APPLICATIONS OF DERIVATIVES

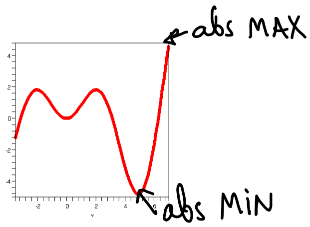
NEW to high school
Required only for some programs not on AP exam

pg	Topics	HW
2-7	**4.1: MAXIMUM AND MINIMUM VALUES** • How to find local maxima and minima of functions; • Apply the Closed Interval Method to find absolute maxima and minima.	**4.1: 3, 13, 33, 43, 53, 69** [If you feel you need more practice, # 5, 7, 9, 11, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41, 49, 51, 55, 59, 61, 73, 75]
8-11	**4.2: THE MEAN VALUE THEOREM** • The Mean Value Theorem (and Rolle's Theorem) and associated applications; • Use of MVT together with IVT to prove that an equation has only one root.	**4.2: 1, 15, 17, 19, 23, 25** [If you feel you need more practice, # 3, 5, 7, 11, 13]
12-15	**4.3: HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH** • Locating intervals on which a function is increasing/decreasing; • How to determine if a function is concave up/concave down on a given interval, and finding points of inflection; • Use of the first and second derivative tests to classify local max/min; • How to accurately sketch the graph of a function.	**4.3: 1, 9, 17, 23, 27, 37, 51** [If you feel you need more practice, # 5, 7, 11, 13, 15, 19, 21, 25, 29, 31, 33, 35, 39, 41, 45, 47, 49, 63, 67]
16-19	**4.4: INDETERMINATE FORMS AND L'HOSPITAL'S RULE** • Understanding of L'Hospital's rule and when it applies. • How to apply L'Hospital's Rule to evaluate limits.	**4.4: 3, 13, 21, 25, 29, 35** [If you feel you need more practice, # 1, 5, 7, 9, 11, 15, 17, 19, 23, 27, 31, 33, 37, 39, 41, 43, 45, 53, 55, 57, 59, 63, 65, 73, 79, 81, #71 is messy so don't necessarily do it, you might want to read it for an interesting historical note.]
20-22	**4.5: SUMMARY OF CURVE SKETCHING** • How to accurately sketch the graph of a function (by finding domain, intercepts, symmetry, asymptotes, first and second derivatives, critical points, and points of inflection, etc. as appropriate).	**4.5: 5, 9, 27, 37, 41** [If you feel you need more practice, # 1, 3, 7, 11, 15, 17, 19, 21, 23, 33, 35, 37, 43, 45, 47]
23-28	**4.7: OPTIMIZATION PROBLEMS** • Set up of various optimization problems and solving these problems using calculus; • Demonstrating that the solution to an optimization problem is indeed an absolute max or min.	**4.7: 3, 9, 13, 33, 35, 49, 59, 69, 73a** [If you feel you need more practice, # 1, 5, 9, 11, 15, 17, 19, 21, 25, 27, 31, 37, 47] **Hint: you may find that you need some formulas for areas and volumes of various shapes…some of these can be found on the reference page at the front of your text.**
29-33	**4.9: ANTIDERIVATIVES** • Understand the basic concept of antidifferentiation; • Finding general and particular antiderivatives of simple functions; • Finding a position function given an object's acceleration; • Sketching the graph of an antiderivative given the graph of the function.	**4.9: 9, 33, 49, 57, 75** [If you feel you need more practice, # 1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41, 43, 45, 47, 51, 53, 59, 61, 63, 67, 69, 71, 73, 77] **For some of the questions in 4.9 (e.g. 77), note that to convert from mi/h to ft/s, multiply by 5280/3600.**

Maximum and Minimum Values (4.1)

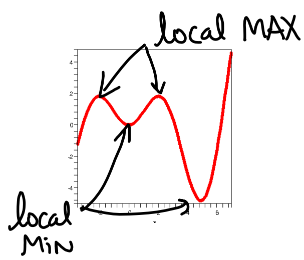
Definition: A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain. We call $f(c)$ the **maximum value**. Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in the domain; $f(c)$ is then the **minimum value**.

Example:

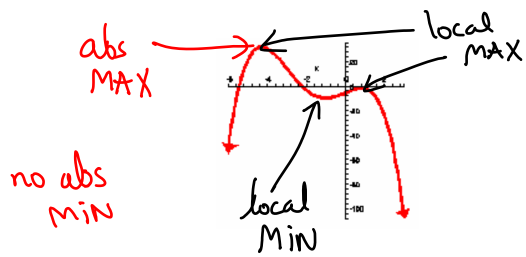


Definition: A function f has a **local maximum** at c if $f(c) \geq f(x)$ when x is near c. Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c.

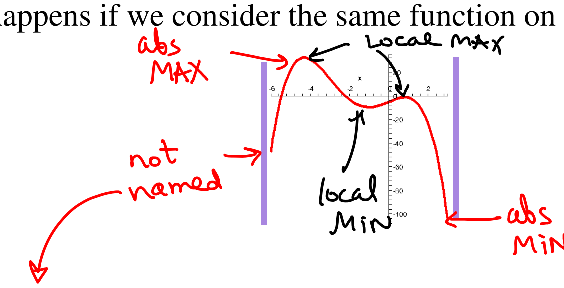
Example:



Example:



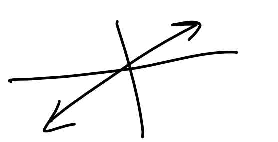
Question: What happens if we consider the same function on a closed interval?



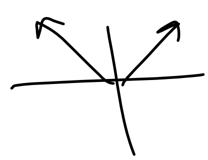
- **NOTE:** Endpoints can NOT be local max/min.

Question: Do all functions have an absolute max and min?

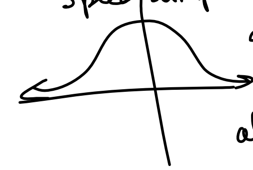
NO linear has neither



abs. val has abs MIN but no abs MAX

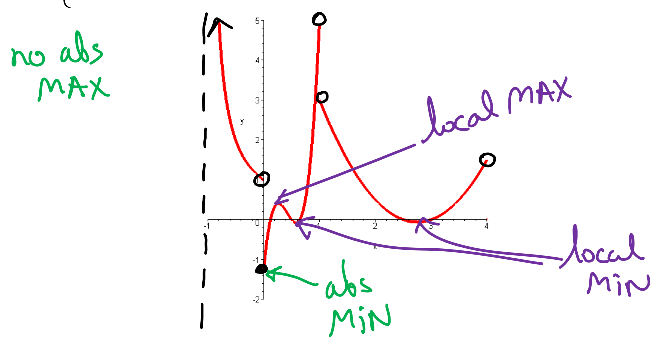


"speed bump" has abs MAX but no abs. MIN

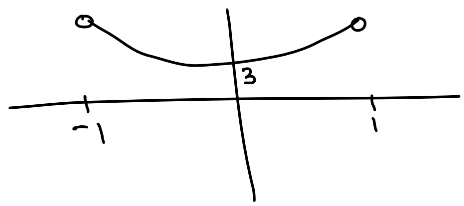


quad with no zeros

Example: $f(x)=\begin{cases} \dfrac{1}{x+1} & -1<x<0 \\ (4x-2.6)(4x-0.5)(2x-1) & 0\le x<1 \\ (x-2.5)(x-3) & 1<x<4 \end{cases}$

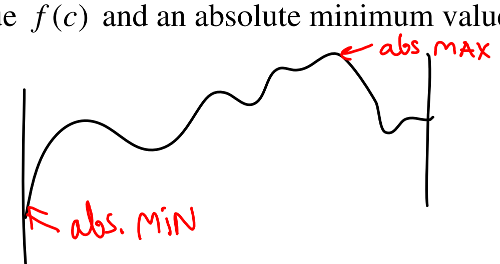


Example: Sketch a graph of a continuous function *f* such that the absolute maximum of $f(x)$ on the interval (-1, 1) does not exist and the absolute minimum equals 3.



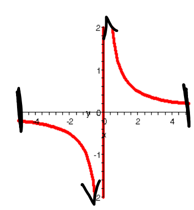
Question: How do we know when a function will definitely have an absolute max/min?

Extreme Value Theorem: If *f* is <u>continuous</u> on a <u>closed</u> interval [*a*, *b*], then *f* attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers *c* and *d* in [*a*, *b*].



Question: What happens if the function isn't continuous? Can we still apply the Extreme Value Theorem?

