Big idea

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. In this unit, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century.

Feedback & Assessment of Your Success

Date	Pages	Topics	Finished assignment pages? Made corrections?	Summarized notes in a journal? Added your own explanations?	How many extra practice questions did you try in each topic?	Tentative TEST date: _____ Questions to ask the teacher:
2.5days	2-7	Max/Min + EVT (MCV) + Other Theorems (AP) Journal #1				
1.5days	8-11	Function Analysis (MCV) Journal #2				
2days	12-18	Sketching (MCV) Journal #3				
3days	19-25	Velocity & Acceleration (MCV) Journal #4				
	26-28	Other Application (MCV) Journal #5				

ASSIGNMENT Maxima/Minima + EVT (MCV) + Other Theorems (AP)

Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task, or figuring out how to perform a task most productively under a given budget and time schedule are some examples of practical real-world applications of Calculus. The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval. This procedure depends on the nature of the interval.

Label all local and absolute extreme values.

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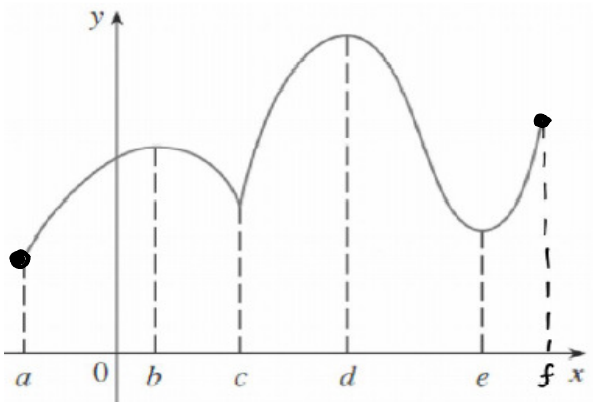

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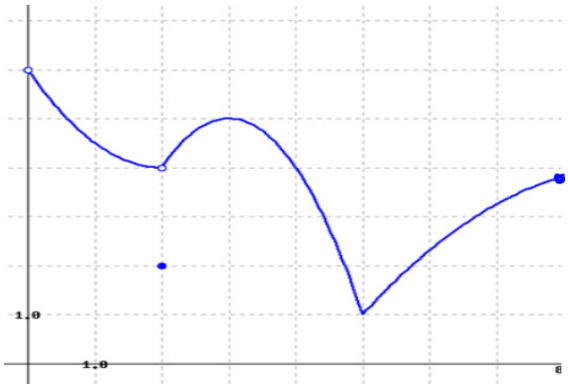

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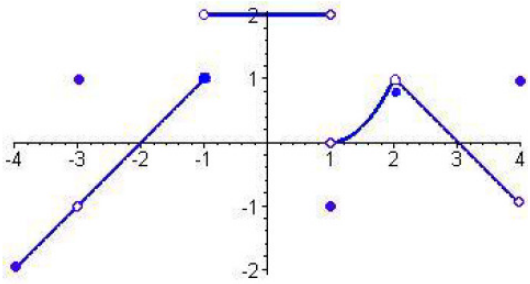

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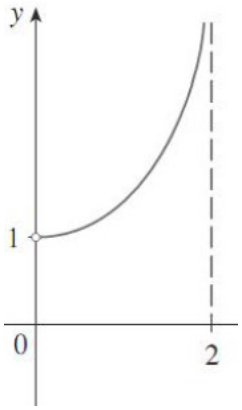

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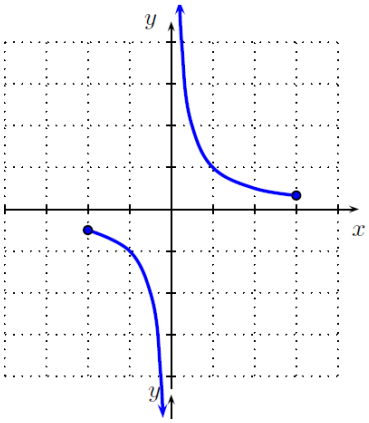

