Big idea

In primary school, we learned how to find areas of shapes with straight sides (e.g. area of a triangle or rectangle). But how do you find areas when the sides are curved? How do you use integration to find the volume of an object with curved sides, e.g. wine barrels. At this point I hope you are comfortable with curve sketching of quadratics, exponentials, logs and conics.

Feedback & Assessment of Your Success

Date	Pages	Topics	**Finished assignment pages?** **Made corrections?**	**Summarized notes in a journal?** **Added your own explanations?**	**How many extra practice questions did you try in each topic?**	**Tentative TEST date:** ____________________ **Questions to ask the teacher:**
2.5days	2-7	U-Substitution Method (AB) Journal #1				
	8-9	Area between Curves (AB) Journal #2				
2days	10-14	Volumes (AB) Journal #3				
1.5days	15-18	Arc Length & Area of Surface of (BC) Journal #4				
2days	19-24	Applications (BC) **NO** Journal				

ASSIGNMENT U-Substitution Method (AB)

1. Evaluate and compare:

$$\frac{d}{dx}\left[\tan\left(e^{4x^2}\right)\right]$$

$$\int 8x\cdot e^{4x^2}\cdot\sec^2\left(e^{4x^2}\right)dx$$

Chain rule DERIVATIVE RULE integrated:

U-Substitution (Change of variable):

If $u = g(x)$ is a differentiable function whose range is an interval *I*, and *f* is continuous on *I*, then

$$\int f'\left(g(x)\right)g'(x)dx = \int f'(u)du$$

STEPS:

1) Let u=(inside quantity) usually
2) Find du=______dx
3) Change variables; Balance constants
4) Integrate with respect to u
5) Back substitute to original variable

2. $\int\sqrt{3x-1}\,dx$

3. $\int \sin(8x)dx$

4. $\int 6x^7\left(5x^8+3\right)^9 dx$

5. $\int 2e^x\sqrt{6e^x+1}dx$

6. **Definite integrals with a change of variables.** $\int_{-1}^{1} x^2(x^3+2)^2\,dx$

$$\int_a^b f\left(g(x)\right)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

- ❖ Deal with the indefinite integral for convenience, and then return to the definite integral after we've finished integrating.
- ❖ Change the limits of integration over to u, then use the u integral.

MEMORIZE! OR develop from scratch – no formulas provided on AP exam

7. (a) $\int \sin x dx$ (b) $\int \cos x dx$ (c) $\int \sec^2 x dx$ (d) $\int \csc^2 x dx$ (e) $\int \sec x \cdot \tan x dx$ (f) $\int \csc x \cdot \cot x dx$

8. (a) $\int \tan x dx$ (b) $\int \cot x dx$

9. (c) $\int \sec x dx$ (d) $\int \csc x dx$

10. $\int \sin^2 x dx$

11. $\int \sin^3 x dx$

12. Derivatives

a) $\frac{d}{dx}[\arcsin x] =$ b) $\frac{d}{dx}[\arctan x] =$ c) $\frac{d}{dx}[\operatorname{arc}\sec x] =$

Note: arccos, arccot, arccsc are negatives of above so do not need them for antiderivatives.

d) Recall how to develop these from scratch:

13. Now do integrals with constants:

a) $\int \frac{du}{\sqrt{a^2 - u^2}}$

b) $\int \frac{du}{a^2 + u^2}$

c) $\int \frac{du}{|u|\sqrt{u^2 - a^2}}$

14. Compare and Contrast

(a) $\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\int \frac{x}{\sqrt{4-x^2}}dx$ (c) $\int \frac{dx}{2+9x^2}$ (d) $\int \frac{x}{2+9x^2}dx$

15. Two of the Three integrals you can do. Evaluate the two you can do, and understand why you cannot do the third one.

(a) $\int \frac{3x^3}{\sqrt{5-x^4}}dx$ (b) $\int \frac{3x^2}{\sqrt{5-x^4}}dx$ (c) $\int \frac{3x}{\sqrt{5-x^4}}dx$

16. Factoring or Completing Square Strategy

(a) $\int \frac{3}{x^2+6x+9}dx$ (b) $\int \frac{3}{x^2+6x+10}dx$ (c) $\int \frac{3}{x^2+6x+8}dx$

