MCV4U1

# Unit #2 – Algebraic Vectors

## Progress Check

Part A

1. Complete the following table:

| Geometric Vector | Algebraic Vector Coordinate Form | Algebraic Vector Unit Vector Form |
| --- | --- | --- |
| u = 35 cm [S40ºW] | | |
| | | $v = 10\vec{i} + 7\vec{j}$ |
| | w = (-3, 0, 5) | |

2. Find the vector $\overrightarrow{UV}$ where U(3, -2, 9) and V(21, 25, -6).

3. Graph the following vectors:

a) $\vec{u} = (3, -5)$

b) $\vec{v} = (4, -3, 5)$

4. If $\vec{a} = 15\vec{i} - 20\vec{j} + 7\vec{k}$ and $\vec{b} = 5\vec{j} + 13\vec{k}$, find $|4\vec{a} - 3\vec{b}|$.

5. If the points A(x, 10, -2), B(25, 2, 15) and C(69, -14, z) are collinear, solve for x and z.

6. Determine whether or not $\vec{u} = (10, 20, 30)$ and $\vec{v} = (2, -1, 1)$ are perpendicular.

7. Calculate the dot product of the following:

a) $\vec{u} = (-10, 13, -4)$ and $\vec{v} = (2, 0, 15)$

b) $|\vec{u}| = 80$, $|\vec{v}| = 25$, $\theta = 78^{o}$

8. Calculate the cross product of $\vec{a} = (100, 56, 243)$ and $\vec{b} = (-30, 28, 95)$.

9. Calculate $|4\vec{a} \times 3\vec{b}|$ where $|\vec{a}| = 15$, $|\vec{b}| = 38$ and $\theta = 11^{o}$.

10. Calculate the area of a parallelogram with sides $\vec{a} = (4, -20, 15)$ and $\vec{b} = (9, 12, -5)$.

11. Calculate the area of triangle ABC, where A(0, 2, 7), B(11, -14, 8), and C(-7, 7, 1).

12. Miss Pick pushes a box of textbooks 3 metres across the floor with her foot. If she has to work against a frictional force of 80 N, how much work must she do to move the box of books.

(Part B)

Part B

1. Three forces act on an object. A 35-N force acts at an angle of 20° relative to a 60-N force. A 40-N force acts at an angle of 75° relative to the 60-N force on the opposite side of the 35-N force. Determine the equilibrant of the three forces.

2.

a) Convert to geometric form by quoting magnitude and direction angles for the vector $\vec{u} = (1, 8, -3)$

b) Convert to 2-D algebraic component form
$\vec{u} = 12[\theta = 150^\circ]$

c) Convert to 3-D algebraic component form
$\vec{u} = \sqrt{3}[\alpha = 30^\circ, \gamma = 60^\circ]$

3) An airplane is travelling N30°E at an airspeed of 340 km/h when it encounters a wind blowing north at 100 km/h. Determine the resultant vector of the airplane.

4) Given P(–1, 0, 5), Q(5, –4, 12), R(–19, 12, –16), and S(2, –2, 5), is it possible to express $\overrightarrow{PQ}$ as a linear combination of $\overrightarrow{PR}$ and $\overrightarrow{PS}$? Use your result to find if the given points are coplanar/collinear/neither

5. The vector $\overrightarrow{XY} = [-3, -1, 4]$ has its initial point at X(5,7,4). Determine the coordinates of Y.

6. Given A(–2, 1, 3), B(3, –5, –1), and C(0, 2, 0).

**a)** Determine the coordinates of midpoints D and E of lengths AB and AC, respectively.

**b)** Express $\overrightarrow{DE}$ and $\overrightarrow{BC}$ as position vectors. State their relationship.

**c)** Calculate the lengths of DE and BC. Explain how they are related.

**d)** Determine the direction cosines and corresponding angles for $\overrightarrow{BC}$.

7) a. The set of vectors $\{(1, 0, 0), (0, 1, 0)\}$ spans a set in $R^3$. Describe this set.

b. Write the vector $(-2, 4, 0)$ as a linear combination of these vectors.

c. Explain why it is not possible to write $(3, 5, 8)$ as a linear combination of these vectors.

d. If the vector $(1, 1, 0)$ were added to this set, what would these three vectors span in $R^3$?

8. A sailor climbs a mast at 0.5 m/s on a ship travelling north at 12 m/s, while the current flows east at 3 m/s. What is the speed of the sailor relative to the ocean floor?

9. A car is 260 m north and a truck 170 m west of an intersection. They are both approaching the intersection, the car from the north at 80 km/h, and the truck from the west at 50 km/h. Determine the velocity of the truck relative to the car.

10. Suppose $\vec{c}$ = 60 km/h represents the velocity of an eastbound car, $\vec{t}$ = 80 km/h represents the velocity of a westbound truck, and $\vec{b}$ = 40 km/h represents the velocity of a westbound bus. For each situation, state and draw the velocity vector.

**a)** the velocity of the car as it appears relative to the driver of the truck

**b)** the velocity of the bus as it appears relative to the driver of the truck

**c)** the velocity of the bus as it appears relative to the driver of the car

11. A boat on a lake is heading at a bearing of 30°. The boat is travelling at a velocity of 40 km/h and the current is flowing from the southeast at 9 km/h. Determine the heading of the boat and the resultant velocity relative to the ground

12. Does $\bar{a} \downarrow (\bar{b} + \bar{c}) = \bar{a} \downarrow \bar{b} + \bar{a} \downarrow \bar{c}$ ? Prove your answer.

13. Suppose that $|\vec{c}| = 8$, $|\vec{d}| = 3$, and the angle between these two vectors is $\theta = 60°$.

**a.** Find $\vec{c} \cdot \vec{d}$.

**b.** Determine the dot product of the vectors $\vec{c} - \vec{d}$ and $\vec{c} + 2\vec{d}$.

**c.** What value of $k$ would cause the vectors $\vec{c} - \vec{d}$ and $\vec{c} + k\vec{d}$ to be perpendicular?

14. Consider any two vectors in $R^3$, $\vec{a}$ and $\vec{b}$.

**a.** Describe in your own words what the scalar projections of $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$ represent geometrically. Draw these projections in your diagram. What does it mean if this scalar projection is positive? What does it mean if this scalar projection is negative? What does it mean if this scalar projection is zero?

**b.** Describe in your own words what the vector projections of $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$ represent geometrically.

15. For each of the following computations, state whether the result will be a scalar, a vector, or if the computation is meaningless.

**a.** $|\vec{a} \times \vec{b}|\vec{c}$

**b.** $((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}$

**c.** $\vec{a} \times \vec{b} + \vec{a} \cdot \vec{b}$

**d.** $(\vec{a} \cdot (\vec{b} \times \vec{c})) \cdot \vec{d}$

**e.** $(((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d})|\vec{a} \times \vec{b}|\vec{c}$

**f.** $(((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}) - |\vec{a} \times \vec{b}|\vec{c}$

(16) Consider the following vectors $\vec{a} = 5\vec{i} - \vec{j} + 2\vec{k}$ , $\vec{b} = \vec{i} - 4\vec{k}$ and $\vec{c} = -\vec{i} + 6\vec{j}$ . Compute the following

a) $\vec{a} \bullet \vec{b}$
b) $(2\vec{c} + \vec{b}) \times \vec{a}$
c) $\vec{b} \downarrow \vec{a}$
d) $\vec{b} \times \vec{c}$
e) $(\vec{a} - 3\vec{b}) \bullet \vec{c}$
f) $|\vec{a} \downarrow \vec{c}|$
g) Find angle between vectors $\vec{a}$ and $\vec{b}$
h) If a triangle is defined by vectors $\vec{b}$ and $\vec{c}$ , find the area of this triangle.