NO! eg. $y=\frac{1}{x}$ on $[-5,5]$
has no abs. max/min
since VA is at $x=0$



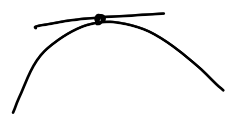
Question: What if we aren't given the graph of a function?

we will need a way to locate max/min, then compare it to values at these points and at the end points

Recall: Earlier we discussed the Extreme Value Theorem which states that on a closed interval, a continuous function attains both an absolute max and an absolute min.

We had notices that abs max/min
occur at either → end points
or → local max/min * lets explore this one more

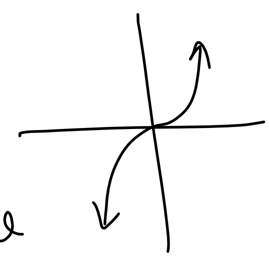
Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. slope = 0



Caution: Some Important Points

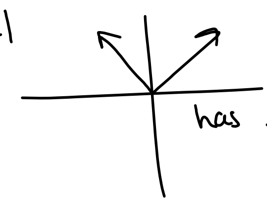
Are all places where the derivative is zero a local max/min?

NO! ex. $f(x) = x^3$
$f'(x) = 3x^2$
so at $x=0$ $f'(x)=0$
BUT look at the graph that point is not a local max/min



Do local max/min occur only at places where the derivative is zero?

No! ex. $y = |x|$

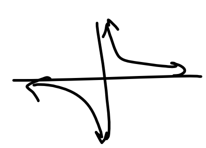


has local MIN but $f'(0)=$ DNE

Definition: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Key

ex. $y = \frac{1}{x}$



$f'(0)$ DNE but $x=0$ is not in the domain of original function so for this ex. there are no critical numbers

So, rephrasing **Fermat's Theorem**:
If f has a local maximum or minimum at c, then c is a critical number of f.

so, to locate local max/min, we find all critical numbers (CN)
ie. places where $f'(x)=0$ or $f'(x)=$ DNE that are in the domain of $f(x)$. These are candidates for max/min.

Example: Find the critical numbers of $f(x)=-2x^3+3x^2$.

$f'(x) = -6x^2+6x$
defined all the time no $f'(x) = $ D.N.E anywhere

but $f'(x)=0$ at $x=0$ and $x=1$
$0=-6x(x-1)$

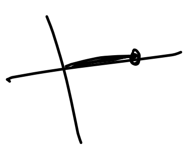
$\therefore$ critical numbers are $x=0, 1$

Example: Find the critical numbers of $f(x)=x-\sin x$.

$f'(x) = 1-\cos x$ defined everywhere

$0 = 1-\cos x$

$1=\cos x$



$x = 0, \pm 2\pi, \pm 4\pi, \ldots$
or $x = 2\pi n$, $n \in \mathbb{Z}$ (n is an integer)

Question: How can we use the idea of critical points to help us find absolute max/min?

CN's helps us locate local max/min since abs max/min occur @ either local max/min or endpoints.
All we need to do to find abs. max/min is to find CN + compare their y-values to the y-values of the endpoints

The Closed Interval Method: To find the absolute max and min values of a continuous function *f* on a closed interval [*a*, *b*]:

1. Find the critical numbers of *f* in (a, b).

2. Find the values of *f* at the critical numbers.

3. Find the values of *f* at the endpoints of the interval.

4. Select the largest value of *f* as the absolute max, and the smallest as the absolute min.

Example: Find the absolute max and min of $f(x) = \dfrac{x}{x^2+1}$ on [0, 2].

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$ never undefined

$$0 = \frac{1-x^2}{(x^2+1)^2}$$

$$0 = 1-x^2$$

$$0 = (1-x)(1+x)$$

$\therefore$ CN are $x = 1, -1$

but only $x=1$ is in $[0,2]$

now compare:

CN $\rightarrow$ $f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$ $\leftarrow$ abs MAX

endpt $\rightarrow$ $f(0) = 0$ $\leftarrow$ abs MIN

endpt $\rightarrow$ $f(2) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$

$\therefore$ abs MAX value $= \frac{1}{2}$ at $x=1$

abs MIN value $= 0$ at $x=0$

Application: The risk of exposure to harmful fungi that thrive in buildings appears to increase in damp environments. Researchers have discovered that by controlling both the temperature and the relative humidity in a building, the growth of the fungus *A. versicolor* can be limited. The relationship between temperature and relative humidity, which limits growth, can be described by

$$R(T) = -0.00007T^3 + 0.0401T^2 - 1.6572T + 97.086 \text{ for } 15 \le T \le 46$$

where $R(T)$ is the relative humidity (in %) and T is the temperature (in °C). Find the temperature at which the relative humidity is minimized.

[Source: "Calculus for the Life Sciences", Greenwell, Ritchey, and Lial, 2003.]

$R'(T) = -0.00021T^2 + 0.0802T - 1.6572$

$R'(T)$ is defined everywhere

$0 = R'(T)$ solve using quadratic formula

$$T = \frac{-0.0802 \pm \sqrt{(0.0802)^2 - 4(-0.00021)(-1.6572)}}{2(-0.00021)}$$

$T = 21.92166416$ or $T = 3\cancel{59}.9830977$

not in $[15, 46]$

CN $\rightarrow$ $R(21.9216646) \doteq 79.3$

end pt $\rightarrow$ $R(15) \doteq 81$

end pt $\rightarrow$ $R(46) \doteq 98.9$

$\therefore$ Relative humidity is minimized at $T \doteq 22$ °C with min humidity $\doteq 79.3$ %

The Mean Value Theorem (4.2)

Example: If policewoman Betsy sees Paula enter the 407 at the QEW in Burlington (just outside Hamilton) at precisely 12:00 noon, and then her husband policeman Bobby sees Paula 107 km away exiting the 407 at HWY 7 in Pickering at precisely 12:45 pm, can they give her a speeding ticket?

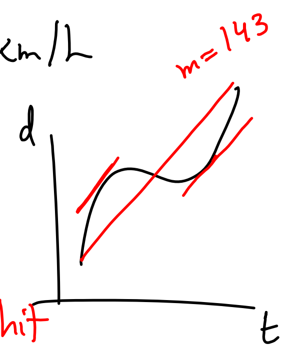
$$\frac{\Delta d}{\Delta t} = \frac{107}{0.75} \doteq 143 \text{ km/hr}$$

Yes she can get a ticket.

Actually, if we assume that the position function (distance vs. time) is continuous and differentiable at all times, then there must have been at least one time during the trip when Paula was traveling at 143 km/hour. Why?

Since if I travelled on average at 143km/h then it isn't possible to always be above or to always be below that average value

ex



there must be at least once that I hit the same speed.

The Mean Value Theorem: Let f be a function which has the following properties:

1. f is continuous on the closed interval [a, b].
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

for above ex: $t_1 = 0$ $t_2 = 45 \text{ min} = 0.75 h$
$d_1 = 0$ $d_2 = 107 \text{ km}$

$$f'(c) = \frac{d(t_2) - d(t_1)}{t_2 - t_1} = \frac{107 - 0}{0.75} \doteq 143 \text{ km/h}$$

A special case occurs when $f(a) = f(b)$:

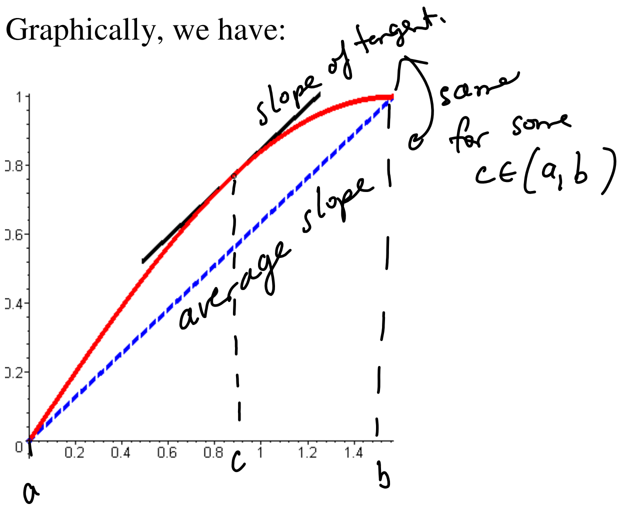
Rolle's Theorem: If $f(a) = f(b)$, and f satisfies the properties 1 and 2 listed above, Then there exists some number c in (a, b) such that $f'(c) = 0$.

