac{3}{4}\right)=0\Rightarrow x=3, \quad y=4$

44. $D^2=(x-4)^2+(y-2)^2=\left(\dfrac{y^2}{8}-4\right)^2+(y-2)^2$ $\qquad (D^2)'=\dfrac{4y^3}{64}-4=0\Rightarrow y=4$ $\qquad$ *when* $y=4, \quad x=2$

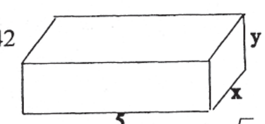
45 Profit $=P\ (20+x)(30-x)-10(30-x)=300+20x-x^2$ $\qquad P'=20-2x=0\Rightarrow x=10$ $\qquad$ *new price is* \$30

46. *Let x be the number of additional trees.* $\qquad y=(20+x)(100-2x)=2000+60x-2x^2$

$y'=60-4x=0\Rightarrow x=15$ $\qquad$ *35 trees should be planted*

47. *Let the length be x and the height be y* $\qquad V=5xy=250$

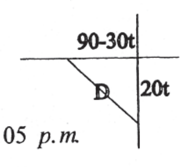
$C=(2)(5x)+(1)(10+2x)(y)=10x+(10+2x)\left(\dfrac{250}{5x}\right)$ $\qquad C'=0\Rightarrow x=5\sqrt{2}\quad C=\$241.42$



48. Let W be the width, D be the depth and S be the strength. $\qquad W^2+D^2=20^2=400$

$S=kWD^2$ for some constant k. $\qquad S=kW(400-w^2),\ \ S'=k(400-3W^2)=0,$ $\qquad W^2=\dfrac{400}{3},\ W=\dfrac{20}{\sqrt{3}}=\dfrac{20\sqrt{3}}{3}$

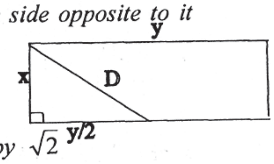
$D^2=400-W^2=400-\dfrac{400}{3}=\dfrac{400\times 2}{3}$ $\qquad D=\dfrac{20\sqrt{2}}{\sqrt{3}}=\dfrac{20\sqrt{6}}{3}$



49. At noon, ship B is 90 km west of the dock. Let t be the time in hours after noon.

$D^2=(90-30t)^2+(20t)^2,\ \ (D^2)'=2(90-30t)(-30)+2(20t)(20)=0$ $\qquad t=2.077$ $\qquad$ Ans: 2:05 *p.m.*

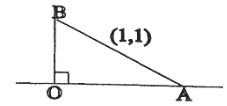
50. *Let x be length of the adjacent side of that vertex and y be the length of the side opposite to it*



$xy=4$ $\qquad D^2=x^2+\left(\dfrac{y}{2}\right)^2=x^2+\dfrac{1}{4}y^2=x^2+\dfrac{4}{x^2}$

$(D^2)'=2x-\dfrac{8}{x^3}=0\Rightarrow x=\sqrt{2}$ $\qquad$ *when* $x=\sqrt{2},\ \ y=2\sqrt{2}$ $\qquad$ dim*ension is* $2\sqrt{2}$ *by* $\sqrt{2}$

51. *Let the equation be* $y-1=m(x-1)$ $\quad x=0 \Rightarrow OB=1-m \quad y=0 \Rightarrow OA=\dfrac{m-1}{m}$



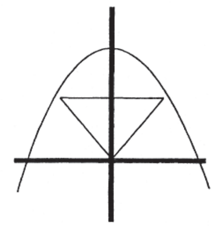
$$L^2=AB^2=OA^2+OB^2=\left(\frac{m-1}{m}\right)^2+(1-m)^2$$

$$(L^2)'=(2)\left(\frac{m-1}{m}\right)\left(\frac{m-(m-1)(1)}{m^2}\right)+(2)(1-m)(-1)=0 \Rightarrow m=-1 \quad OA=OB=2, \quad AB=2\sqrt{2}$$