(17) Find word done by the force $\vec{F} = (-2, 3, -5)$ when an object is moved from point $A(0,1,4)$ to point $B(-3,0,5)$

(18) Find a unit vector perpendicular to both $\vec{a} = (2, -3, 0)$ and $\vec{b} = (0, 1, -2)$

(19) Prove the following relation involving vectors:

$$\vec{a} \cdot \vec{b} = \frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$$

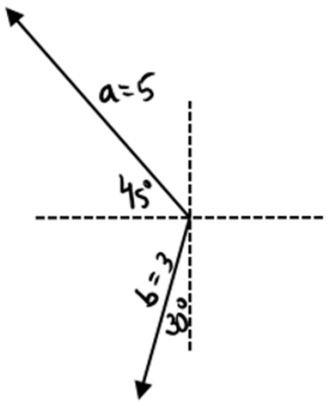
(20) If $\vec{a}$ and $\vec{b}$ are unit vectors, and $\|\vec{a} - \vec{b}\| = \sqrt{2}$ , determine $(2\vec{a} - 3\vec{b}) \cdot (\vec{a} + 2\vec{b})$

(21) If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ prove that $\vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{c} + \vec{c} \bullet \vec{a} = -\frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)$

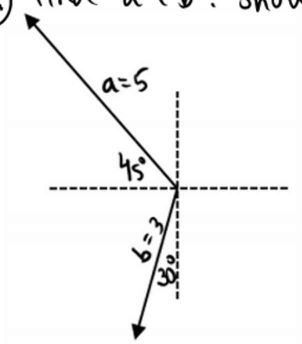
(22) A special bicycle computer reports the magnitude of the torque exerted by a cyclist during a strenuous portion of the ride to be $\frac{600}{\sqrt{3}}$ J at the top of his pedal-stroke. If the force exerted by the cyclist at this point is in a direction that is 60° relative to the pedal, and the pedals are 25 cm in length, how much force, in N, must the cyclist be exerting at this moment?

(23) Suppose a box on a frictionless ramp is being pulled by a rope with a tension of 350 N making an angle of 45° to the horizontal ground. If the angle of incline of the ramp is 15°, and the box is pulled 50 m, determine the amount of work done.

24. Find $\vec{b} - \vec{a}$ . Use both methods




25. Find $\vec{a} + \vec{b}$ . Show both methods.



ReviewSol

August 6, 2018 3:01 PM

Solutions

MCV4U1

Part A

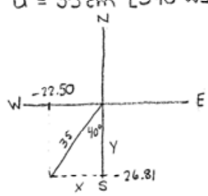
# Unit #2 – Algebraic Vectors

Progress Check

1. Complete the following table:

| | Geometric Vector | Algebraic Vector Coordinate Form | Algebraic Vector Unit Vector Form |
|---|---|---|---|
| ① | $\vec{u} = 35$ cm [S40°W] | $\vec{u} = (-22.50, -26.81)$ | $\vec{u} = -22.50\vec{i} - 26.81\vec{j}$ |
| ② | $\vec{v} = \sqrt{149}$ units [N55.01°E] | $\vec{v} = (10, 7)$ | $\vec{v} = 10\vec{i} + 7\vec{j}$ |
| ③ | $\vec{w} = \sqrt{34}$ units [120.96° to x] [90° to y] [30.96° to z] | w = (-3, 0, 5) | $\vec{w} = -3\vec{i} + 5\vec{k}$ |

① $\vec{u} = 35$ cm [S40°W]




$\sin\theta = \frac{o}{h}$

$\sin 40 = \frac{x}{35}$

$35\sin 40 = x$

$\boxed{22.50 \doteq x}$

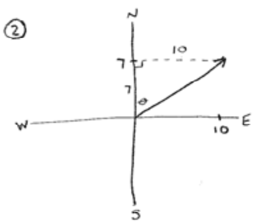
$\cos\theta = \frac{a}{h}$

$\cos 40 = \frac{y}{35}$

$35\cos 40 = y$

$\boxed{26.81 \doteq y}$

②




$a^2 + b^2 = c^2$

$7^2 + 10^2 = c^2$

$\boxed{\sqrt{149} = c}$

$\tan\theta = \frac{o}{a}$

$\theta = \tan^{-1}\left(\frac{10}{7}\right)$

$\boxed{\theta = 55.01°}$

③ $|\vec{w}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(-3)^2 + (0)^2 + (5)^2}$

$\boxed{= \sqrt{34}}$

$\cos A = \frac{x}{|\vec{w}|}$

$A = \cos^{-1}\left(\frac{-3}{\sqrt{34}}\right)$

$\boxed{A = 120.96° \text{ to x-axis}}$

$\cos B = \frac{y}{|\vec{w}|}$

$B = \cos^{-1}\left(\frac{0}{\sqrt{34}}\right)$

$\boxed{B = 90° \text{ to y-axis}}$

$\cos C = \frac{z}{|\vec{w}|}$

$C = \cos^{-1}\left(\frac{5}{\sqrt{34}}\right)$

$\boxed{C = 30.96° \text{ to z-axis}}$

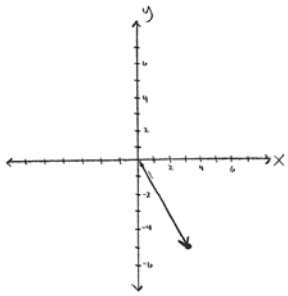
2. Find the vector $\overline{UV}$ where U(3, -2, 9) and V(21, 25, -6).

$\overrightarrow{UV} = (21-3, 25-(-2), -6-9)$

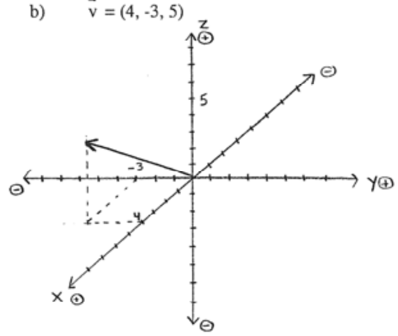
$\boxed{\overrightarrow{UV} = (18, 27, -15)}$

3. Graph the following vectors:

a) $\vec{u} = (3, -5)$




b) $\vec{v} = (4, -3, 5)$




4. If $\vec{a} = 15\vec{i} - 20\vec{j} + 7\vec{k}$ and $\vec{b} = 5\vec{j} + 13\vec{k}$, find $|4\vec{a} - 3\vec{b}|$.

$\vec{a} = (15, -20, 7) \qquad \vec{b} = (0, 5, 13)$

$4\vec{a} - 3\vec{b}$

$= 4(15, -20, 7) - 3(0, 5, 13)$

$= (60, -80, 28) + (0, -15, -39)$

$= (60, -95, -11)$

$|4\vec{a} - 3\vec{b}| = \sqrt{x^2 + y^2 + z^2}$

$|4\vec{a} - 3\vec{b}| = \sqrt{(60)^2 + (-95)^2 + (-11)^2}$

$\boxed{|4\vec{a} - 3\vec{b}| = \sqrt{12746}}$

5. If the points A(x, 10, -2), B(25, 2, 15) and C(69, -14, z) are colinear, solve for x and z.

$\vec{AB} = (25 - x, 2 - 10, 15 - (-2)) \qquad \vec{BC} = (69 - 25, -14 - 2, z - 15)$