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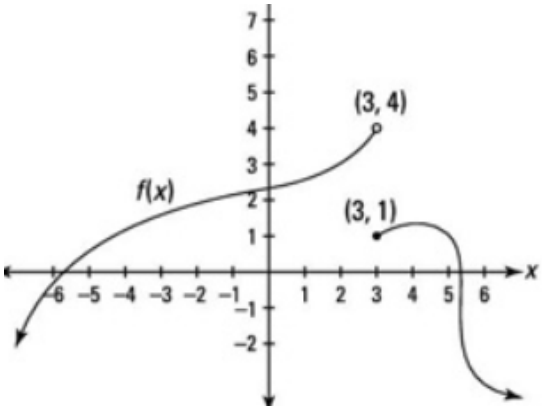

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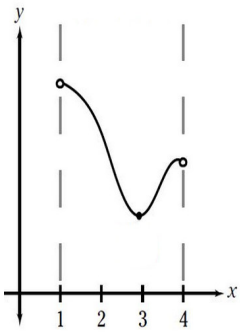

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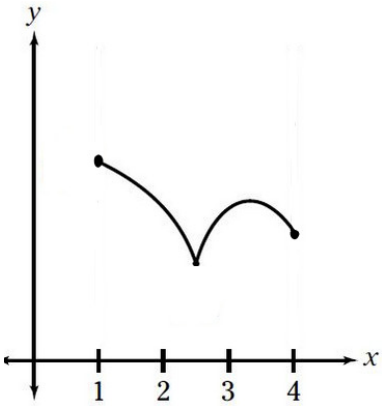

9. Since our goal will be to find absolute extreme values without a graph what are some conditions where absolute extremes can occur?

10. Find the critical numbers of

a) $f(x) = -2x^3 + 3x^{-2}$.

b) $f(x) = x - \sin x$

11. Determine if the EVT applies. If so, find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-8,1]$

Find the absolute max and min of

12. $f(x) = \dfrac{x}{x^2+1}$ on [0, 2].

13. $h(x) = 2\sin\dfrac{\pi}{5}x + 1$ on x ∈ [– 2, 6]

Find the absolute max and min of

14. $k(x)=\begin{cases} x^2+2x+1 & \text{if } x\le 1 \\ -3x+7 & \text{if } x>1 \end{cases}$ on [– 2, 3]

15. $g(x)=4(0.5)^{-x}$ on x ∈ [– 5, – 1]

<u>What if closed interval method doesn't apply</u>?

16. Find extrema on (0, ∞) for $f(x)=\frac{1}{x}+2x$

17. Find extrema on [5, ∞) for $f(x)=\frac{1}{x-2}$

18. Find extrema on [1, 4) for $f(x)=x^2-4x$

Find the absolute max and min of

19. $f(x) = \ln\left|\frac{x}{2+x^2}\right|, \quad -2 \le x \le 2$

20. $f(x) = \begin{cases} 4-2x^2, & x \le 1 \\ x+1, & x > 1 \end{cases}$

21. The population index of a new species of pink elephants is modelled by the equation $P(t) = 2t + \frac{1}{162t+1}, \quad 0 \le t \le 1$. Where t is the time in years. At what time was the population index at its lowest level?

22. Verify that $f(x) = x^3 + x - 1$ satisfies the properties of the MVT on [0, 2], and then find all numbers *c* that satisfy the conclusion of the MVT.

23. Suppose that $f(1) = 7$ and $f'(x) \le 4$ for all values of *x*. How large can $f(3)$ possibly be?

The following is the type of rigorous proof you will have to do at university and on the AP exam.

24. Show that the equation $x^{99}+x^{47}+x-1=0$ has exactly one real root.

ASSIGNMENT Function Analysis (MCV)

Find and classify all the critical points using the 1st derivative

1. $y = (2x+1)^{10}(5x+6)^{8}$

2. $y = (3x+1)(x+1)^{\frac{1}{3}}$

3. $y = x - \sqrt{x}$

4. $y = x^2 e^{2x}$

Choose the best way to analyze the increasing/decreasing intervals for this function.

5. $$y=\sqrt{\frac{1-2x}{x^2-1}}$$

Find the INCREASING intervals and CONCAVE UP intervals, then sketch the polynomials showing all critical and inflection points.