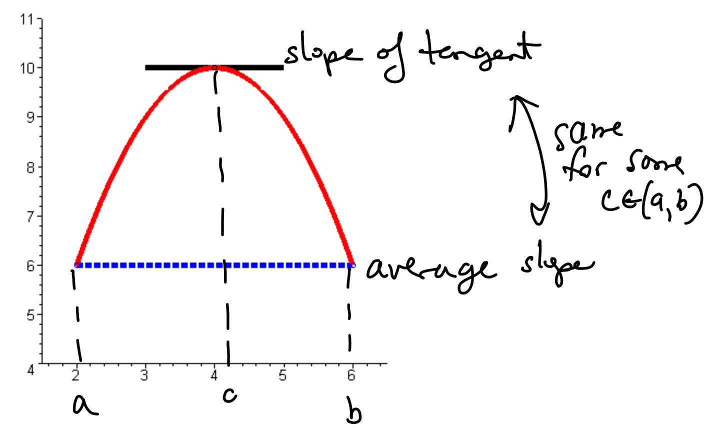
using MVT $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ for $c \in (a,b)$

$= \dfrac{0}{b-a}$ since $f(b)=f(a)$

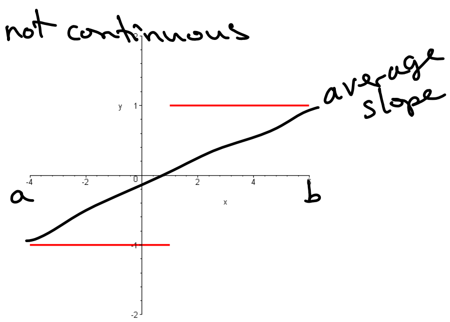
$= 0$

Graphically, we have:

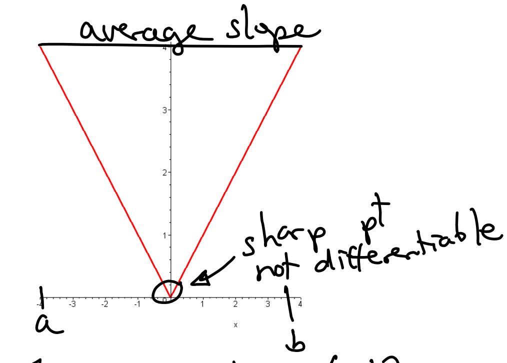




Note: Properties 1 (continuity) and 2 (differentiability) have to hold for both theorems!



but there is no $c \in (a,b)$ with the same slope!



There is no pt $c \in (a,b)$ with same average slope

MVT doesn't apply for discont. functions

Both of these theorems give us an important link between information about a function, and information about the derivative of a function!

Example: Verify that $f(x) = x^3 + x - 1$ satisfies the properties of the MVT on [0, 2], and then find all numbers c that satisfy the conclusion of the MVT.

f(x) is a polyn. so it's cont. and differentiable everywhere (that includes $[\underset{a}{0}, \underset{b}{2}]$)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f(a) = -1$

$f(b) = 9$

$f'(x) = 3x^2 + 1$

$$3c^2 + 1 = \frac{9 - -1}{2 - 0}$$

$3c^2 + 1 = 5$

$c^2 = 4/3$

$c = \pm \frac{2}{\sqrt{3}}$ only $\frac{2}{\sqrt{3}}$ is in $[0, 2]$

∴ at $x = 2/\sqrt{3}$ slope is the same as the average slope on $[0,2]$ aroc $= 5$

Example: Suppose that $f(\underset{a}{1}) = 7$ and $f'(x) \le 4$ for all values of x. How large can $f(\underset{b}{3})$ possibly be?

We know $f'(c) = \frac{f(b) - f(a)}{b - a}$ for $c \in [1, 3]$

$$f'(c) = \frac{f(3) - 7}{3 - 1}$$

$$f'(c) = \frac{f(3) - 7}{2}$$

Since $f'(x) \le 4$ for all x then

$f'(c) \le 4$

$\frac{f(3) - 7}{2} \le 4$

$f(3) - 7 \le 8$

$f(3) \le 15$

∴ $f(3)$ is at most 15.

The Mean Value Theorem can be used to prove various facts of differential calculus. Refer to Theorem 5 (pg. 284) and Corollary 7 (pg. 284) for two examples.

> **5 THEOREM** If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

> **7 COROLLARY** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Very hard to factor without technology Q: How does tech. 'know' what to do? learn later!

Example: Show that the equation $x^{99}+x^{47}+x-1=0$ has exactly one real root.

Plan: (1) use IVT to show at least one root
if $f(a) < N < f(b)$ and f cont on $[a,b]$ then there is $c \in (a,b)$ s.t. $f(c)=N$

(2) show not possible to have 2 roots using Rolles Th.
if $f(a)=f(b)$ and f cont. on $[a,b]$ and different. on (a,b)
then there is $c \in (a,b)$ s.t. $f'(c)=0$

DO (1.) Since $f(x)= x^{99}+x^{47}+x-1$ is a polynomial it's cont. everywhere
choose two x's so that $f(a)=$ pos, $f(b)=$ neg } then there will be a root between

try $f(0)=-1$, $f(1)=2$ $\therefore$ There is at least one $c \in [0,1]$ s.t. $f(c)=0$
ie. there is at least one real root.

(2) Proof by contradiction:
Assume there are 2 or more roots, ie. $f(d)=0$, $f(e)=0$
(why choose Rolle's? $f(d)=f(e)=0$:))
$f(x)$ is cont. and differentiable so Rolles Th. applies
it states that there is $c \in (d,e)$ s.t. $f'(c)=0$

find $f'(x) = 99x^{98}+47x^{46}+1$
this is always pos. $\therefore f'(c) \neq 0$ } contradiction assumption was wrong.

$\therefore$ There can't be 2 or more roots.
and IVT already showed there is at least one $\therefore$ There is exactly ONE :)

How Derivatives Affect the Shape of Graphs(4.3)

Question: Suppose you are given $f(x) = 2x^3 - 3x^2 - 12x$. How would you sketch this?

plotting points from table of values is tedious and depending on x values chosen you may even miss important features of the graph

Our work so far in this course has given us lots of information about functions and their derivatives; combining all of this information, we are able to accurately sketch functions!

ex. – domain
– intercepts
– asymptotes
– max / min

> **Increasing/Decreasing Test:**
> a) If $f'(x) > 0$ on an interval, then f is **increasing** on that interval.
> b) If $f'(x) < 0$ on an interval, then f is **decreasing** on that interval.

$f'(x) > 0$ ↗ inc $\qquad$ $f'(x) < 0$ ↘ dec

Example: Find where the function $f(x) = 2x^3 - 3x^2 - 12x$ is increasing and where it is decreasing.

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$

when there's multiplication you can look at signs only

	$-\infty$ to -1	-1 to 2	2 to ∞
6	+	+	+
$x-2$	−	−	+
$x+1$	−	+	+
f'	+	−	+
f	↗	↘	↗

∴ f is inc on $(-\infty, -1)$ and $(2, \infty)$ and f is dec on $(-1, 2)$

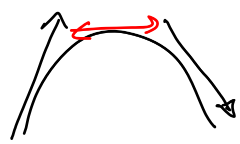
What's happening at the points $x = -1$ and $x = 2$ in our above example?

these are candidates for local max/min
see 1st deriv. test below

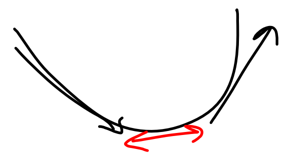
$f'(c) = 0$ or ~~$f'(c) = DNE$~~ not this for continuous

The First Derivative Test

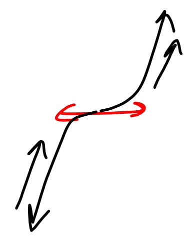
Suppose that c is a critical number of a continuous function f.
a) If f' changes from positive to negative at c, then f has a local maximum at c.
b) If f' changes from negative to positive at c, then f has a local minimum at c.
c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



pos slope then neg slope means local MAX



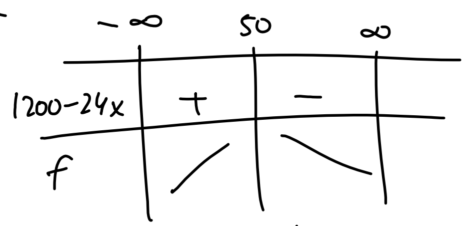
neg then pos means local MiN



pos then pos is a saddle pt.