$\vec{AB} = (25 - x, -8, 17) \qquad \vec{BC} = (44, -16, z - 15)$

$-8 \times k = -16$

$k = \frac{-16}{-8}$

$k = 2$

$(25 - x)(2) = 44$

$50 - 2x = 44$

$-2x = -6$

$\boxed{x = 3}$

$17(2) = z - 15$

$34 = z - 15$

$\boxed{49 = z}$

6. Determine whether or not $\vec{u} = (10, 20, 30)$ and $\vec{v} = (2, -1, 1)$ are perpendicular.

$\vec{u}\cdot\vec{v} = 10(2) + 20(-1) + 30(1)$

$\vec{u}\cdot\vec{v} = 20 - 20 + 30$

$\boxed{\vec{u}\cdot\vec{v} = 30}$

$\therefore$ No, they are not perpendicular

7. Calculate the dot product of the following:

a) $\vec{u} = (-10, 13, -4)$ and $\vec{v} = (2, 0, 15)$

$\vec{u}\cdot\vec{v} = -10(2) + 13(0) + (-4)(15)$

$\vec{u}\cdot\vec{v} = -20 - 60$

$\boxed{\vec{u}\cdot\vec{v} = -80}$

b) $|\vec{u}| = 80$, $|\vec{v}| = 25$, $\theta = 78°$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\vec{u}\cdot\vec{v} = 80(25)\cos 78$

$\boxed{\vec{u}\cdot\vec{v} \doteq 415.82}$

8. Calculate the cross product of $\vec{a} = (100, 56, 243)$ and $\vec{b} = (-30, 28, 95)$.

$\vec{a}\times\vec{b} = (y_1z_2 - y_2z_1,\ z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1)$

$\vec{a}\times\vec{b} = (56(95) - 243(28),\ 243(-30) - 100(95),\ 100(28) - 56(-30))$

$\boxed{\vec{a}\times\vec{b} = (-1484, -16790, 4480)}$

$$\begin{array}{ccc} 56 & & 28 \\ 243 & \times & 95 \\ 100 & \times & -30 \\ 56 & \times & 28 \end{array}$$

9. Calculate $|4\vec{a} \times 3\vec{b}|$ where $|\vec{a}| = 15$, $|\vec{b}| = 38$ and $\theta = 11°$.

$|4\vec{a}| = 4(15)$
$= 60$

$|3\vec{b}| = 3(38)$
$= 114$

$|4\vec{a}\times3\vec{b}| = |4\vec{a}||3\vec{b}|\sin\theta$

$|4\vec{a}\times3\vec{b}| = 60(114)\sin 11$

$\boxed{|4\vec{a}\times3\vec{b}| \doteq 1305.13}$

10. Calculate the area of a parallelogram with sides $\vec{a} = (4, -20, 15)$ and $\vec{b} = (9, 12, -5)$.

$A = |\vec{a} \times \vec{b}|$

$A = \sqrt{x^2 + y^2 + z^2}$

$A = \sqrt{(-80)^2 + (155)^2 + (228)^2}$

$\boxed{A = \sqrt{82409}}$

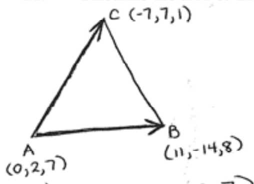
$\vec{a} \times \vec{b} = (100 - 180, 135 + 20, 48 + 180)$

$\vec{a} \times \vec{b} = (-80, 155, 228)$

$\begin{matrix} -20 & \times & 12 \\ 15 & \times & -5 \\ 4 & \times & 9 \\ -20 & \times & 12 \end{matrix}$

$\therefore$ the area is $\sqrt{82409}$ units$^2$

11. Calculate the area of triangle ABC, where A(0, 2, 7), B(11, -14, 8), and C(-7, 7, 1).




$\vec{AB} = (11-0, -14-2, 8-7)$

$\boxed{\vec{AB} = (11, -16, 1)}$

$\vec{AC} = (-7-0, 7-2, 1-7)$

$\boxed{\vec{AC} = (-7, 5, -6)}$

$A = \frac{|\vec{a} \times \vec{b}|}{2}$

$A = \frac{|\vec{AB} \times \vec{AC}|}{2}$

$\boxed{A = \frac{\sqrt{15011}}{2}}$

$\vec{AB} \times \vec{AC} = (96 - 5, -7 - (-66), 55 - 112)$

$= \boxed{(91, 59, -57)}$

$|\vec{AB} \times \vec{AC}| = \sqrt{x^2 + y^2 + z^2}$

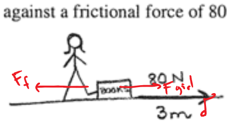
$= \sqrt{(91)^2 + (59)^2 + (-57)^2}$

$= \boxed{\sqrt{15011}}$

$\begin{matrix} -16 & \times & 5 \\ 1 & \times & -6 \\ 11 & \times & -7 \\ -16 & \times & 5 \end{matrix}$

$\therefore$ The area is $\frac{\sqrt{15011}}{2}$ units$^2$.

12. Miss Pick pushes a box of textbooks 3 metres across the floor with her foot. If she has to work against a frictional force of 80 N, how much work must she do to move the box of books.




$W_{person} = \vec{F} \cdot \vec{d}$

$W = |\vec{F}||\vec{d}| \cos\theta$

$W = 80(3) \cos 0$

$W = 240 \text{ J}$

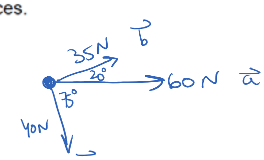
$W_{friction} = 80(3) \cos 180^\circ$

$= -240 \text{ J}$

$\therefore$ She must do at least 240 J of work.

Part B

(1.) Three forces act on an object. A 35-N force acts at an angle of 20° relative to a 60-N force. A 40-N force acts at an angle of 75° relative to the 60-N force on the opposite side of the 35-N force. Determine the equilibrant of the three forces.




Assume 60N points in pos. x axis dir

$\vec{a} = (60N, 0^\circ) = 60(\cos 0^\circ, \sin 0^\circ) = (60, 0)$

$\vec{b} = (35N^\circ, 20^\circ) = 35(\cos 20^\circ, \sin 20^\circ)$

$\vec{c} = (40N, 285^\circ) = 40(\cos 285^\circ, \sin 285^\circ)$

$\vec{F} = \vec{a} + \vec{b} + \vec{c}$

$= (60 + 35\cos 20^\circ + 40\cos 285,\ 0 + 35\sin 20^\circ + 40\sin 285^\circ)$

$\doteq (103.242,\ -26.666)$

$\therefore \vec{F_E} = (-103.242,\ 26.666)$ convert to geo in relation to 60N

26.666, 103.242, $\theta$

$\theta = \tan^{-1}\left(\frac{26.666}{103.242}\right) = 14.48^\circ$ $\quad |\vec{F_E}| = \sqrt{26.666^2 + 103.242^2} \doteq 106.6$

$\therefore \vec{F_E} \doteq 106.6\text{ N}$ [ $165.52^\circ$ from 60N on same side as 35 N force ]