6. $y = 6x^3 + 3x^2 + 5x + 10$

7. $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x - 1$

Find the CONCAVE DOWN intervals and use the 2nd derivative to classify the critical points, then sketch the polynomials showing all critical and inflection points.

8. $y = \frac{1}{2}x^4 + x^3 - 6x^2 - 18x + 36$

9. $y = 2x^3 - 3x^2 - 12x + 5$

ASSIGNMENT Sketching (MCV)

1. Sketch the graph of a continuous function, $y = f(x)$, which has all of the following properties:

- $f(x)$ has a domain of $[1, 7]$
- $f(x)$ has an absolute maximum of 6 when $x = 2$ and an absolute minumum of -1 when $x = 5$.
- $f''(x) > 0$ for all x in the domain of $f(x)$, with the exception of $x = 2$ where $f''(x)$ DNE.

Sketch the derivatives of the following

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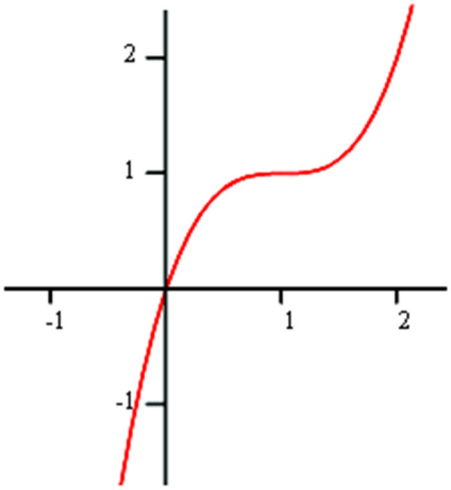

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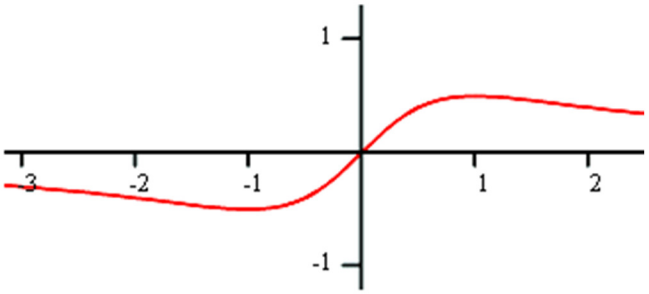

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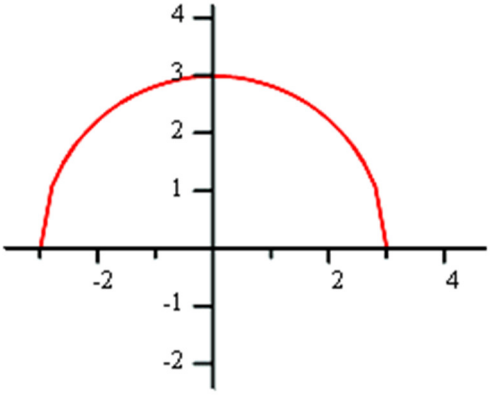

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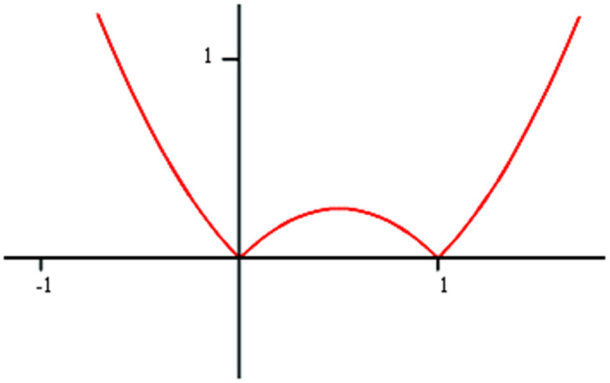


Sketch and label all intercepts, asymptotes, critical points and inflection points

6. $$y = \frac{x}{x^2 + 1}$$

Sketch and label all intercepts, asymptotes, critical points and inflection points