52. *Let the line be* $y-4=m(x-3)$ $\quad x=0 \Rightarrow y=4-3m \quad y=0 \Rightarrow x=\dfrac{3m-4}{m}$

$$(a)\ area=A=\frac{1}{2}\frac{(3m-4)(4-3m)}{m}=-\frac{(3m-4)^2}{m} \qquad \frac{dA}{dm}=0 \Rightarrow -3^2+\frac{4^2}{m^2}=0 \Rightarrow m=\pm\frac{4}{3}$$

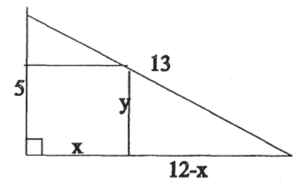
$$(3,4)\ \textit{in first quadrant}, \quad m=-\frac{4}{3} \qquad A=-\frac{1}{2}\left(\frac{\left(3\left(-\frac{4}{3}\right)-4\right)^2}{\left(-\frac{4}{3}\right)}\right)=24$$



53. Let $(x,\ 12-x^2)$ be the vertex of the triangle that lies on the curve in the first quadrant

$$A=\frac{1}{2}(2x)\left(12-x^2\right)=12x-x^3 \qquad A'=12-3x^2=0 \Rightarrow x=2 \qquad A=16\ unit^2$$

54. Let x be the length of the side of the rectangle along the side of 12
Let y be the length of the side of the rectangle along the side of 5



$$\frac{12-x}{y}=\frac{12}{5} \Rightarrow y=\frac{60-5x}{12} \qquad A=xy=x\left(\frac{60-5x}{12}\right)=5x-\frac{5}{12}x^2$$

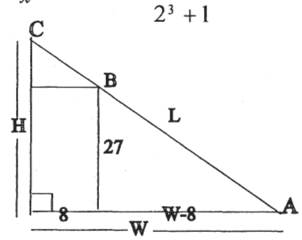
$$A'=5-\frac{10}{12}x=0 \Rightarrow x=6, \qquad when\ x=6,\quad y=\frac{5}{2} \qquad A=5(6)-\frac{5(6)^2}{12}=15$$

55. $D=y_1-y_2=x^2-\left(-(x-2)^2\right)=x^2+(x-2)^2 \qquad D'=2x+2(x-2)=0 \Rightarrow x=1 \qquad D=1+1=2$

56. *Let* $AP=x, \quad I=I_A+I_B=\dfrac{2k}{x^2}+\dfrac{k}{(100-x)^2}$

$$\frac{dI}{dx}=-\frac{4k}{x^3}+\frac{2k}{(100-x)^3}=0 \qquad \frac{4k}{x^3}=\frac{2k}{(100-x)^3} \Rightarrow \left(\frac{x}{100-x}\right)^3=2 \qquad \frac{x}{100-x}=2^{\frac{1}{3}}=x=\frac{2^{\frac{1}{3}}(100)}{2^{\frac{1}{3}}+1}=55.75$$

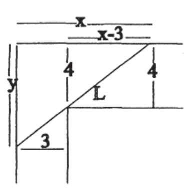
57. $\dfrac{H}{W}=\dfrac{27}{W-8} \Rightarrow H=\dfrac{27W}{W-8} \qquad L^2=W^2+H^2=W^2+\left(\dfrac{27W}{W-8}\right)^2$



$$(L^2)'=2W+(27)^2(2)\left(\frac{W}{W-8}\right)\left(\frac{(W-8)(1)-W(1)}{(W-8)^2}\right)=0 \Rightarrow W=26$$

$$H=\frac{27(26)}{26-8}=39 \qquad L=\sqrt{26^2+39^2}=\sqrt{2197}=46.9\,m$$

58. Similar to # 56 Let L be length of the rod



$$\frac{y}{x}=\frac{4}{x-3}, \qquad L^2=x^2+y^2 \qquad L=9.87\,m$$