(2)

a) Convert to geometric form by quoting magnitude and direction angles for the vector $\vec{u} = (1, 8, -3)$

b) Convert to 2-D algebraic component form $\vec{u} = 12[\theta = 150^\circ]$

$\hat{u} = (\cos\theta, \sin\theta)$ in 2D

$\hat{u} = (\cos\alpha, \cos\beta, \cos\gamma)$ in 3D

c) Convert to 3-D algebraic component form $\vec{u} = \sqrt{3}[\alpha = 30^\circ, \gamma = 60^\circ]$

(a) $|\vec{u}| = \sqrt{1^2 + 8^2 + 3^2}$

$= \sqrt{1 + 64 + 9}$

$= \sqrt{74}$

$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$

$(\cos\alpha, \cos\beta, \cos\gamma) = \frac{(1, 8, -3)}{\sqrt{74}}$

$\therefore \alpha = \cos^{-1}\left(\frac{1}{\sqrt{74}}\right) \doteq 83^\circ$

$\beta = \cos^{-1}\left(\frac{8}{\sqrt{74}}\right) \doteq 22^\circ$

$\gamma = \cos^{-1}\left(\frac{-3}{\sqrt{74}}\right) \doteq 110^\circ$

(b) $\vec{u} = |\vec{u}|\hat{u}$

$= |\vec{u}|(\cos\theta, \sin\theta)$

$= 12(\cos 150^\circ, \sin 150^\circ)$

$= 12\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\therefore \vec{u} = (-6\sqrt{3}, 6)$

2, 1, $-\sqrt{3}$, $150^\circ$

(c) $\vec{u} = |\vec{u}|\hat{u}$

$= |\vec{u}|(\cos\alpha, \cos\beta, \cos\gamma)$

$= \sqrt{3}(\cos 30^\circ, \cos\beta, \cos 60^\circ)$ $\rightarrow$ $|\hat{u}| = 1$

$= \sqrt{3}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$

$\therefore \vec{u} = \left(\frac{3}{2}, 0, \frac{\sqrt{3}}{2}\right)$

$\sqrt{\cos^2 30^\circ + \cos^2\beta + \cos^2 60} = 1$

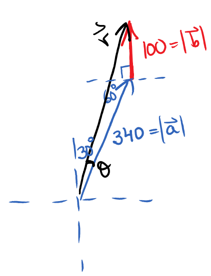
$\left(\frac{\sqrt{3}}{2}\right)^2 + \cos^2\beta + \left(\frac{1}{2}\right)^2 = 1$

$\frac{3}{4} + \cos^2\beta + \frac{1}{4} = 1$

$\cos^2\beta = 0$

$\beta = 90^\circ$

(3) An airplane is travelling N30°E at an airspeed of 340 km/h when it encounters a wind blowing north at 100 km/h. Determine the resultant vector of the airplane.




Geometric Way: (Sine Cosine Laws)

$|\vec{r}|^2 = 340^2 + 100^2 + 2(340)(100)\cos 150^\circ$ $\quad +\frac{\sqrt{3}}{2}$

$|\vec{r}| = \sqrt{125600 + 34000\sqrt{3}}$

$\frac{\sin\theta}{100} = \frac{\sin 150^\circ}{\sqrt{125600 + 34000\sqrt{3}}}$

$\theta \doteq 6.7^\circ$

$\therefore \vec{r} \doteq 429.5\text{ km/h}$ [N 23.3° E]

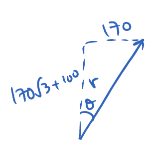
Algebraic Way: (Components)

$\vec{a} = 340(\cos 60^\circ, \sin 60^\circ) = 340\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (170, 170\sqrt{3})$

$\vec{b} = 100(\cos 90^\circ, \sin 90^\circ) = 100(0, 1) = (0, 100)$

$\therefore \vec{r} = \vec{a} + \vec{b} = (170 + 0,\ 170\sqrt{3} + 100) = (170,\ 170\sqrt{3} + 100)$

convert to geo




mag: $|\vec{r}| = \sqrt{170^2 + (170\sqrt{3} + 100)^2} \doteq 429.5$

dir: $\theta = \tan^{-1}\left(\frac{170}{170\sqrt{3} + 100}\right) = 23.3^\circ$

$\therefore \vec{r} = 429.5\text{ km/h}$ [N23.3°E]

(4) Given P(–1, 0, 5), Q(5, –4, 12), R(–19, 12, –16), and S(2, –2, 5), is it possible to express $\overrightarrow{PQ}$ as a linear combination of $\overrightarrow{PR}$ and $\overrightarrow{PS}$? Use your result to find if the given points are coplanar/collinear/neither

$\overrightarrow{PQ} = (5 - -1, -4 - 0, 12 - 5)$
$= (6, -4, 7)$

$\overrightarrow{PR} = (-19 - -1, 12 - 0, -16 - 5)$
$= (-18, 12, -21)$

$\overrightarrow{PS} = (2 - -1, -2 - 0, 5 - 5)$
$= (3, -2, 0)$

can it be written as a linear comb? $\boxed{\overrightarrow{PQ} \stackrel{?}{=} a\overrightarrow{PR} + b\overrightarrow{PS}}$

$(6, -4, 7) = a(-18, 12, -21) + b(3, -2, 0)$

seperate components: $\frac{6}{3} = \frac{-18a}{3} + \frac{3b}{3}$ $\quad \frac{-4}{2} = \frac{12a}{2} - \frac{2b}{2}$ $\quad 7 = -21a + 0b$

as a linear comb?

$(6,-4,7) = a(-18,12,-21) + b(3,-2,0)$

separate components: $\frac{6}{3} = \frac{-18a}{3} + \frac{3b}{3}$ ① $2 = -6a + b$

$\frac{-4}{2} = \frac{12a}{2} - \frac{2b}{2}$ ② $-2 = 6a - b$

$7 = -21a + 0b$ $\boxed{-\frac{1}{3} = a}$

$b = 6a + 2$ sub in

$b = 6(-\frac{1}{3}) + 2$

$\boxed{b = 0}$ check in ①

$2 \overset{?}{=} -6(-\frac{1}{3}) + 0$ ✓ works ☺

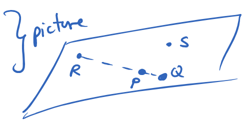
$\therefore \overrightarrow{PQ}$ can be written as linear combination of $\overrightarrow{PR}$ and $\overrightarrow{PS}$

$\overrightarrow{PQ} = -\frac{1}{3}\overrightarrow{PR} + \phi\overrightarrow{PS}$

Since there was no contradiction in linear combination vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar

Since $\overrightarrow{PQ} = -\frac{1}{3}\overrightarrow{PR}$ (without $\overrightarrow{PS}$) $\overrightarrow{PQ}$ is collinear/parallel to $\overrightarrow{PR}$

} picture




⑤ The vector $\overrightarrow{XY} = [-3, -1, 4]$ has its initial point at X(5,7,4). Determine the coordinates of Y.

$\overrightarrow{XY} = \overrightarrow{XO} + \overrightarrow{OY}$

$= -\overrightarrow{OX} + \overrightarrow{OY}$

$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$

$\overrightarrow{OX} = (5,7,4)$

$\therefore \overrightarrow{OY} = \overrightarrow{XY} + \overrightarrow{OX}$

$= (-3,-1,4) + (5,7,4)$

$= (2,6,8)$ $\therefore$ pt. $Y = (2,6,8)$

⑥ Given A(–2, 1, 3), B(3, –5, –1), and C(0, 2, 0).

a) Determine the coordinates of midpoints D and E of lengths AB and AC, respectively.

b) Express $\overrightarrow{DE}$ and $\overrightarrow{BC}$ as position vectors. State their relationship.

c) Calculate the lengths of DE and BC. Explain how they are related.

d) Determine the direction cosines and corresponding angles for $\overrightarrow{BC}$.

