Calculus UNIT 4

APPLICATION of DERIVATIVES Part 2 (MCV + AP)– journal questions

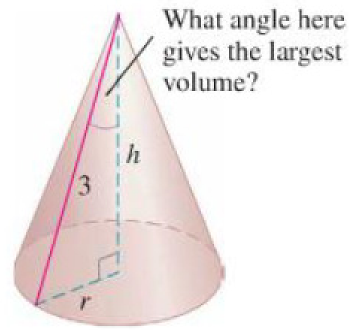
Summarize everything you need to know about these topics. Use examples and concise (not long – but with enough detail) explanations. Include definitions and diagrams if necessary

1. OPTIMIZATION (MCV)

   a. Copy/Paste the steps for solving optimization word problems:

- Understand the problem. Draw a diagram, introduce notation (variables must be defined with let statements or with the use of a diagram with labels). Figure out the variable to be maximized, and use the information in the question to express this variable in terms of one other variable.
- Find domain (in terms of independent variable)
- Find the critical points
- Find the absolute maximum or minimum of the function on its domain using one of the following:
  - 1^{st} or 2^{nd} derivative test
  - EVT on Closed Interval.
  - Using limits on open interval
- Answer the question. Ask yourself if the answer makes sense.

   b. The slant height of the cone is 3m. How large should the indicated angle be to maximize the cone's volume? Ensure to verify that the answer you give is a maximum.



2. RELATED RATES (MCV)

   a. Copy/Paste the following formulas that you may need to reference

Pythagoras	$a^2+b^2=c^2$
Trig. Formulas	SOH, CAH, TOA
Sine Law	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
Cosine Law	$c^2=a^2+b^2-2ab\cos C$
Area of triangle	$\frac{1}{2}bh$ or $\frac{1}{2}ab\sin\theta$
Circle	$A=\pi r^2$, $C=2\pi r$
Sphere	$V=\frac{4}{3}\pi r^3$, $SA=4\pi r^2$
Cylinder	$V=\pi r^2 h$, $SA=2\pi r^2+2\pi rh$
Cone	$V=\frac{1}{3}\pi r^2 h$, $SA=\pi r\sqrt{r^2+h^2}$
Prisms	$V=(Area\ of\ base)\times(Distance\ between\ bases)$
Similar Triangles	Ratios of corresponding sides are equal.

- Read the problem to determine what is given and what is required. (Pay attention to units "per" for rate of change) Introduce notation. If possible, draw a diagram, indicate with arrows what is changing with respect to time?
- Write an equation relating the variables in the problem. There can be more than one equation needed.
- Differentiate both sides of the equation with respect to time. (Ask yourself if the quantity actually changes with time or if it's constant).
- Substitute in the given info AFTER taking the derivative and solve for the unknown rate of change. Use another equation if there are too many unknowns still.
- Answer the question, include units

   b. Review the importance of Leibniz notation and find the following derivatives for $a^2+3a=b^3-ab^2$ (no need to simplify)

      i. $\frac{d}{da}\left[a^2+3a=b^3-ab^2\right]$ assume b is constant

      ii. $\frac{d}{db}\left[a^2+3a=b^3-ab^2\right]$ assume a is constant

      iii. $\frac{d}{dx}\left[a^2+3a=b^3-ab^2\right]$ if $a(x)$ and $b(x)$ ie. both a and b are functions of x.

   c. Water flows at 8cm^3/min into a cone with a radius of 4cm and height of 12cm. How fast is the water level rising when the water is 2cm deep?

3. L'HOPITAL's RULE (AP - AB)

a. Copy/Paste the following rule

L'Hospital's Rule:

Suppose *f* and *g* are differentiable and $g'(x) \neq 0$ near *a* (except possibly at *a*). If you get the indeterminate forms of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ if the limit on the right side exists or even if it approaches ∞ or $-\infty$

Use L'Hopital's Rule to find the limits to these indeterminate forms. Ensure it's clear when L'H rule is applied.