7. $y = x^{\frac{4}{5}}(5-x)$

Sketch and label all intercepts, asymptotes, critical points and inflection points

8. $y = e^x - x - 1$

Sketch and label all intercepts, asymptotes, critical points and inflection points

9. $y = \sin x + \sqrt{3}\cos x$

10. Find the points on $x^2 - 2x + 4y^2 + 16y + 1 = 0$ where the line tangent to the curve is vertical.

11. Determine what you know about the shape of $x^2 y - 2x = 3y$ at point $(1, -1)$

12. Find the critical points and possible inflection points for $2x^3 - 3y^2 = 16$

ASSIGNMENT Velocity & Acceleration (MCV)

1. What is the difference between displacement and distance? Or velocity and speed? Give examples.

2. If $x(t)$ represents the position of a particle along the x-axis at any time, t, then the following statements are true.

(a) "Initially" means when _____

(b) "At the origin" means _____

(c) "At rest" means _____

(d) If the velocity of the particle is positive, then the particle is moving to the _____.

(e) If the velocity of the particle is _____, then the particle is moving to the left.

(f) To find average velocity over a time interval, divide the change in _____ by the change in time.

(g) _____ velocity is the velocity at a single moment (instant) in time.

(h) If the acceleration of the particle is positive, then the _____ is increasing.

(i) If the acceleration of the particle is _____, then the velocity is decreasing.

(j) In order for a particle to change directions, the _____ must change signs.

(k) One way to determine _____ _____ _____ over a time interval, when given the position function or graph, is to find the sum of the absolute values of the differences in position between all resting points.

3. Fill in the blanks.

(a) If velocity is negative and acceleration is positive, then speed is _____.

(b) If velocity is positive and speed is decreasing, then acceleration is _____.

(c) If velocity is positive and decreasing, then speed is _____.

(d) If speed is increasing and acceleration is negative, then velocity is _____.

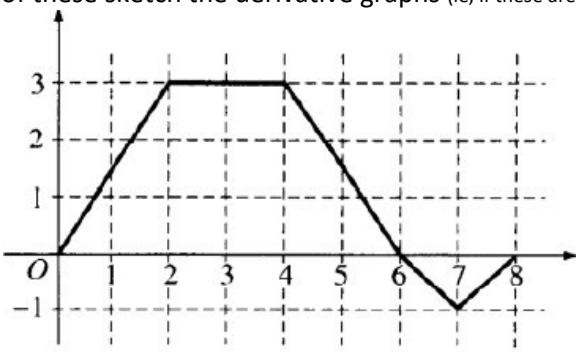
(e) If velocity is negative and increasing, then speed is _____.

(f) If the particle is moving to the left and speed is decreasing, then acceleration is _____.

4. If the position of a particle along a horizontal line is given by $x(t)=\frac{1}{3}t^3-t^2-3t+4$, find the total distance traveled on the interval $0 \le t \le 6$.

For each of these sketch the derivative graphs (ie, if these are displacement-time graphs, what will be the velocity-time graphs?)

5.



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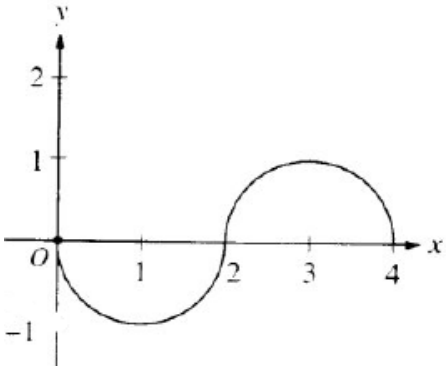

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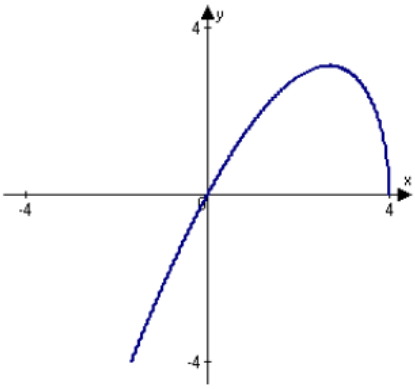