Application: An autocatalytic chemical reaction is one in which the product being formed causes the rate of formation to increase. The rate of a certain autocatalytic reaction is given by $V(x) = 12x(100 - x)$ where x is the quantity of the product present and 100 represents the quantity of chemical present initially. For what value of x is the rate of the reaction a maximum?
[Source: Calculus for the Life Sciences by Greenwell, Ritchey and Lial, 2003]

$V(x) = 1200x - 12x^2$
$V'(x) = 1200 - 24x$ set $V'(x) = 0$ to find C.N.
$0 = 1200 - 24x$
$24x = 1200$
$x = 50$

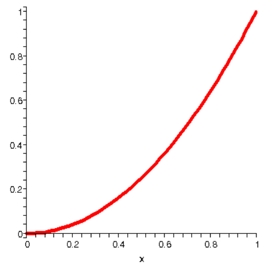


$\therefore$ at $x = 50$ local MAX.

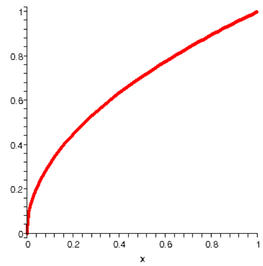
$\therefore$ Rate of reaction reaches a max at $x = 50$

Question: So far, we have used the first derivative to tell us if a function is increasing or decreasing, but how do we learn more about the shape?

Example:

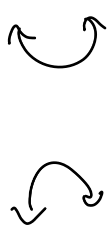


as compared to



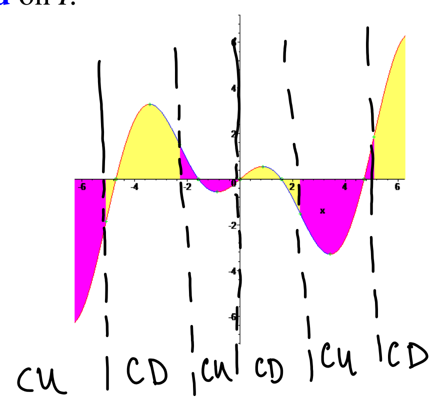
These functions are both increasing but the curvature is different.

Definition: If the graph of f lies above all of its tangents on an interval *I*, then it is called **concave upward** on *I*. If the graph of f lies below all of its tangents on *I*, it is called **concave downward** on *I*.



CU

CD



Definition: A point *P* on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or vice versa) at *P*. $f''(x) = 0$ solves for possible P.O.Inf. don't know until check each side!

> **Definition:** **(Concavity test)**
> a) If $f''(x) > 0$ for all x in *I*, then the graph of f is concave upward on *I*.
> b) If $f''(x) < 0$ for all x in *I*, then the graph of f is concave downward on *I*.

Example: $f(x) = 2x^3 - 3x^2 - 12x$ (again! ☺)

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$0 = 6(2x - 1)$

possible Point of Inflection $x = \frac{1}{2}$

check

	$-\infty$ to $\frac{1}{2}$	$\frac{1}{2}$ to ∞
6	+	+
$2x-1$	−	+
f''	−	+
f	CD	CU

$\therefore$ CD on $(-\infty, \frac{1}{2})$
CU on $(\frac{1}{2}, \infty)$

and $x = \frac{1}{2}$ is a P.O.Inf.

In addition, information about the second derivative can also help us to classify local max/min! ie. If already found f'' it's easier to do 2nd deriv. test and not 1st deriv. test.

The Second Derivative Test: Suppose f'' is continuous near c.
If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Example: $f(x)=2x^3-3x^2-12x$

$f'(x)=6x^2-6x-12=6(x+1)(x-2)$ CN's $x=-1,2$

$f''(x)=12x-6=6(2x-1)$

$f''(-1)=12(-1)-6=-18<0$ ∴ ∩ CD ∴ local MAX at $x=-1$

$f''(2)=12(2)-6=18>0$ ∴ ∪ CU ∴ local MIN at $x=2$

→ faster than doing chart on bottom of pg. 12 of these notes!

Example: So, finally, what does $f(x)=2x^3-3x^2-12x$ look like?

Domain: $x \in \mathbb{R}$

intercepts y-int $y=2(0)^3-3(0)^2-12(0)$ ∴ $y=0$

x-int $0=x(2x^2-3x-12)$

$0 \doteq x(2)(x-3.3)(x+1.8)$ using quad. formula

x-int $x=0, 3.3, -1.8$

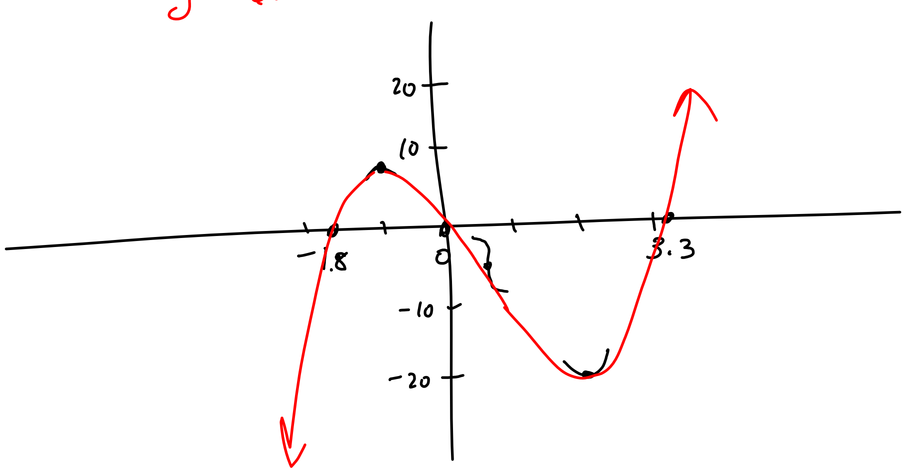
no asymptotes

CN $x=-1$ was local MAX, $x=2$ was local MIN

$y=7$ $y=-20$

POInf $x=\frac{1}{2}$ CD first then CU

$y=-6.5$



Indeterminate Forms and L'Hospital's Rule (4.4)

Recall: In the past, we've encountered limits of the form "$\frac{0}{0}$" and developed techniques to evaluate them.

The strategies we applied were
- factoring
- LCD
- rationalizing
- change of variable

Question: What about other examples of the form "$\frac{0}{0}$" where our techniques don't work? ex. $\lim_{x\to 1} \frac{\ln x}{x-1}$

This limit is called an **indeterminate form of type "$\frac{0}{0}$"**.

Also, we had techniques for evaluating limits of the form "$\frac{\infty}{\infty}$" we divided by highest power of the denominator

ex. $\lim_{x\to\infty} \frac{3x^2+4x}{7x^2-2x+1} = \lim_{x\to\infty} \frac{3+\cancel{\frac{4}{x}}^{0}}{7-\cancel{\frac{2}{x}}^{0}+\cancel{\frac{1}{x}}^{0}} = \frac{3}{7}$

Question: But what about examples where our techniques no longer apply?

ex. $\lim_{x\to\infty} \frac{\ln x}{x-1}$

This limit is called an **indeterminate form of type "$\frac{\infty}{\infty}$"**.

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

ie have $\frac{0}{0}$ or $\frac{\infty}{\infty}$ FORMS only!!

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is ∞ or $-\infty$.

Example: $\lim_{x\to 1} \frac{\ln x}{x-1}$ "$\frac{0}{0}$" form

must indicate where you use L'Hopital's Rule!

$\overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1}$

$= 1$

:) so fast (if only we told you this sooner)

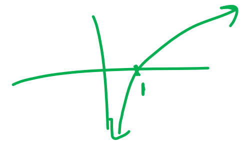
Example: $\lim_{x\to\infty}\frac{\ln x}{x-1}$

"$\frac{\infty}{\infty}$" form $\therefore$ can apply L'H

$\overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1}$

$= 0$

visualize $\ln x$

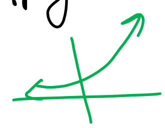


Example: $\lim_{x\to\infty}\frac{e^x+3x}{\ln x}$

"$\frac{\infty}{\infty}$" for so can apply L'H

$\overset{L'H}{=} \lim_{x\to\infty}\frac{e^x+3}{\frac{1}{x}}$

visualize e^x

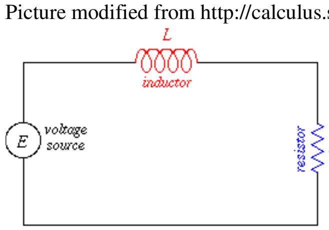


$= \lim_{x\to\infty} x(e^x+3) = \infty$ (D.N.E but does go to ∞ not $-\infty$)

Application: Consider an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t=0$, then the current I flowing through the circuit at time t is given by $I=\frac{V}{R}(1-e^{-Rt/L})$. What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e. $R\to 0^+$)?