ⓐ midpt. $D = \left(\frac{-2+3}{2}, \frac{1+-5}{2}, \frac{3+-1}{2}\right)$

$= \left(\frac{1}{2}, -2, 1\right)$

midpt. $E = \left(\frac{-2+0}{2}, \frac{1+2}{2}, \frac{3+0}{2}\right)$

$= \left(-1, \frac{3}{2}, \frac{3}{2}\right)$

ⓑ $\overrightarrow{DE} = \left(-1-\frac{1}{2}, \frac{3}{2}--2, \frac{3}{2}-1\right)$

$= \left(-\frac{3}{2}, \frac{7}{2}, \frac{1}{2}\right)$

$\overrightarrow{BC} = (0-3, 2--5, 0--1)$

$= (-3, 7, 1)$

$\therefore \overrightarrow{DE}$ is half of $\overrightarrow{BC}$

ⓒ $|\overrightarrow{DE}| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{9+49+1}{4}}$

$= \frac{\sqrt{59}}{2}$

$|\overrightarrow{BC}| = \sqrt{3^2 + 7^2 + 1^2}$

$= \sqrt{59}$

ⓓ direction cosines = unit vector = $\widehat{BC}$

$\widehat{BC} = \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{(-3,7,1)}{\sqrt{59}}$

$= \left(\frac{-3}{\sqrt{59}}, \frac{7}{\sqrt{59}}, \frac{1}{\sqrt{59}}\right)$

$\cos\alpha$ $\cos\beta$ $\cos\gamma$

$\therefore$ angles $\alpha \doteq 113°$ $\beta \doteq 24°$ $\gamma = 83°$

⑦ a. The set of vectors $\{\vec{u}(1, 0, 0), \vec{v}(0, 1, 0)\}$ spans a set in $R^3$. Describe this set.

b. Write the vector $\vec{x}$ $(-2, 4, 0)$ as a linear combination of these vectors.

c. Explain why it is not possible to write $\vec{y}$ $(3, 5, 8)$ as a linear combination of these vectors.

d. If the vector $\vec{z}$ (1, 1, 0) were added to this set, what would these three vectors span in $R^3$?

ⓐ vectors $\vec{u}, \vec{v}$ are not multiples $\therefore$ not collinear they span a plane in $\mathbb{R}^3$.

ⓑ $\vec{x} = a\vec{u} + b\vec{v}$

$(-2,4,0) = a(1,0,0) + b(0,1,0)$

$-2 = 1a + 0b$ $\boxed{-2 = a}$

$4 = 0a + 1b$ $\boxed{4 = b}$

$\therefore \vec{x} = -2\vec{u} + 4\vec{v}$

ⓒ $\vec{y} = a\vec{u} + b\vec{v}$

$(3,5,8) = a(1,0,0) + b(0,1,0)$

ⓓ $\vec{z}$ can be written as linear comb of $\vec{u}$ and $\vec{v}$

$\vec{z} = a\vec{u} + b\vec{v}$

$(1,1,0) = a(1,0,0) + b(0,1,0)$

$1 = 0a + 1b$

(c) $\vec{y} = a\vec{u} + b\vec{v}$

$(3,5,8) = a(1,0,0) + b(0,1,0)$

$8 = 0a + 0b$

$8 \neq 0$ contradiction $\therefore \vec{y} = (3,5,8)$ not in the span of $\vec{u}, \vec{v}$

$\vec{z} = a\vec{u} + b\vec{v}$

$(1,1,0) = a(1,0,0) + b(0,1,0)$

$1 = 1a + 0b \qquad 1 = 0a + 1b$

$a = 1 \qquad 1 = b$

$\therefore \vec{z} = 1\vec{u} + 1\vec{v}$

$\therefore \vec{z}, \vec{u}, \vec{v}$ are coplanar, still span a plane in $\mathbb{R}^3$ (need only 2 of them.)

(8) A sailor climbs a mast at 0.5 m/s on a ship travelling north at 12 m/s, while the current flows east at 3 m/s. What is the speed of the sailor relative to the ocean floor?

3-D since we have North, East, climb up!

Assume coordinate system is placed with origin at MAST base.

$\therefore$ pos. z-axis = MAST

pos. x-axis = North

pos. y-axis = East.

So ... $\vec{V}_{sailor} = (0,0,0.5)$

$\vec{V}_{ship} = (12,0,0)$

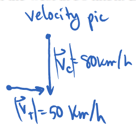
$\vec{V}_{current} = (0,3,0)$

$\vec{V}_{resultant}$ (rel. to ground) $= \vec{V}_{sailor} + \vec{V}_{ship} + \vec{V}_{current} = (12, 3, 0.5)$

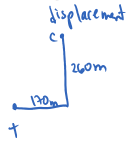
$\therefore$ Speed = only mag. $= \sqrt{12^2 + 3^2 + 0.5^2} \doteq 12.38$ m/s

(9) A car is 260 m north and a truck 170 m west of an intersection. They are both approaching the intersection, the car from the north at 80 km/h, and the truck from the west at 50 km/h. Determine the velocity of the truck relative to the car.

velocity pic




displacement pic

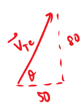



$\vec{V}_{TC} = \vec{V}_T - \vec{V}_C$ ← from pos. x-axis

$= 50(\cos 0°, \sin 0°) - 80(\cos 270°, \sin 270°)$

$= (50, 0) - (0, -80)$

$= (50, 80)$ convert form




$|\vec{V}_{TC}| = \sqrt{80^2 + 50^2}$

$= \sqrt{8900}$

$= 10\sqrt{89}$ km/h

$\theta = \tan^{-1}\left(\frac{80}{50}\right) \doteq 58°$

$\therefore \vec{V}_{TC} = 10\sqrt{89}$ km/h [N 32° E]

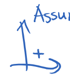
(10) Suppose $\vec{c}$ = 60 km/h represents the velocity of an eastbound car, $\vec{t}$ = 80 km/h represents the velocity of a westbound truck, and $\vec{b}$ = 40 km/h represents the velocity of a westbound bus. For each situation, state and draw the velocity vector.

**a)** the velocity of the car as it appears relative to the driver of the truck

**b)** the velocity of the bus as it appears relative to the driver of the truck

**c)** the velocity of the bus as it appears relative to the driver of the car

Assume




$\vec{c} = 60$ km/h [E] $= 60(\cos 0°, \sin 0°) = (60,0)$

$\vec{t} = 80$ km/h [W] $= 80(\cos 180°, \sin 180°) = (-80,0)$

$\vec{b} = 40$ km/h [W] $= 40(\cos 180°, \sin 180°) = (-40,0)$

(a) $\vec{V}_{ct} = \vec{c} - \vec{t}$

in truck looking at car.