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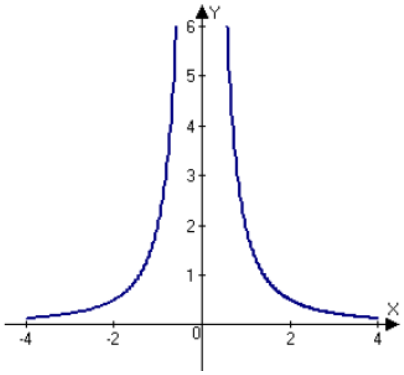

For each of these sketch the possible anti-derivative graphs (ie, if these are velocity-time graphs, what will be the displacement-time graphs?)

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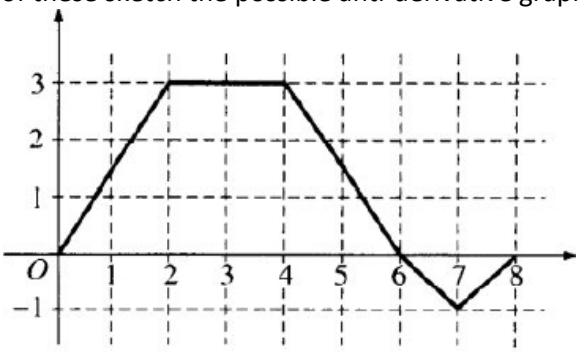

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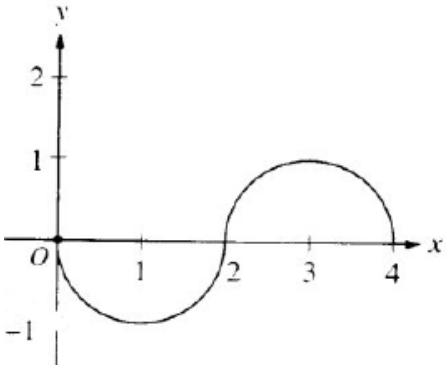

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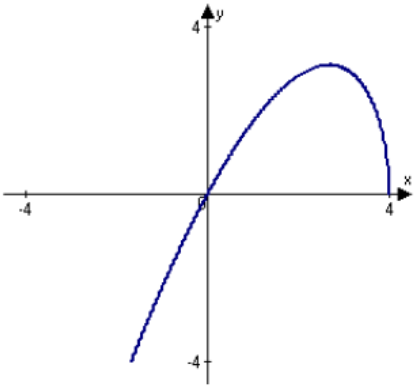

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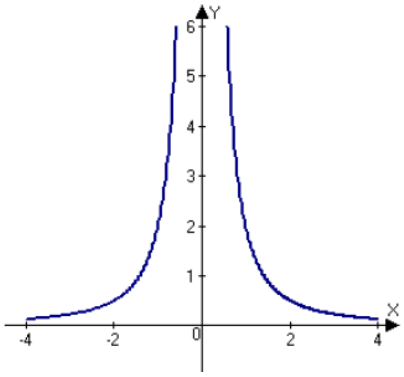
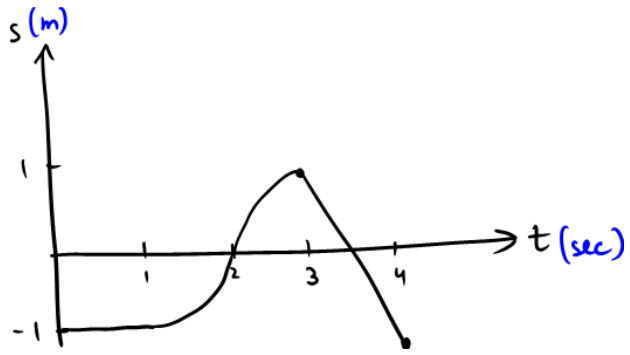


13. Explain what is going on in the graph
Include words like
-stop
-speed up/slow down
-position in relation to origin
-north/right direction
-south/left direction
-turn around
-go past the origin