[Source: Calculus: Early Transcendentals, Single Variable, 8th ed. By H. Anton, I. Bivens, S. Davis, 2005.]

Picture modified from http://calculus.sjdccd.cc.ca.us/ODE/7-A-4/7-A-4-h.html



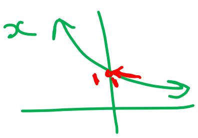
$\lim_{R\to 0^+}\frac{V}{R}\left(1-e^{-\frac{Rt}{L}}\right)$

$= \lim_{R\to 0^+}\frac{V-Ve^{-\frac{Rt}{L}}}{R}$ "$\frac{0}{0}$" form $\therefore$ can apply L'H

$\overset{L'H}{=} \lim_{R\to 0^+}\frac{0-Ve^{-\frac{Rt}{L}}\left(-\frac{t}{L}\right)}{1}$

$= \lim_{R\to 0^+}\frac{Vt}{L}e^{-\frac{Rt}{L}} \to 1$

visualize e^{-x}



$= \frac{Vt}{L}$ $\therefore$ at fixed time current is similar to $I=\frac{Vt}{L}$

Example: $\lim_{x\to 0}\frac{4x}{\tan x}-\cos x$

$= \lim_{x\to 0}\frac{4x}{\tan x} - \lim_{x\to 0}\cos x$ can just sub in.

"$\frac{0}{0}$" form

$\overset{L'H}{=} \lim_{x\to 0}\frac{4}{\sec^2 x} - 1 = \frac{4}{1^2}-1 = 3$

can also do LCD 1st

$\lim_{x\to 0}\frac{4x-\tan x\cos x}{\tan x}$ $\sin x$

$\overset{L'H}{=} \lim_{x\to 0}\frac{4-\cos x}{\sec^2 x}$

$= \frac{4-1}{1^2} = 3$

Now, what if it looks like the answer is something like "$\infty * 0$"?
This limit is called an **indeterminate form of type "$\infty * 0$"**.

Example: $\lim_{x\to -\infty} x^2 e^x$ "$\infty \cdot 0$" form try to get it to be "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" form so that L'H can be applied.

$= \lim_{x\to -\infty} \frac{x^2}{e^{-x}}$ now "$\frac{\infty}{\infty}$"

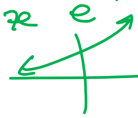
visualize e^{-x}

$\overset{L'H}{=} \lim_{x\to -\infty} \frac{2x}{e^{-x}(-1)}$

now $\frac{-\infty}{-\infty} = \frac{\infty}{\infty}$ form again

$\overset{L'H}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}}$

visualize e^x



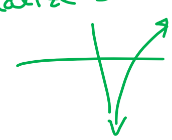
goes to zero

$= \lim_{x\to -\infty} 2e^x = 0$

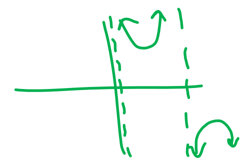
Example: $\lim_{x\to 0^+} \sin x \cdot \ln x$ "$0 \cdot \infty$" form

$= \lim_{x\to 0^+} \frac{\ln x}{\csc x}$ now "$\frac{\infty}{\infty}$"

visualize $\ln x$



$\csc x$



$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\underbrace{\csc x}_{\frac{1}{\sin x}}\underbrace{\cot x}_{\frac{\cos x}{\sin x}}}$

$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x}$ now "$\frac{0}{0}$"

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-2\cancel{\sin x}^{0} \cdot \cancel{\cos x}^{1}}{x(-\cancel{\sin x})^{0} + \cancel{\cos x}(1)^{1}} = \frac{0\cdot 1}{0+1} = \frac{0}{1} = 0$

Finally, what if it looks like the answer is something like "0^0" or "∞^0" or "1^∞"? These can be converted to a limit of the form "$\infty * 0$".

↳ if you take ln of both sides then exponent can come down

BE CAREFUL at the end go back to an answer without this ln !!

Example: $\lim_{x\to\infty} x^{1/x}$ "∞^0" form

let $y = x^{1/x}$

take ln of both sides

$\ln y = \ln x^{1/x}$

$\ln y = \frac{1}{x}\ln x$

now apply limit

$\lim_{x\to\infty} \ln y = \underbrace{\lim_{x\to\infty} \frac{\ln x}{x}}_{\text{work with this side for now}}$ "$\frac{\infty}{\infty}$" form

$\lim_{x\to\infty} \ln y \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$

$\lim_{x\to\infty} \ln y = 0$

now remove ln by exponentiating both sides

$e^{\lim_{x\to\infty} \ln y} = e^0$

$\lim_{x\to\infty} e^{\ln y} = e^0$

$\lim_{x\to\infty} y = 1$

$\therefore \lim_{x\to\infty} x^{1/x} = 1$ ◡̈

Summary of Curve Sketching (4.5)

Calculus helps us to sketch curves.
Even if technology is used for a sketch
you may miss important info of a function
(you may not know to zoom in/out in a
particular spot to see more detail ...)
won't know these key spots without calculus

Summary of Curve Sketching:

• domain	• regions of concavity
• asymptotes	• actual P.O.Inf.
• intercepts	• local max/min — use 2nd deriv. test
• critical numbers + y values	• regions of increasing/decreasing
• possible P.O.Inf. + y values	• symmetry

may not be necessary to finish the graph
do only if asked

Example: $f(x)=\dfrac{x^3-1}{x^3+1}$

Hint: You may use the fact that $f'(x)=\dfrac{6x^2}{(x^3+1)^2}$ and $f''(x)=\dfrac{-12x(2x^3-1)}{(x^3+1)^3}$

Domain: $x^3+1 \neq 0$ $\boxed{x \neq -1 \quad \text{VA}}$
solve denom≠0

HA: $\lim\limits_{x\to\pm\infty}\dfrac{x^3-1}{x^3+1}=\lim\limits_{x\to\pm\infty}\dfrac{1-\cancel{1/x^3}^{\,0}}{1+\cancel{1/x^3}^{\,0}}=1$ $\therefore$ HA $y=1$ if get ∞ no HA exists
divide by highest power of denom.

y-int: $y=-1$
set x=0

x-int: $0=\dfrac{x^3-1}{\cancel{x^3+1}}$
set y=0

$0=x^3-1$

$1=x$

CN's: $\dfrac{6x^2}{(x^3+1)^2}=0$
solve $f'(x)=0$ or undefined

$\therefore$CN $x=0, -1$
$y=-1$ $y=$VA

poss P.O.Inf:
solve $f''(x)=0$ or undefined

$$0 = \frac{-12x\left(2x^3-1\right)}{\left(x^3+1\right)^3}, \quad x=0,\ \frac{1}{\sqrt[3]{2}},\ -1$$

$y=-1$, $y=-\frac{1}{3}$, $y=\text{VA}$

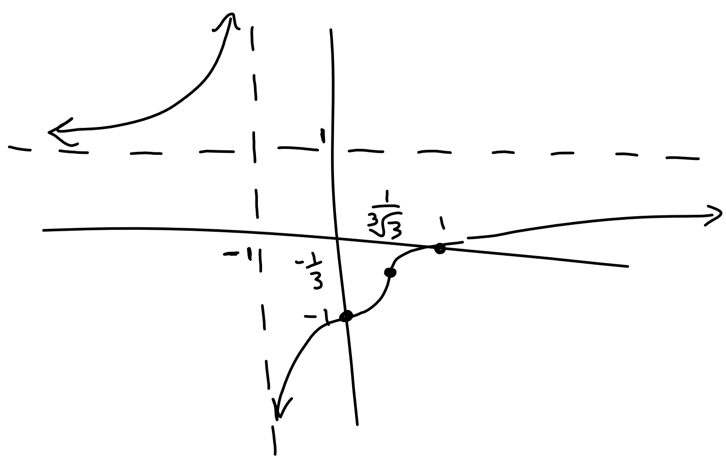
	$-\infty$ to -1	-1 to 0	0 to $\frac{1}{\sqrt[3]{2}}$	$\frac{1}{\sqrt[3]{2}}$ to ∞
$-12x$	+	+	−	−
$2x^3-1$	−	−	−	+
$(x^3+1)^3$	−	+	+	+
f''	+	−	+	−
f	CU	CD		

VA

P.O.Inf.