$= (60,0) - (-80,0)$

$= (140,0)$

$= 140$ km/h [E]


(b) $\vec{V}_{bt} = \vec{b} - \vec{t}$

$= (-40,0) - (-80,0)$

$= (40,0)$

$= 40$ km/h [E]

(c) $\vec{V}_{bc} = \vec{b} - \vec{c}$

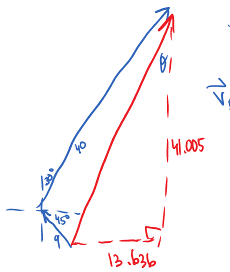
$= (-40,0) - (60,0)$

$= (-100,0)$

$= 100$ km/h [W]

$= (-100, 0)$
$= 100\text{ km/h } [W]$

(11.) A boat on a lake is heading at a bearing of 30°. The boat is travelling at a velocity of 40 km/h and the current is flowing from the southeast at 9 km/h. Determine the heading of the boat and the resultant velocity relative to the ground




$\vec{v}_{BW} = (40\text{ km/h}, 60°) = 40(\cos 60°, \sin 60°) = (20, 20\sqrt{3})$

$\vec{v}_{WG} = (9\text{ km/h}, 135°) = 9(\cos 135°, \sin 135°) = \left(-\frac{9}{2}\sqrt{2}, \frac{9\sqrt{2}}{2}\right)$

$\vec{v}_{BG} = \vec{v}_{BW} + \vec{v}_{WG}$
$= \left(20 - \frac{9}{2}\sqrt{2},\ 20\sqrt{3} + \frac{9}{2}\sqrt{2}\right)$
$\doteq (13.636, 41.005)$

$\theta = \tan^{-1}\left(\frac{13.636}{41.005}\right) = 18.394°$

$|\vec{v}_{BG}| = \sqrt{41.005^2 + 13.636^2}$
$\doteq 43.213$

$\therefore \vec{v}_{BG} = 43.2\text{ km/h } [N\ 18.4°E]$

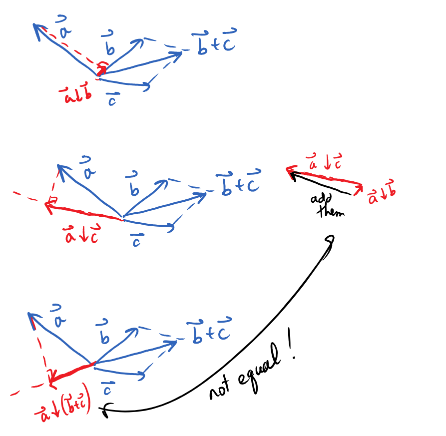
(12.) Does $\vec{a} \downarrow (\vec{b} + \vec{c}) = \vec{a} \downarrow \vec{b} + \vec{a} \downarrow \vec{c}$ ? Prove your answer.

$\frac{\vec{a} \cdot (\vec{b} + \vec{c})}{|\vec{b} + \vec{c}|^2} (\vec{b} + \vec{c})$

$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} + \frac{\vec{a} \cdot \vec{c}}{|\vec{c}|^2} \vec{c}$

not look like can make equal

try counterexample with pics




(13.) Suppose that $|\vec{c}| = 8$, $|\vec{d}| = 3$, and the angle between these two vectors is $\theta = 60°$.

**a.** Find $\vec{c} \cdot \vec{d}$.

**b.** Determine the dot product of the vectors $\vec{c} - \vec{d}$ and $\vec{c} + 2\vec{d}$.

**c.** What value of $k$ would cause the vectors $\vec{c} - \vec{d}$ and $\vec{c} + k\vec{d}$ to be perpendicular?

(a) $\vec{c} \cdot \vec{d} = |\vec{c}||\vec{d}|\cos\theta$
$= (8)(3)\cos 60°$

(b) $(\vec{c} - \vec{d}) \cdot (\vec{c} + 2\vec{d})$
$= \vec{c} \cdot \vec{c} + \underbrace{2\vec{c} \cdot \vec{d} - \vec{d} \cdot \vec{c}}_{\text{like terms since dot prod is commutative}} - 2\vec{d} \cdot \vec{d}$

ⓐ $\vec{c}\cdot\vec{d} = |\vec{c}||\vec{d}|\cos\theta$
$= (8)(3)\cos 60^\circ$
$= 12$

$= \vec{c}\cdot\vec{c} + \underbrace{2\vec{c}\cdot\vec{d} - \vec{d}\cdot\vec{c}}_{\text{like terms since dot prod is commutative}} - 2\vec{d}\cdot\vec{d}$
$= |\vec{c}|^2 + \vec{c}\cdot\vec{d} - 2|\vec{d}|^2$
$= 8^2 + 12 - 2(3)^2$
$= 58$

ⓒ $(\vec{c} - \vec{d})\cdot(\vec{c} + k\vec{d}) = 0$ for $\perp$

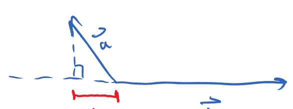
$\vec{c}\cdot\vec{c} + k\vec{c}\cdot\vec{d} - \vec{d}\cdot\vec{c} - k\vec{d}\cdot\vec{d} = 0$
$|\vec{c}|^2 + (k-1)\vec{c}\cdot\vec{d} - k|\vec{d}|^2 = 0$
$8^2 + (k-1)(12) - k(3)^2 = 0$
$64 + 12k - 12 - 9k = 0$
$3k = -52$
$k = -\frac{52}{3}$

⑭ Consider any two vectors in $R^3$, $\vec{a}$ and $\vec{b}$.

a. Describe in your own words what the scalar projections of $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$ represent geometrically. Draw these projections in your diagram. What does it mean if this scalar projection is positive? What does it mean if this scalar projection is negative? What does it mean if this scalar projection is zero?

b. Describe in your own words what the vector projections of $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$ represent geometrically.

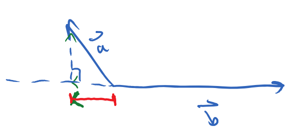
ⓐ $|\vec{a}\downarrow\vec{b}|$ is the length of the "shadow that vector $\vec{a}$ has on line upon which vector $\vec{b}$ is located
"floor"




* value will be negative for obtuse angles
* positive for acute angles
* zero for 90° (no shadow)

→ locate the "floor" extend if needed (ignore $\vec{b}$'s direction or magnitude)
→ drop a perpendicular from head of $\vec{a}$ to the "floor"
→ connect tail of $\vec{a}$ to the point where the perpendicular meets the "floor"

ⓑ $\vec{a}\downarrow\vec{b}$ : vector projections
add an arrow onto the "shadow" → in direction of $\vec{b}$ except maybe opposite if angle is obtuse

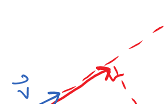



OR → break vector $\vec{a}$ into components $\parallel$ and $\perp$ to the floor, then take only the vector component on the floor as the vector projection.

for $\vec{b}\downarrow\vec{a}$

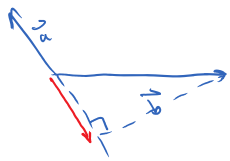
obtuse angle

acute angle

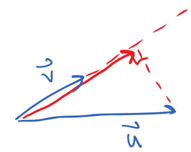


for $\vec{b} \downarrow \vec{a}$

obtuse angle




acute angle




$\vec{u} \downarrow \vec{v}$ (in red)

15. For each of the following computations, state whether the result will be a scalar, a vector, or if the computation is meaningless.