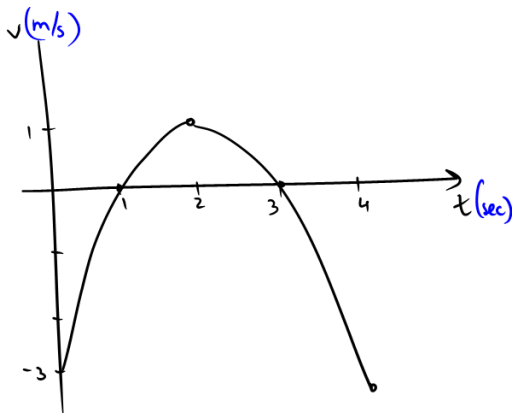
Include times and speeds as necessary



Draw the velocity-time graph from the above displacement graph.

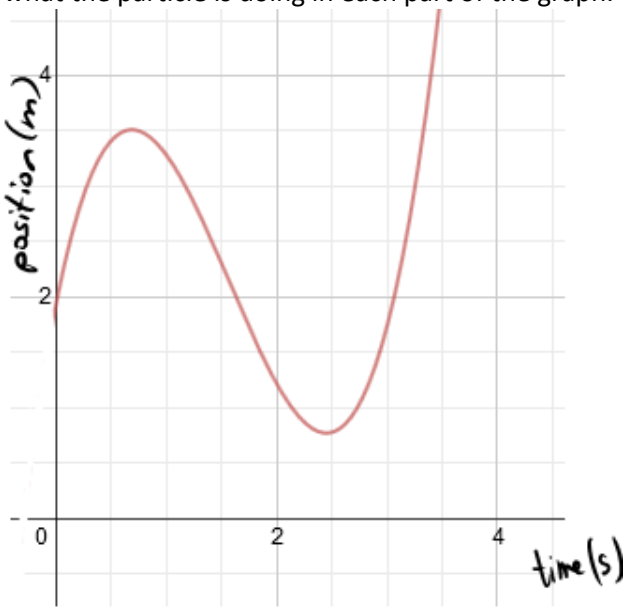
14. Explain what is going on in the graph
Include words like
-north/right direction
-south/left direction
-speed up/slow down
-stop
-turn around
-what is unknown to you

Include times and speeds as necessary



Draw the displacement-time graph from the above velocity graph.

15. Suppose this graph represents position of a particle travelling along the north-south line. Describe what the particle is doing in each part of the graph.



16. Sketch 1st and 2nd derivatives on separate grids. Describe what each characteristic of the graph tells you.

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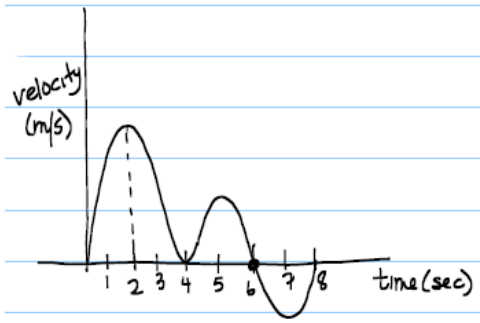
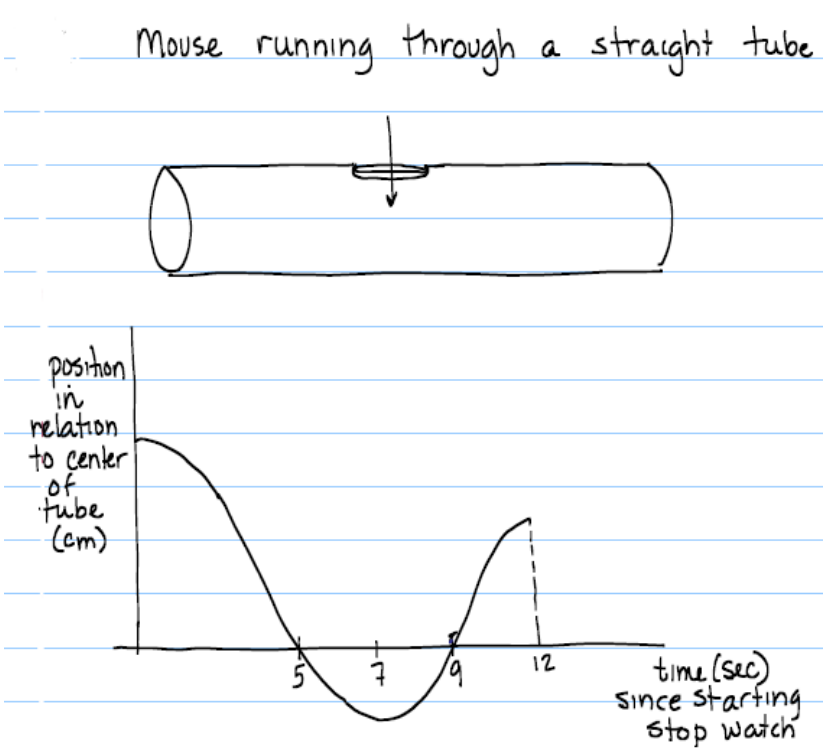