CN's $x=-1$ and 0 already qualified ∴ no local max/min

(if they weren't see if they fall in CU or CD region for MIN or MAX)



Example: Sketch a graph of $y = f(x)$ with all of the following properties:

(1) Domain: $(-\infty, -1) \cup (1, \infty)$ gap betw

(2) $f(2) = 0$ pt (2,0)

(3) $f(x) = -f(-x)$ odd symmetry about origin

(4) $\underbrace{\lim_{x\to\infty} f(x) = 4}_{\text{HA}}$ and $\underbrace{\lim_{x\to 1^+} f(x) = -\infty}_{\text{VA}}$

(5) $f'(3) = 0$, $f'(5) = 0$ slopes zero ie. max/min/~~saddle~~ pts.

(6) $f'(x) \geq 0$ on $(1, 3) \cup (5, \infty)$ increasing intervals

(7) $f'(x) \leq 0$ on $(3, 5)$ decreasing

(8) local maximum at 3

(9) local minimum at 5

(10) $f''(x) \geq 0$ on $(4, 6)$ CU

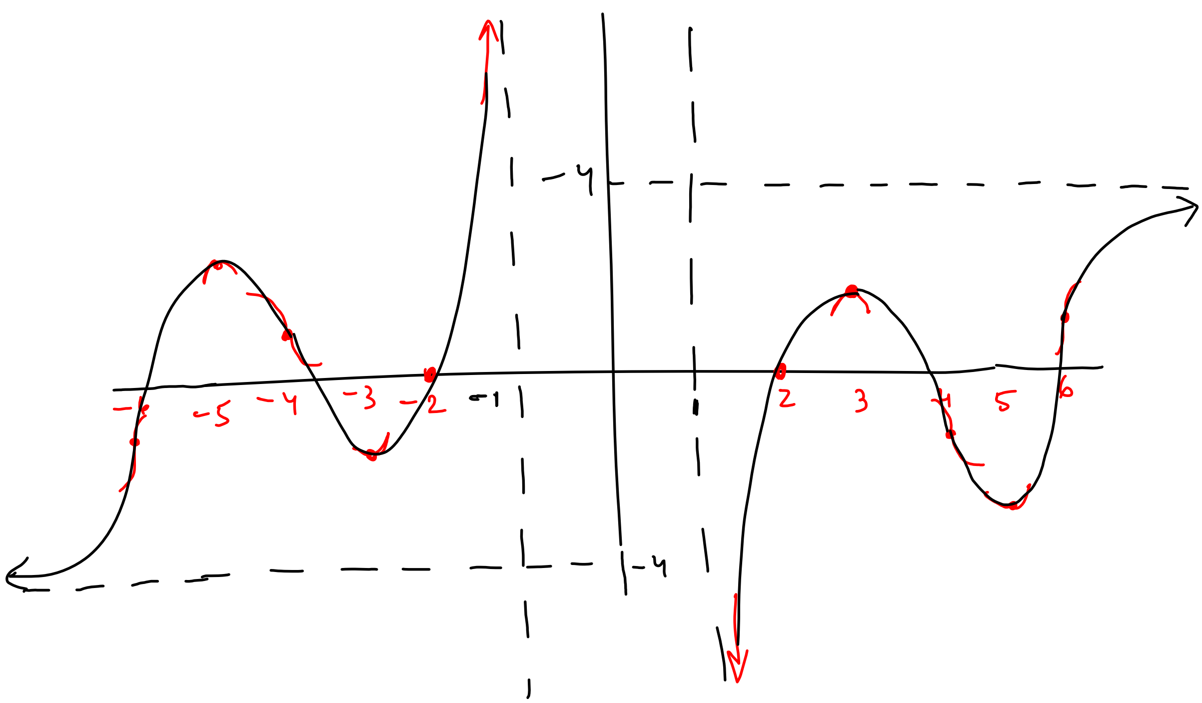
(11) $f''(x) \leq 0$ on $(1, 4) \cup (6, \infty)$ CD

(12) $f''(4) = 0$, $f''(6) = 0$ actual P.O.Inf.

because of symmetry VA at $x = -1$ and 1

HA $y = 4$ ($x \to \infty$) and HA $y = -4$ ($x \to -\infty$)

pt. (2,0) and pt (-2,0)



Optimization Problems (4.7)

Optimization problems are essentially problems of finding the absolute maximum or minimum of a function (which we already know how to do!), but we must first be able to set up the problem.

Steps in Solving Optimization Problems:

1. Understand the problem. Draw a diagram Introduce notation (variables).
2. Figure out the variable to be maximized, and use the information in the question to express this variable in terms of one other variable.
3. Find domain
4. Find the absolute maximum or minimum of the function on its domain. Show that the critical point is indeed an absolute max or min!

- 1st or 2nd derivative test • Closed Interval Method. • using limits on open interval

5. Answer the question

Example: Find two numbers whose difference is 100 and whose product is a minimum.

let x and y be the numbers and P be product

then $y - x = 100$

and $P = yx$ ← this is to be minimized ∴ need to be in terms of either x or y only

→ solve for $y = 100 + x$ and sub in P

$P = (100 + x)(x)$

$P = 100x + x^2$ → Domain $x \in \mathbb{R}$ as a polyn function AND x is not restricted to be in a particular range in ques.

Minimum will occur if $P'(x) = 0$

$P' = 100 + 2x$

$0 = 2(50 + x)$ CN is $x = -50$

to show MINIMUM can use 2nd deriv. test

$P'' = 2 > 0$

∴ P always CU

∴ $x = -50$ is MIN

Now answer the question

∴ The two numbers are -50 and 50 to get min Prod $= -2500$

Example: What angle θ between two edges of length 3 will result in an isosceles triangle with the largest area?

$A = \frac{1}{2}bh$

$\cos\left(\frac{\theta}{2}\right) = \frac{y}{3}$

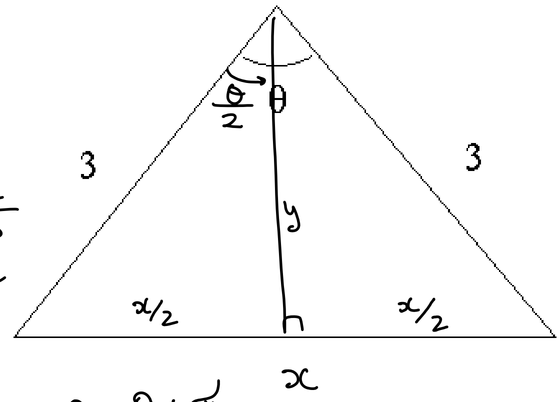
$A = \frac{1}{2}xy$

$\therefore 3\cos\frac{\theta}{2} = y$

convert to be in terms of one variable θ

$\sin\left(\frac{\theta}{2}\right) = \frac{x/2}{3}$

$\therefore 6\sin\frac{\theta}{2} = x$



$A = \frac{1}{2}\left(6\sin\left(\frac{\theta}{2}\right)\right)\left(3\cos\left(\frac{\theta}{2}\right)\right)$

domain $0 \le \theta \le \pi$

$A = 9\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$A' = \frac{dA}{d\theta} = 9\sin\left(\frac{\theta}{2}\right)\left(-\sin\left(\frac{\theta}{2}\right)\right)\left(\frac{1}{2}\right) + 9\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)$

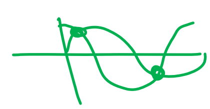
$0 = -\frac{9}{2}\sin^2\left(\frac{\theta}{2}\right) + \frac{9}{2}\cos^2\left(\frac{\theta}{2}\right)$ for max A find $A' = 0$

$\frac{9}{2}\sin^2\left(\frac{\theta}{2}\right) = \frac{9}{2}\cos^2\left(\frac{\theta}{2}\right)$

$\sin^2\left(\frac{\theta}{2}\right) = \cos^2\left(\frac{\theta}{2}\right)$

$\sin\left(\frac{\theta}{2}\right) = \pm\cos\left(\frac{\theta}{2}\right)$

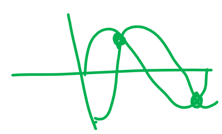
should know



$\sin\alpha = \cos\alpha$ if $\alpha = \pi/4$ or $\frac{5\pi}{4}$

$\therefore \frac{\theta}{2} = \pi/4, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

and $\sin\alpha = -\cos\alpha$ then $\alpha = 3\pi/4, \frac{7\pi}{4}$



$\therefore \theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}}$

not in domain $0 \le \theta \le \pi$

now show max at $\theta = \pi/2$ using closed interval method:

$A(0) = 9\cancel{\sin(0/2)}\cos\left(\frac{0}{2}\right) = 0$

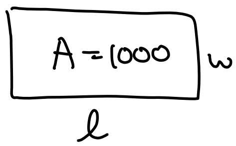
$A(\pi/2) = 9\sin\left(\frac{\pi/2}{2}\right)\cos\left(\frac{\pi/2}{2}\right) = 9\sin\pi/4\cos\pi/4 = 9\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{9}{2}$

$A(\pi) = 9\sin\left(\frac{\pi}{2}\right)\cancel{\cos\left(\frac{\pi}{2}\right)} = 0$

this shows on $x = \pi/2$ is max.