a. $|\vec{a} \times \vec{b}|\vec{c}$ = scalar $\vec{c}$ = vector

b. $((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}$ = $(\text{vector} \times \vec{c}) \cdot \vec{d}$ = vector $\cdot \vec{d}$ = scalar

c. $\vec{a} \times \vec{b} + \vec{a} \cdot \vec{b}$ = vector + scalar = ?? meaningless

d. $(\vec{a} \cdot (\vec{b} \times \vec{c})) \cdot \vec{d}$ = $(\vec{a} \cdot \text{vector}) \cdot \vec{d}$ = scalar $\cdot$ vector = ?? meaningless

e. $(((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d})|\vec{a} \times \vec{b}|\vec{c}$ = $(\text{vector} \times \vec{c}) \cdot \vec{d}$ scalar $\vec{c}$ = $(\text{vector} \cdot \vec{d})$vector = (scalar)(vector) = vector

f. $(((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}) - |\vec{a} \times \vec{b}|\vec{c}$ = $(\text{vector} \times \vec{c}) \cdot \vec{d}$ − scalar $\vec{c}$
= (vector $\cdot \vec{d}$) − vector
= scalar − vector = ?? meaningless

16. Consider the following vectors $\vec{a} = 5\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} - 4\vec{k}$ and $\vec{c} = -\vec{i} + 6\vec{j}$. Compute the following

$\vec{a} = (5,-1,2)$ $\vec{b} = (1,0,-4)$ $\vec{c} = (-1,6,0)$

a) $\vec{a} \bullet \vec{b}$
b) $(2\vec{c} + \vec{b}) \times \vec{a}$
c) $\vec{b} \downarrow \vec{a}$
d) $\vec{b} \times \vec{c}$
e) $(\vec{a} - 3\vec{b}) \bullet \vec{c}$
f) $|\vec{a} \downarrow \vec{c}|$
g) Find angle between vectors $\vec{a}$ and $\vec{b}$
h) If a triangle is defined by vectors $\vec{b}$ and $\vec{c}$, find the area of this triangle.

(a) $\vec{a} \cdot \vec{b} = 5(1) + -1(0) + 2(-4) = -3$

(b) $[2(-1,6,0) + (1,0,-4)] \times \vec{a}$
$= (-2+1, 12+0, 0-4) \times (5,-1,2)$
$= (-1, 12, -4) \times (5,-1,2)$
$= (24-4, -20+2, 1-60) = (20, -18, -59)$

$\begin{matrix} -1 & 12 & -4 & -1 & 12 \\ 5 & -1 & 2 & 5 & -1 \end{matrix}$

(c) $\vec{b} \downarrow \vec{a} = \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a}$
$= \dfrac{-3}{(\sqrt{5^2+1^2+2^2})^2}(5,-1,2)$
$= \dfrac{-3}{30}(5,-1,2)$
$= \left(-\frac{1}{2}, \frac{1}{10}, -\frac{1}{5}\right)$

(d) $\begin{matrix} 1 & 0 & -4 & 1 & 0 \\ -1 & 6 & 0 & -1 & 6 \end{matrix}$

$\vec{b} \times \vec{c} = (0+24, +4-0, 6-0)$
$= (24, 4, 6)$

(e) $[(5,-1,2) - 3(1,0,-4)] \cdot \vec{c}$
$= (5-3, -1-0, 2+12) \cdot (-1,6,0)$
$= (2,-1,14) \cdot (-1,6,0)$
$= 2(-1) + -1(6) + 14(0)$
$= -8$

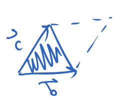
(f) $|\vec{a} \downarrow \vec{c}| = \dfrac{\vec{a} \cdot \vec{c}}{|\vec{c}|}$
$= \dfrac{(5,-1,2) \cdot (-1,6,0)}{\sqrt{1^2+6^2+0^2}}$
$= \dfrac{5(-1) + -1(6) + 2(0)}{\sqrt{37}}$
$= \dfrac{-11}{\sqrt{37}}$

(g) $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{-3}{\sqrt{30}\sqrt{1^2+0^2+4^2}}$

$\cos\theta = \dfrac{-3}{\sqrt{510}}$

$\theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{510}}\right) \doteq 98°$

(h) 


$A_\triangle = \frac{1}{2} A_\square$
$= \frac{1}{2}|\vec{b} \times \vec{c}|$
$= \frac{1}{2}|(24,4,6)|$
$= \frac{1}{2}\sqrt{24^2+4^2+6^2} = \frac{1}{2}\sqrt{628} = \sqrt{157}$

17. Find work done by the force $\vec{F} = (-2,3,-5)$ when an object is moved from point $A(0,1,4)$ to point $B(-3,0,5)$

$\vec{d} = \vec{AB} = (-3-0, 0-1, 5-4)$
$= (-3,-1,1)$

$W = \vec{d} \cdot \vec{F}$
$= (-3,-1,1) \cdot (-2,3,-5)$
$= -3(-2) + -1(3) + 1(-5)$

$= -3(-2) + -1(3) + 1(-5)$
$= 6 - 3 - 5$
$= -2$
or 2 Joules

negative work since $\vec{F}$ and $\vec{d}$ at obtuse angles


(18.) Find a unit vector perpendicular to both $\vec{a} = (2, -3, 0)$ and $\vec{b} = (0, 1, -2)$

$\perp$ to both $\leadsto$ use cross product for normal $\vec{n}$
$\leadsto$ convert to unit vector.

$$\begin{matrix} 2 & -3 & 0 & 2 & -3 & \cancel{0} \\ \cancel{0} & 1 & -2 & 0 & 1 & \cancel{-2} \end{matrix}$$

$(6-0, 0+4, 2-0)$

$\therefore \vec{n} = \pm(6, 4, 2)$

can switch order of cross prod.

$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\pm(6,4,2)}{\sqrt{6^2+4^2+2^2}} = \pm\frac{(6,4,2)}{2\sqrt{14}} = \pm\left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

are the 2 vectors $\perp$ to both $\vec{a}$ and $\vec{b}$

(19.) Prove the following relation involving vectors:

$\vec{a} \cdot \vec{b} = \frac{1}{4}|\vec{a}+\vec{b}|^2 - \frac{1}{4}|\vec{a}-\vec{b}|^2$

$\frac{1}{4}\left[(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) - (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})\right]$

$\frac{1}{4}\left[(\vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b})\right]$

$\frac{1}{4}\left[\cancel{|\vec{a}|^2} + 2\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2} - \cancel{|\vec{a}|^2} + 2\vec{a}\cdot\vec{b} - \cancel{|\vec{b}|^2}\right]$

$\frac{1}{4}\left[4(\vec{a}\cdot\vec{b})\right]$

$\vec{a}\cdot\vec{b}$

$\therefore$ LS = RS

(20.) If $\vec{a}$ and $\vec{b}$ are unit vectors, and $\|\vec{a}-\vec{b}\| = \sqrt{2}$, determine $(2\vec{a}-3\vec{b})\cdot(\vec{a}+2\vec{b})$

$|\vec{a}-\vec{b}|^2 = 2$

$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = 2$

$\vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = 2$

$|\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = 2$

$1 - 2\vec{a}\cdot\vec{b} + 1 = 2$

$0 = \vec{a}\cdot\vec{b}$

use this there.