Represents the velocity of a model train moving along a straight piece of track.

a) Draw a possible s-t graph (hints: zeros are t.p., t.p. are inf.pts, neg vel is opposite dir)
b) What do you know about position and velocity when t=0, 2, 4, 7
c) When is the train farthest from the initial position?

18.



a) When is the mouse running to the left?
b) When is the mouse farthest from where he was at t=0?
c) Did the mouse make it back to the centre during this 12sec period?
d) When is the mouse moving most quickly?
e) Sketch v-t and a-t graphs on separate coordinate systems

19. The position of a robot chicken in a straight hallway is given by $s(t) = t^3 - 12t^2 + 36t$ measured in meters from the doorway of an office and t is time in seconds since it was placed in the hallway. What do we know about the motion of the robot at t=5sec? (position left or right of origin? Moving left or right? Slowing down or speeding up?)

20. What do we know about the object at t=2 moving along a straight line given that $x(t) = \sqrt{3t+4} - 4t^2$

21. The data in the table below gives selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

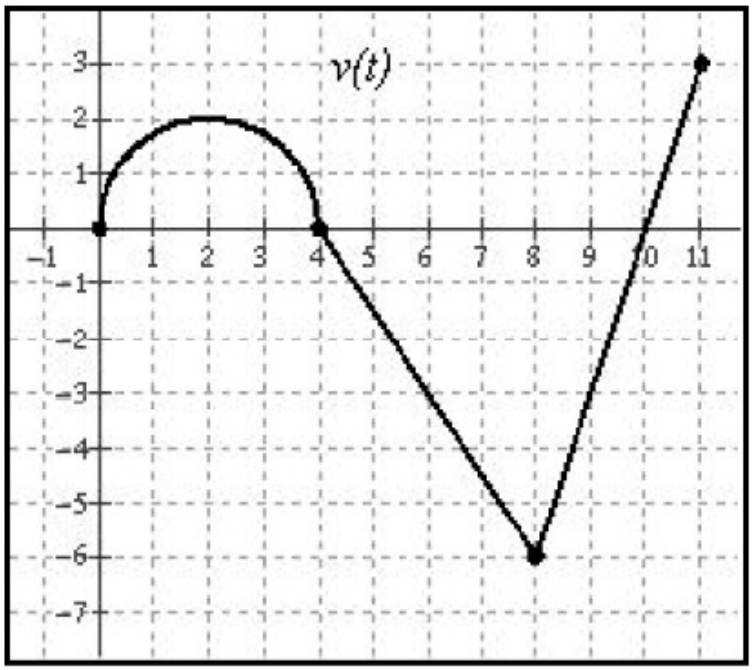
(a) At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

(b) Is there a time during the time interval $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer.

(c) Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computations that lead to your answer (this means show the difference quotient!!!!!), and indicate units of measure.

(d) (Genius Question) Let $a(t)$ denote the acceleration of the particle at time t. Is there guaranteed to be a time $t = c$ in the interval $0 \le t \le 12$ such that $a(c) = 0$? Justify your answer.