17. Long Division Strategy

(a) $\int \frac{x^2+x+1}{x^2+1}dx$

(b) $\int \frac{x^3-3x^2+4x-9}{x^2+3}dx$

Other tricks

18. Distribute denominator

$\int \frac{x+2}{\sqrt{4-x^2}}dx$

19. Do implicit derivative of "let u=" statement

$\int x\sqrt{2x-1}dx$

20. Not quite arcsecant…

$$\int \frac{dx}{\sqrt{e^{2x}-1}}$$

21. Recognize 'inside' for u when hidden

$$\int \frac{e^{2x}}{16+e^{4x}}dx$$

Try yourself

22. $$\int_0^{\sqrt{\pi}} x\sin x^2 dx$$

23. $$\int_1^2 \frac{1}{3+(x-2)^2}dx$$

24. $$\int \frac{x}{\pi\left(x^2+1\right)^3}dx$$

25. $$\int \frac{e^x}{1+e^x}dx$$

26. $$\int \frac{2x+3}{x^2+3x}dx$$

27. $$\int \frac{\sec^2 x}{14+\tan x}dx$$

28. $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$

29. $\int 7\sin 2x \sqrt[3]{\cos^2 2x}\, dx$

30. $\int \frac{dx}{\sqrt{4x - x^2}}$

31. $\int \frac{x^2}{\sqrt{2x-1}} dx$

32. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

33. $\int \frac{3x^2 - 7x}{3x+2} dx$

34. $\int \frac{dx}{1+\sqrt{2x}}$

35. $\int \frac{dx}{x^2 + 2x + 3}$

ASSIGNMENT Area Between Curves (AB)

1. Find the area bounded by the curves $y=\sqrt{x}$ and $y=1$, and between the lines $x=1$, and $x=7$.
Using VERTICAL slices

Using HORIZONTAL slices

2. Find the area enclosed by $y=5-2x$ and $y=4x-x^2$.

3. Find the area enclosed by $y=x$ and $y=x^3$.

4. Find the area of the region bounded by the graphs of $x+4y^2=4$ and $x+y^4=1$ for $x\ge 0$

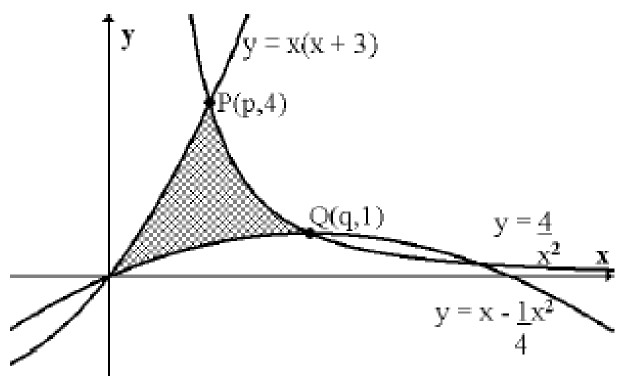
5. Find the area of the region bounded by the graphs of $x=3\sin y\sqrt{\cos y}$ and $x=0$, $0\le y\le\frac{\pi}{2}$

6. The diagram opposite shows an area enclosed by 3 curves:

$y=x(x+3)$, $y=\frac{4}{x^2}$ and $y=x-\frac{1}{4}x^2$

(a) P and Q have coordinates (p,4) and (q,1). Find the values of p and q.

(b) Calculate the shaded area.



ASSIGNMENT Volumes (AB)

1. Using the Disk/Washer method find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

2. Using the Disk/Washer method, find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines y = 1, x = 4 about the line y = 1. Would the answer change if it's rotated about the line y = -1?

3. Repeat above question with the SHELL method

4. Repeat the above question with the SHELL method

5. The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid using the Disk/Washer method.

6. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid using the Disk/Washer method.

7. Repeat above question with the SHELL method

8. Why is SHELL method impossible for the above question?

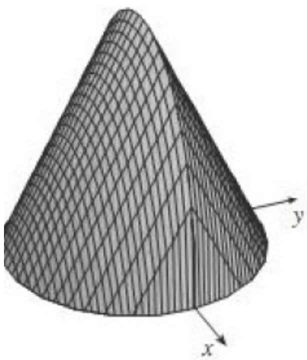
9. Decide on the best method to find volume generated by rotating $x = 12(y^2 - y^3)$ around the x-axis for $0 \le y \le 1$

10. Decide on the best method to find volume generated by rotating $x = 12(y^2 - y^3)$ around the y-axis for $0 \le y \le 1$

Note: expect a different answer than 9. Since this will be a different shape.