$\therefore$ at $x = \pi/2$ you have max Area at $\frac{9}{2}$ of the isosceles Δ.

Example: A farmer wants to fence off a rectangular field with an area of 1000m^2. What dimensions should it be so that fencing costs are minimized?



$1000 = lw$

minimize cost
is like minimizing perimeter

$P = 2l + 2w$ want in terms of l or w

$\frac{1000}{w} = l$

$P = 2\left(\frac{1000}{w}\right) + 2w$

$P = \frac{2000}{w} + 2w$ domain $w > 0$ or $w \in (0, \infty)$

$P' = -\frac{2000}{w^2} + 2$ minimum at $P' = 0$

$0 = -\frac{2000}{w^2} + 2$

$2000 = 2w^2$

$l = \frac{1000}{\sqrt{1000}} = \sqrt{1000}$

$\pm\sqrt{1000} = w$

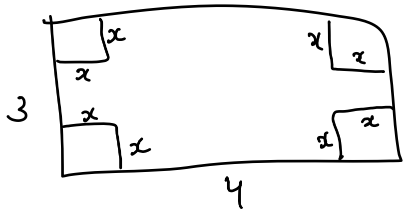
$\sqrt{1000} = w$ CN now to show minimum will use limits this time to mimic closed interval method

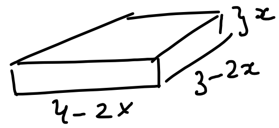
- $P(0)$ is undefined but $\lim\limits_{w \to 0} \frac{2000}{w}^{\nearrow \infty} + 2w^{\nearrow 0} = \infty$
- $P(\sqrt{1000}) = 2(\sqrt{1000}) + 2(\sqrt{1000}) \doteq 126.49$
- $P(\infty)$ can only be done with a limit
  $\lim\limits_{w \to \infty} \frac{2000}{w}^{\nearrow 0} + 2w^{\nearrow \infty} = \infty$

this shows CN $w = \sqrt{1000}$ is a minimum

$\therefore$ to minimize cost use
$l = w = \sqrt{1000}$ m

Example: A sheet of cardboard 3 ft. by 4 ft. will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume?





$$V = x(3-2x)(4-2x)$$
$$V = x(12-14x+4x^2)$$
$$V = 12x - 14x^2 + 4x^3$$

domain $x \geq 0$ $\quad 4-2x \geq 0$, $4 \geq 2x$, $2 \geq x$ $\quad 3-2x \geq 0$, $3 \geq 2x$, $1.5 \geq x$

$\therefore$ domain $x \in [0, 1.5]$

$$V' = 12 - 28x + 12x^2$$
$$0 = 4(3x^2 - 7x + 3)$$
$$x = \frac{7+\sqrt{13}}{6} \quad \text{or} \quad x = \frac{7-\sqrt{13}}{6}$$
$$x \doteq \cancel{1.77} \quad \text{Too big for domain} \qquad x \doteq 0.565741454$$

now show MAX. using closed interval method

$$\left.\begin{array}{l} V(0) = 0 \\ V(0.565741454) \doteq 3.032 \\ V(1.5) = 0 \end{array}\right\}$$

$\therefore$ MAX at
$x \doteq 0.5657\ldots$ ft
with Volume $= 3.032$ ft^3

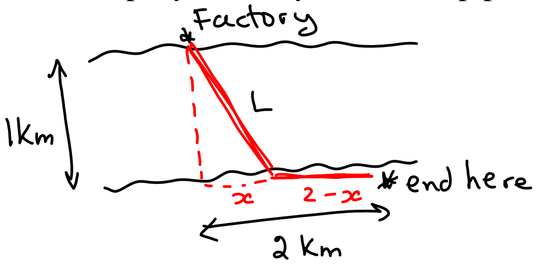
dimensions are
$h \doteq 0.57$ $\quad x$
$l \doteq 2.87$ $\quad 4-2x$
$w \doteq 1.87$ $\quad 3-2x$

OR use 2nd deriv. test

$$V'' = -28 + 24x$$
$$V''(0.56) = \text{neg} \quad \therefore CD \quad \therefore MAX$$

choose a method that's fastest.

Application: A factory needs to run a pipeline across a 1 km wide river to a point that is also 2 km east of its factory. It costs \$5 million/km of pipeline built under the river, and \$3 million/km of pipeline built on land. In order to minimize costs, how far to the east of the company's factory should the pipeline be when it crosses out of the river?



$$x^2+1^2=L^2$$

$$\sqrt{x^2+1}=L$$

$$\text{Cost}=5L+3(2-x)$$

$$C=5\left(\sqrt{x^2+1}\right)+6-3x$$

domain $x\in[0,2]$

$$C'=5\left(\frac{1}{\cancel{2}}\right)(x^2+1)^{-1/2}(\cancel{2}x)-3$$

$$3=\frac{5x}{\sqrt{x^2+1}}$$

$$3\sqrt{x^2+1}=5x$$

sq. root

$$9(x^2+1)=25x^2$$

$$9x^2+9=25x^2$$

$$9=16x^2$$

$$\frac{9}{16}=x^2$$

$$\pm\frac{3}{4}=x$$

only $x=3/4$ in domain

show minimum on $[0,2]$

$$C(0)=5\sqrt{1+0^2}+3(2-0)=11$$

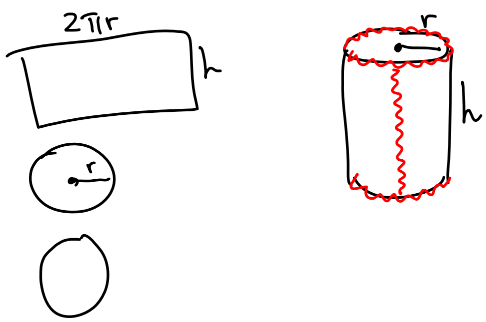
$$C(3/4)=5\sqrt{1+(3/4)^2}+3(2-3/4)=10$$

$$C(2)=5\sqrt{1+2^2}+3(2-2)\doteq 11.2$$

this shows min at $x=3/4$

$\therefore$ pipeline should run $3/4$ km east of the factory when it crosses the river.

Application: Consider a cylindrical tin can which is to be constructed by joining the ends of a rectangular piece of metal to form the cylindrical side, and then attaching circular pieces to form the top and bottom. There are seams around the perimeter of the top and bottom and there is one seam down the side surface (where the ends of the rectangle join together). Suppose the volume of the can is 1000cm^3. Also suppose that the cost of the material is \$1.00 per m^2 and the cost of the seam is \$0.20 per meter. Find the dimensions of the can that will minimize the cost.



$V = \pi r^2 h$

$1000 = \pi r^2 h$

$SA = (\underbrace{2\pi r}_{l})(\underbrace{h}_{w}) + 2(\pi r^2)$

seam $= h + 2(2\pi r)$

$\frac{\$0.20}{m} \times \frac{1m}{100cm} = \frac{\$0.002}{cm}$

Cost = 1.00 (Surface Area) + 0.002(length of seam)

$C = 2\pi r h + 2\pi r^2 + 0.002h + 0.008\pi r$

need one variable

$\frac{1000}{\pi r^2} = h$

$C = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 + 0.002\left(\frac{1000}{\pi r^2}\right) + 0.008\pi r$

$C = \frac{2000}{r} + 2\pi r^2 + \frac{2}{\pi r^2} + 0.008\pi r$

domain $r > 0$

$h > 0$

$\frac{1000}{\pi r^2} > 0$

get nothing new from here.