$2\vec{a}\cdot\vec{a} + 4\vec{a}\cdot\vec{b} - 3\underbrace{\vec{a}\cdot\vec{b}}_{\text{ok to switch commutative}} - 6\vec{b}\cdot\vec{b}$

$= 2|\vec{a}|^2 + \vec{a}\cdot\vec{b} - 6|\vec{b}|^2$

$= 2(1) + \emptyset - 6(1)$

$= -4$

(21.) If $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ prove that $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)$

same as proving $-2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{c}) = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c}$

$\vec{a}\cdot(-\vec{b}-\vec{c}) + \vec{b}\cdot(-\vec{a}-\vec{c}) + \vec{c}\cdot(-\vec{a}-\vec{b})$

$-\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$

$-2\vec{a}\cdot\vec{b} - 2\vec{a}\cdot\vec{c} - 2\vec{b}\cdot\vec{c}$

$\therefore$ LS = RS

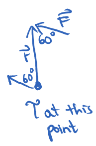
(22.) A special bicycle computer reports the magnitude of

22. A special bicycle computer reports the magnitude of the torque exerted by a cyclist during a strenuous portion of the ride to be $\frac{600}{\sqrt{3}}$ J at the top of his pedal-stroke. If the force exerted by the cyclist at this point is in a direction that is 60° relative to the pedal, and the pedals are 25 cm in length, how much force, in N, must the cyclist be exerting at this moment?

$|\vec{\tau}| = \frac{600}{\sqrt{3}}$ $\quad |\vec{\tau}| = |\vec{r} \times \vec{F}|$

$|\vec{F}| = ?$ $\quad |\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta$

$|\vec{r}| = 0.25\text{m}$ $\quad \frac{600}{\sqrt{3}} = (0.25)|\vec{F}| \sin 60°$

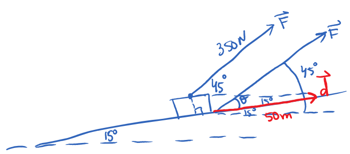



$\frac{600}{\sqrt{3}} = |\vec{F}| \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$

$\frac{(2)(600)(4)}{\sqrt{3}\sqrt{3}} = |\vec{F}|$

$1600 = |\vec{F}|$ $\quad \therefore$ exert 1600 N of force

23. Suppose a box on a frictionless ramp is being pulled by a rope with a tension of 350 N making an angle of 45° to the horizontal ground. If the angle of incline of the ramp is 15°, and the box is pulled 50 m, determine the amount of work done.



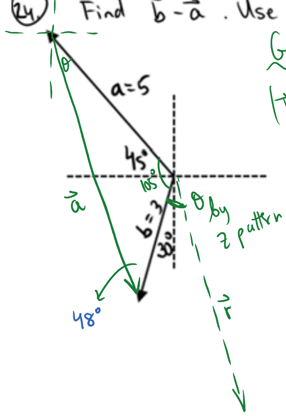

$\therefore \theta = 30°$

$W = \vec{d} \cdot \vec{F}$

$= |\vec{d}||\vec{F}| \cos\theta$

$= (50)(350)\cos 30° \doteq 15155.4$ Joules of Work

24. Find $\vec{b} - \vec{a} = \vec{r}$. Use both methods




Geometric Way

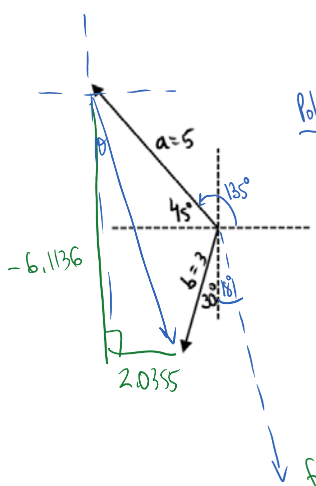
$|\vec{b}-\vec{a}|^2 = 5^2 + 3^2 - 2(5)(3)\cos 105°$

$|\vec{b}-\vec{a}| \doteq 6.46255$

$\frac{\sin\theta}{3} = \frac{\sin 105°}{6.46255}$

$\theta \doteq 26.64077°$

$|\vec{b}-\vec{a}| = 6.5$, $\angle(\vec{r}, \vec{a}) = 153°$, $\angle(\vec{r}, \vec{b}) = 48°$




Polar Form / Algebraic Way:

Assume origin placed at tails $\quad \vec{a} = (5, 135°) \quad \vec{b} = (3, 240°)$ polar forms

$\vec{a} = 5(\cos 135°, \sin 135°) = 5\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(-\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}\right)$

$\vec{b} = 3(\cos 240°, \sin 240°) = 3\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(-\frac{3}{2}, -\frac{3}{2}\sqrt{3}\right)$

$\therefore \vec{b} - \vec{a} = \left(\frac{-3+5\sqrt{2}}{2}, \frac{-3\sqrt{3}-5\sqrt{2}}{2}\right) \doteq (2.0355, -6.1336)$ algebraic forms

$\theta = \tan^{-1}\left(\frac{2.0355}{6.1336}\right) \doteq 18°$

$\vec{b} - \vec{a} = (6.462, 288.359°)$

then convert to geo form since original question didn't have a coordinate system defined.

$|\vec{b}-\vec{a}| = 6.5$, $\angle(\vec{r}, \vec{a}) = 153°$, $\angle(\vec{r}, \vec{b}) = 48°$

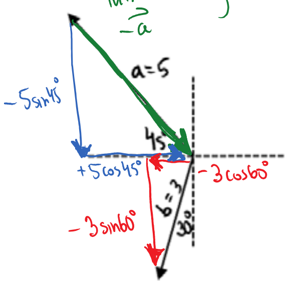
for θ use final components see picture in green

$|\vec{b}-\vec{a}| = \sqrt{2.0355^2 + 6.1336^2}$

components
see picture
in green

$$|\vec{b}-\vec{a}| = \sqrt{2.0355^2+6.1336^2}$$
$$\doteq 6.462$$

Think "add neg. vector" $-\vec{a}$




Physics Way: (must worry about signs!)

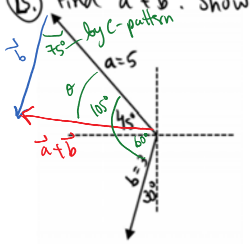
Assume pos. directions are ↑+ →

| x components | y components |
| --- | --- |
| $a_x = 5\cos45°$ | $a_y = -5\sin45°$ |
| $b_x = -3\cos60°$ | $b_y = -3\sin60°$ |

neg. $\vec{a}$ components → since add $-\vec{a}$ is like subtract $\vec{a}$

$\vec{b}-\vec{a} = (2.0355, -6.1336)$ like before :)

(15.) Find $\vec{a}+\vec{b} = \vec{r}$. Show both methods.




Geometric Way:

$|\vec{a}+\vec{b}|^2 = 3^2+5^2-2(3)(5)\cos75°$

$|\vec{a}+\vec{b}| \doteq 5.122$

$$\frac{\sin\theta}{3} = \frac{\sin75°}{5.122}$$

$\theta \doteq 34.5°$

$\therefore |\vec{a}+\vec{b}| = 5.122 \quad \angle(\vec{r},\vec{a}) = 34.5° \quad \angle(\vec{r},\vec{b}) = 70.5°$

Algebraic Way:

$\vec{a} = (5, 135°) = 5(\cos135°, \sin135°)$

$\vec{b} = (3, 240°) = 3(\cos240°, \sin240°)$

$\vec{a}+\vec{b} = (5\cos135°+3\cos240°,\ 5\sin135+3\sin240°)$

$= (-5.036, 0.937)$


convert: $\beta = \tan^{-1}\left(\frac{0.937}{5.036}\right)$

$\beta \doteq 10.5°$

$|\vec{a}+\vec{b}| = \sqrt{5.036^2+0.937^2}$

$\doteq 5.122$

$\therefore |\vec{a}+\vec{b}| = 5.122,\ \angle(\vec{r},\vec{a}) = 34.5°,\ \angle(\vec{r},\vec{b}) = 70.5°$

$45°-10.5° \qquad 60+10.5°$