11. The region enclosed by the x-axis and the parabola $f(x) = 3x - x^2$ is revolved about the line x = – 1 to generate a solid of revolution resembling a Bundt cake. What is the volume of the cake? What would happen if the graph was rotated about the line x = 4 instead?

12.

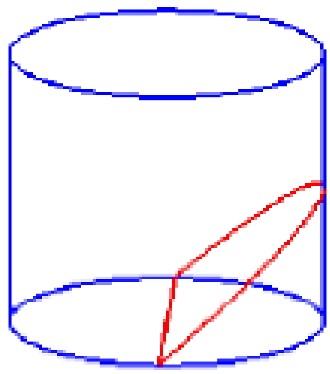


The figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

13.

A region R, defined by the intersections of the graphs of $y = 5x$, $y = \frac{-x}{5} + 3$, and $y = 0$, is the base of the solid. For this solid, at each y, the cross section perpendicular to the y-axis has area $A(y) = y^2 + 1$. Find the volume of the solid.

14. The Department of Public Works intends to cut down a diseased elm tree one meter in diameter. They first cut out a wedge bounded below by a horizontal plane and bounded above by a plane that meets the horizontal plane at a 45° angle. What is the volume of the wedge if the planes meet along a diameter of the tree? Show two different methods (triangle slices and rectangle slices)



ASSIGNMENT Arc Length & Area of Surface of Revolution (BC) – OPTIONAL

1. **The Arc Length Formula:** If f' is continuous on [*a*, *b*], then the length of the curve $y = f(x),\ a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Explanation of the formula

2. Find the length of the curve $y = \frac{x^3}{12} + \frac{1}{x},\ \ 1 \le x \le 4$ using the above formula

3. Find the length of the curve $y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1,\ \ 0 \le x \le 1$

4. Because the Arc Length formula relies on the derivative of a function as part of the integrand, any time the derivative of a function **fails to exist** in our interval of integration (vertical tangent, cusp, discontinuity), we need to be very careful.

Find the length of the curve

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}}, \quad 0 \le x \le 2$$

5. Find the length of the curve

$$y = \left(\frac{3}{2}x\right)^{\frac{2}{3}} + 1, \quad 0 \le x \le 18$$

6. I'm walking along a path defined by $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$ miles. At the end of my stroll, how far have I travelled? Give an exact and approximate answer.

7. If *f* is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the *x*-axis as

$$SA = \int_a^b 2\pi\, f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$SA = \int_a^b 2\pi\, y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

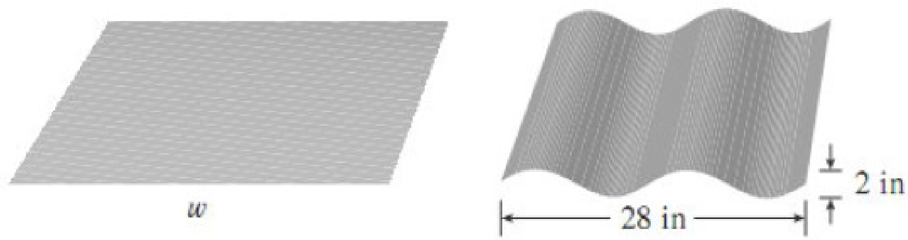
Explanation of formula

8. Find the area of the surface generated by revolving the curve $y = \sqrt{x}$, $1 \le x \le 2$, about the x-axis.

9. The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone. Find its lateral surface area (which excludes the base area)

(Calculator Permitted)

10. A manufacturer of corrugated panel roofing wants to produce "C" panels that are 28 inches wide and 2 inches thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape the sine wave $y = \sin\left(\frac{\pi}{7}x\right)$. Find the width w of a flat metal sheet that is needed to make a 28-inch panel. Assume the process does not stretch the material. Round to three decimals.