22. The graph below shows the velocity, $v(t)$, of a particle moving along the x-axis for $0 \le t \le 11$. It consists of a semi circle and two line segments. Use the graph and your knowledge of motion to answer the following questions.



(a) At what time, $0 \le t \le 11$, is the speed of the particle the greatest?

(b) At which of the times, $t = 2$, $t = 6$, or $t = 9$, is the acceleration of the particle greatest? Explain your answer.

(c) Over what time intervals is the particle moving to the left? Explain your answer.

(d) Over what time intervals is the speed of the particle decreasing? Explain your answer.

(e) Find the area of the semicircle on the interval $0 \le t \le 4$ bounded by the curve and the x-axis, then find the area of the triangle on the interval $4 \le t \le 10$ bounded by the curve and the x-axis, and finally, find the area of the triangle on the interval $10 \le t \le 11$ bounded by the curve and the x-axis. If all of these areas were positive and added together, propose what quantity this might be in terms of the particle's movement.

ASSIGNMENT Other Applications (MCV)

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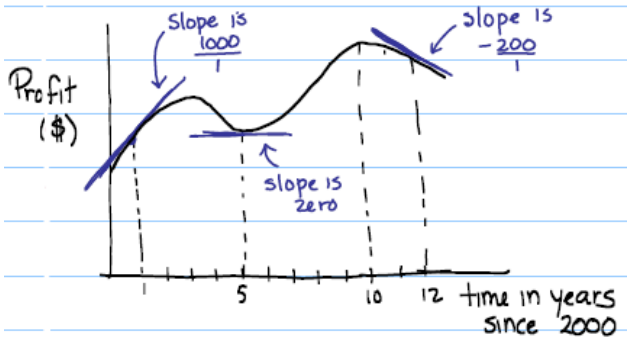

Interpret the derivate at t=1, 5, 12

2. Cost of extracting T tones of copper ore from a mine is given by C(T) dollars. What do $C'(200) = 100$ and $C(1000) = 500\ 000$ mean?

3. You are told that water is flowing through a pipe at a rate of $10\ \mathrm{m}^3/\mathrm{min}$. Write this information as a derivative.

4. The relationship between the number of species in a genus (*x*) and the number of general (*y*) comprising *x* species is given by $x\,y^a = k$ where *a* and *k* are constants. Find $\frac{dy}{dx}$

5. In a model of the nervous system, the intensity of excitation, *I*, of a nerve pathway is given by

$$I = E\left(1 - e^{-a(S-h)/E}\right)$$

where *E* is the maximum possible excitation, *S* is the intensity of a stimulus, *h* is a threshold stimulus, and *a* is a constant. Find the rate of change of the intensity of the excitation with respect to the intensity of the stimulus.

6. A leather coat manufacturer find that the revenue in hundreds of dollars, for producing and selling x coats is given by $R(x)=150+15\sqrt{x}$. Find and interpret the following:

a) $R(100)$

b) $R'(100)$

c) $A(x)=\dfrac{R(x)}{x}$ find $A(100)$

d) $A'(100)$

7. A health food supplement company find s the profit from selling x kits for a stress remedy is given by $P(x)=7.5x^{1.2}$ Find and interpret the following:

a) $P(50)$

b) $A(50)$ where $A(x)=\dfrac{P(x)}{x}$

c) $P'(50)$

d) $A'(50)$

8. The memory recall of unrelated facts without review was tested every week.
$A(t) = 75 - 5\ln(t+1); t \geq 0$, where t is number of weeks since memorizing facts, and A is the average marks of students
   a) What is the average mark the first time students wrote the test
   b) Find the rate of change of the average mark at any time t, what would be the units?
   c) Determine how quickly the average mark is changing 1 month after memorizing the facts
   d) When would the average mark reach 50%?
   e) When is magnitude of the memory loss changing at a rate less than 0.5%?

9. The spread of a virus is given by $P(t) = \dfrac{800}{1+e^{6-t}}$
where t is number of days since the first case, and P is number of people who have been infected.
   a) How many people where initially infected?
   b) About how many people will be infected in the long run?
   c) How fast is the virus spreading after 4 days?
   d) When will the virus be spreading to more than 20people/day?