$\frac{\infty}{\infty}$ b. $\lim_{x\to\infty}\frac{3^x}{x^2+x-1}$

$\frac{0}{0}$ c. $\lim_{x\to 0}\frac{5x-\tan 5x}{x^3}$

$0\cdot\infty$ d. $\lim_{x\to 0+}(\sin x \ln x)$

$\infty-\infty$ e. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

∞^0 f. $\lim_{x\to\pi/2}(\tan x)^{2x-\pi}$

0^0 g. $\lim_{x\to 0+}(\tan 5x)^x$

1^∞ h. Prove $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

4. APPROXIMATIONS (AP - AB)

a. Copy/Paste the following

Draw any curve and a tangent at one of the points.

What do you notice about the function and the tangent near the point of tangency?

The **linear approximation** or **tangent line approximation** of $f(x)$ at the point $x = a$ is

$f(x) \approx f(a) + f'(a)(x-a)$.

The linear function whose graph is this tangent line, i.e.,

$L = f(a) + f'(a)(x-a)$

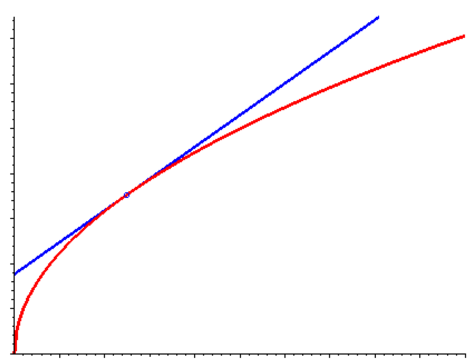
is called the **linearization** of $f(x)$ at the point $x = a$.

Show the reasoning behind this formula:

Instead of linearization one can use differentials to find approximations. Recall that $f'(x) = \frac{dy}{dx}$

We call dx and dy **differentials,** and they are related through $dy = f'(x)dx$.

Show differentials geometrically:



b. If a population is growing at a rate $\ln(x^2+1)$ insects/week, where x is the time in weeks since the beginning of March, and P(4)=100, estimate the population of insects after 6 weeks and determine if the actual population would be more or less than your estimate. Do the calculations in both Linearization and Differentials methods.

5.

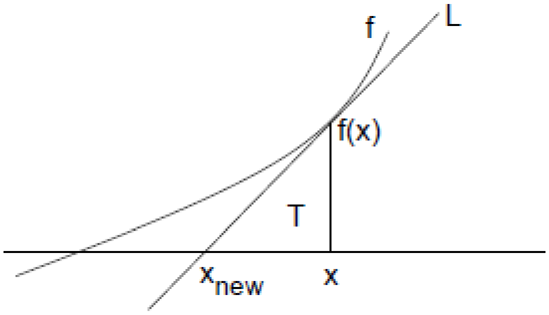
a. NEWTON'S METHOD for finding zeros (AP - BC)

As you already have seen not all equations are solvable using algebraic methods. **Newton's method** is an approximation method for finding a root of $f(x)=0$ under ideal conditions, where x_1 is your initial guess

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Explain where formula comes from:

Show Newton's Method geometrically:



b. Sometimes Newton's Method fails, can you think of when?

c. Find solution to the following using Newton's Method accurate to 5 decimal places.
Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line y = 1.

d. TAYLOR Series to approximate any function with an infinite polynomial. Copy/Paste the following (AP - BC)

WHY?
These sorts of approximations are especially useful for functions that you may have trouble finding the antiderivative or solutions to differential equations (since some of these cannot be found by direct methods but only by approximations).

Taylor series

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

If $a=0$ then Taylor's series is called **Maclaurin Series.**

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

The fact is that the only functions our brains and computers can evaluate directly are those involving the basic operations of $+,-,\times,\div$. Anything else we or calculators evaluate must be ultimately reducible to these 4 operations. But the only functions directly expressible in such terms are polynomial and rational functions.

For the *transcendental* functions like trigonometric, exponential and logarithmic the calculator is programed to do something like the following question and to achieve the accuracy to 11 digits that fit on the calculator screen you need to find a term that is smaller than 10^{-11} and add up the series up to that term.

e. Show how the value of $\ln(2) = 0.69314718\ldots$ can be found using Taylor series with x=2 and a=1.