a) Find width w

b) Find surface area if the length of the sheet is 8 ft

11. (Calculator Permitted) Find the length of the curve $y = x^2 - 4|x| - x$, $-4 \le x \le 4$

ASSIGNMENT PHYSICS Applications (BC) - OPTIONAL

Work done by a CONSTANT force:
When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F *in the direction of motion*, we define the work W (measured in Newton-meters or Joules) done by the force on the body with the formula

$$W = Fd$$

Newton's Second Law of Motion: The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force (measured in kg m/s^2 or Newtons)

$$F = ma$$

Work done by a VARIABLE force:

$$W = \int_a^b F(x)dx$$

where $F(x)$ is in the direction of motion along the x-axis from $x = a$ to $x = b$

Hooke's Law for Springs: the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x .

$$F = kx$$

The constant k , measured in force units per unit length, is a characteristic of the spring, called the force constant (or spring constant) of the spring

Which of the following problems represents Work done by a Constant Force, and which by a Variable Force? Solve both questions

1. a) How much work is done in lifting a mass of 1.2 kg a distance of 12 meters?

b) A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed. The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?

2. Find the work required to compress a spring from its natural length of 1 m to a length of 0.75 m if the force constant is k = 16 N /m.

3. A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.
a) Find the force constant k.
b) How much work will it take to stretch the spring 2 m beyond its natural length?
c) How far will a 45-N force stretch the spring?

The **mass** m of fluid is density times volume

$$m = \rho V$$

For water

$$\rho g = (1000\frac{kg}{m^3})(9.8\frac{m}{s^2}) = 9800\frac{N}{m^3} \text{ or } 62.5\frac{\text{lb}}{ft^3}$$

In a fluid that is standing still, the **pressure** P :

$$P = \rho g h$$

Where ρ (rho) is density of fluid, g is acceleration of gravity, h is the height (depth) of fluid above the object.

Fluid Force on a Constant-Depth Surface:

$$\begin{aligned} F &= \text{Total Force} \\ &= [\text{Force per unit area}][\text{Area}] \\ &= [\text{Pressure}][\text{Area}] \\ &= PA \end{aligned}$$

$$F = \rho g h A$$

However pressure increases with depth that's why dams are built thicker at the bottom than the top, break up the vertical dam plate into thin horizontal slices, find force on each one, then add up (integrate)

Fluid Force against a Vertical Flat Plate:

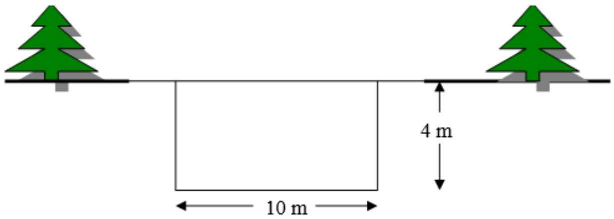
Suppose that a plate is submerged vertically in fluid of weight-density w runs from y = a to y = b on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \rho g[h][\text{Area}]$$

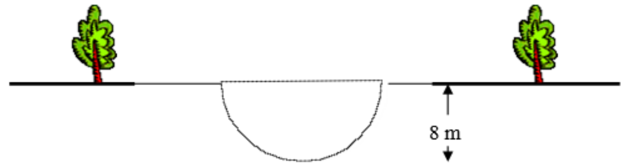
$$F = \rho g[h][\text{Length}][\text{Thickness}]$$

$$F = \rho \text{g}\int_a^b h(y)\, L(y) dy$$

4. For the dam shown, express the force as an integral, and evaluate it.



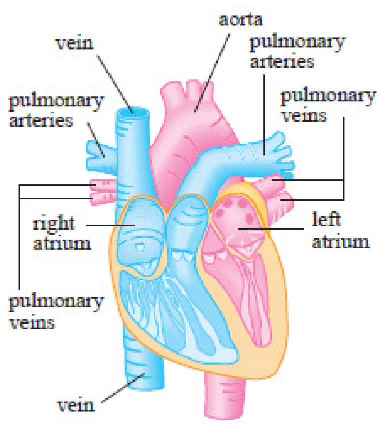
5. For the dam shown, express the force as an integral, and evaluate it.



6. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

BIOLOGY Application

The *dye dilution* method is used to measure cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval until the dye has cleared.



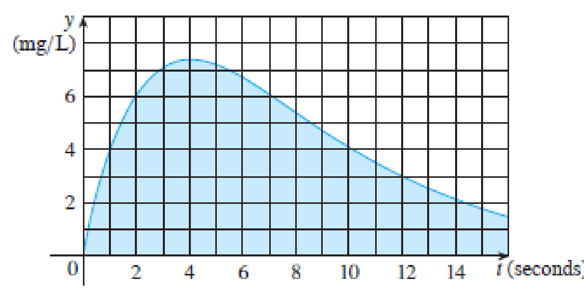
Cardiac Output: This is the volume of blood pumped by the heart per unit time.

$$F = \frac{A}{\int_0^T c(t)\,dt}$$

where F is cardiac output or rate of flow, in L/s, A is amount of dye injected, in mg, $c(t)$ is concentration of dye in blood in mg sec/L

7. If the dye-dilution method is used with 5 mg of dye, and the concentration of the dye (in mg/L) can be modelled by $c(t) = \frac{1}{3}t(9-t)$ for $0 \le t \le 9$ seconds, find the cardiac output.

8. The graph of the concentration function $c(t)$ is shown after a 7-mg injection of dye into a heart. Use Simpson's Rule to estimate the cardiac output with 8 subintervals.



ECONOMICS Application

10.

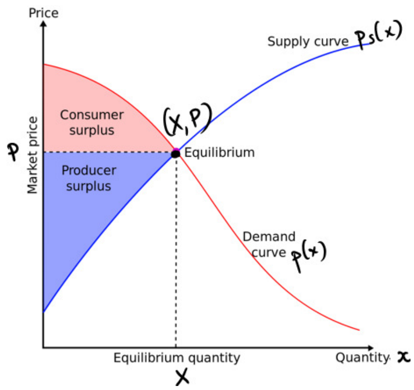
The **demand function** is the price $p(x)$ that a company has to charge in order to sell x units of a commodity. Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function.

The **consumer surplus**

$$CS = [\text{savings per unit}][\text{number of units}]$$

$$CS = \int_0^X [p(x) - P]dx$$

represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X. You can interpret the consumer surplus as the area under the demand curve and above the line $p = P$.



The **supply function** $p_s(x)$ for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, manufacturers will produce more units, so supply function is an increasing function. Let X be the amount of the commodity currently produced and let $P = p_s(x)$ be the current price. Some producers would be willing to make and sell the commodity for a lower selling price and are therefore receiving more than their minimal price. The excess is called the **producer surplus**:

$$PS = \int_0^X [P - p_s(x)]dx$$

9. The demand for a product, in dollars, is $p = 1200 - 0.2x - 0.0001x^2$ Find the consumer surplus when the sales level is 500. Illustrate by drawing the demand curve and identifying the consumer surplus as an area.

10. The **marginal cost** function $C'(t)$ is defined to be the derivative of the cost function.
The marginal cost of producing x gallons of orange juice is $C'(x) = 0.82 - 0.00003x + 0.3\times10^{-8}x^2$ (measured in dollars per gallon). The fixed start-up cost is $C(0) = \$18000$. Use the Net Change Theorem to find the cost of producing the first 4000 gallons of juice.

11. The **marginal revenue** function $R'(t)$ is defined to be the derivative of the revenue function.
A company estimates that the marginal revenue (in dollars per unit) realized by selling units x of a product is $R'(x) = 48 - 0.0012x$. Assuming the estimate is accurate, find the increase in revenue if sales increase from 5000 units to 10,000 units.

12. A movie theater has been charging \$10.00 per person and selling about 500 tickets on a typical weeknight. After surveying their customers, the theater management estimates that for every 50 cents that they lower the price, the number of movie goers will increase by 50 per night. Find the demand function and calculate the consumer surplus when the tickets are priced at \$8.00.

13. If the amount of capital that a company has at time t is $f(t)$, then the derivative, $f'(t)$, is called the **net investment flow**. Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in years). Find the increase in capital (the **capital formation**) from the fourth year to the eighth year.

Other applications include average value integral used in road safety research. Read the following
http://www.intmath.com/applications-integration/hic-head-injury-criterion.php
http://www.intmath.com/applications-integration/hic-part2.php