$C' = -\frac{2000}{r^2} + 4\pi r - \frac{4}{\pi r^3} + 0.008\pi$

$\frac{0}{1} = \frac{-2000\pi r + 4\pi^2 r^4 - 4 + 0.008\pi^2 r^3}{\pi r^3}$

$0 = 4\pi^2 r^4 + 0.008\pi^2 r^3 - 2000\pi r - 4$

need technology to factor quartic

~~$r \doteq -0.001$~~ not in domain

$r \doteq 5.4$

$C(0) \to \lim_{r\to 0} \frac{2000}{r}\nearrow^{\infty} + 2\pi r^2 \nearrow^{0} + \frac{2}{\pi r^2}\nearrow^{\infty} + 0.008\pi r \nearrow^{0} = \infty$

$C(5.4) \doteq 553.74$

$C(\infty) \to \lim_{r\to\infty} \frac{2000}{r}\nearrow^{0} + 2\pi r^2\nearrow^{\infty} + \frac{2}{\pi r^2}\nearrow^{0} + 0.008\pi r\nearrow^{\infty} = \infty$

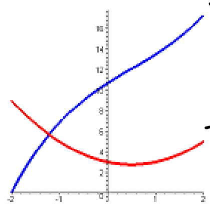
this shows Min Cost of \$553.74 at $r \doteq 5.4$, $h \doteq 10.9$

check $C(6.4) \doteq \$570$, $C(4.4) \doteq 576$

Antiderivatives (4.9)

So far, given a function, we know how to find a rate of change, but what if all we knew was how a function was changing with time, and we wanted to find out about the function itself? ex. given velocity what is the position function?

> **Definition:** A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.



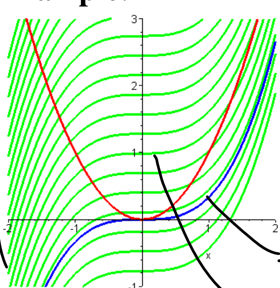
$F(x)$ - antiderivative of a function $f(x)$ is a new function whose slope is given by $f(x)$.

$f(x)$

$$\frac{d}{dx}F(x) = f(x)$$

Example:

family of solutions



$\frac{1}{3}x^3$

$\frac{1}{3}x^2 + 1$

If $f(x) = x^2$

try $F(x) = \frac{1}{3}x^3$

does this work? $F'(x) = \frac{1}{3}(3)(x^2) = x^2$ ✓

but what about $F(x) = \frac{1}{3}x^3 + 1$

$F'(x) = x^2 + 0$ ✓ also works

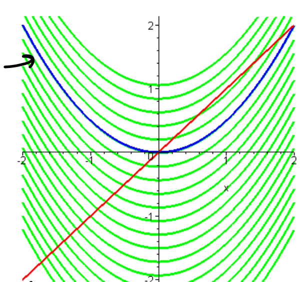
$\therefore$ any shift of $\frac{1}{3}x^3$ will work

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$ any constant

where C is an arbitrary constant.

$F(x)$



$f(x)$

so for above ex.
proper way to write it is $F(x) = \frac{1}{3}x^3 + C$

Here $f(x) = x$

then $F(x) = \frac{1}{2}x^2 + C$

add one to power and divide by that resulting exponent

Examples: 1) $f(x) = x^3$ $\quad F(x) = \frac{1}{4}x^4 + C$

will not work for $\frac{1}{x} = x^{-1}$

this one as antideriv. $\ln|x|$

2) $f(x) = x^{-2}$ $\quad F(x) = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

3) $f(x) = x^{1/4}$ $\quad F(x) = \frac{x^{5/4}}{5/4} + C = \frac{4}{5}x^{5/4} + C$

To deal with more complicated functions, we have the following properties to help us:

- If $F(x)$ is an antiderivative of $f(x)$, then $k \cdot F(x)$ is an antiderivative of $k \cdot f(x)$
- If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$

Examples: 1) $f(x) = 2x^8 + 3x + 1x^0$ $\quad F(x) = \frac{2x^9}{9} + \frac{3x^2}{2} + \frac{1x^1}{1} + C$

2) $f(x) = 5\sin x + 2x^{-1/2}$ $\quad F(x) = -5\cos x + \frac{2x^{1/2}}{\frac{1}{2}} + C = -5\cos x + 4x^{1/2} + C$

Table of Useful Antidifferentiation Formulas:

$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n (for $n \neq -1$)	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln\|x\|$ need to make defined	$\sec x \tan x$	$\sec x$
e^x	e^x	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\sin x$	$-\cos x$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$	$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$

can use instead

Examples:

$f(x) = 7\sec x \tan x + \frac{1}{4}x^{-6}$ $\quad F(x) = 7\sec x + \frac{1}{4}\frac{x^{-5}}{-5} + C = 7\sec x - \frac{1}{20}x^{-5} + C$

$f(x) = \frac{8}{x} - \frac{3}{\sqrt{1-x^2}}$ $\quad F(x) = 8\ln|x| - 3\sin^{-1}x + C$

$$f(x)=\frac{x^2+3x}{\sqrt{x}}$$
$$=x^{3/2}+3x^{1/2}$$

$$\therefore F(x)=\frac{x^{5/2}}{5/2}+\frac{3x^{3/2}}{3/2}+C$$
$$=\frac{2}{5}x^{5/2}+2x^{3/2}+C$$

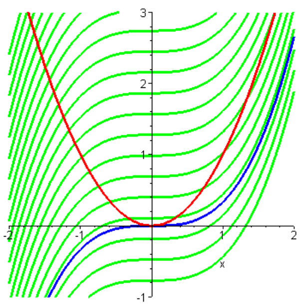
Some More Examples:

If $f'(x)=8\cos x-\dfrac{5}{1+x^2}$, find $f(x)$. $f(x)=8\sin x-5\tan^{-1}x+C$

If $f'(x)=\sqrt{x}(x+1)$, find $f(x)$.
$=x^{3/2}+x^{1/2}$

$$f(x)=\frac{x^{5/2}}{5/2}+\frac{x^{3/2}}{3/2}+C=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C$$

Recall: Because of the arbitrary constant of integration, we end up finding a family of solutions.



ex. Antiderivative of x^2 is $\frac{1}{3}x^3+C$

To find out the value of C (the constant of integration) you'd have to be given additional information

Example: If $f'(x)=x^2$ and $f(0)=5$, find $f(x)$.

$f(x)=\frac{1}{3}x^3+C$ sub $(0,5)$

$5=\frac{1}{3}(0)^3+C$

$5=C$

$\therefore f(x)=\frac{1}{3}x^3+5$

Example: If $f'(x)=\dfrac{x^5+3\sqrt{x}}{x^3}$ and $f(1)=\dfrac{1}{3}$, find $f(x)$.

$=x^2+3x^{-5/2}$

$$f(x)=\frac{1}{3}x^3+\frac{3x^{-3/2}}{-3/2}+C$$

$\frac{1}{3}=\frac{1}{3}(1)^3-2(1)^{-3/2}+C$

$\frac{1}{3}=\frac{1}{3}-2+C$ $\therefore C=2$

$\therefore f(x)=\frac{1}{3}x^3-2x^{-3/2}+2$

Example: If $f''(x) = \sin x + x$ with $f(0) = 7$ and $f'(0) = 2$, find $f(x)$.

$f'(x) = -\cos x + \frac{1}{2}x^2 + C$ sub in $f'(0) = 2$

$2 = -\cos(0) + \frac{1}{2}(0)^2 + C \quad \therefore C = 3$

$\therefore f'(x) = -\cos x + \frac{1}{2}x^2 + 3$

$f(x) = -\sin x + \frac{1}{6}x^3 + 3x + D$ sub in $f(0) = 7$

$7 = -\sin(0) + \frac{1}{6}(0)^3 + 3(0) + D \quad \therefore D = 0$

$$\boxed{\therefore f(x) = -\sin x + \frac{1}{6}x^3 + 3x}$$

Sometimes, it is difficult or impossible to find the antiderivative of a function, but we can still gather info about it graphically.

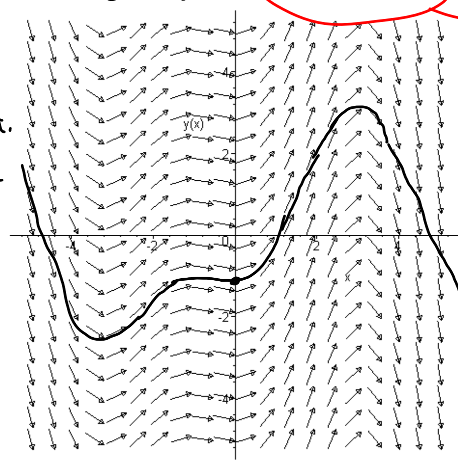
graphical representation of a solution to 1st order diff. eqtn (ch.9)

A **direction field**, which shows the slope at given points, can be used to sketch a graph of the antiderivative of a function. E.g. If $f'(x) = \sin x \cdot (x+1)$ and $f(0) = -1$, sketch $f(x)$.

need integration by parts method (Ch7)

This shows slope that f(x) has to have @ each pt.

To draw f(x) must go through the given point of $f(0) = -1$ and stay parallel to the slope arrows



$f'(0) = 0$ slope

$f'(2) = \sin 2(2+1) \doteq 2.7$ slope

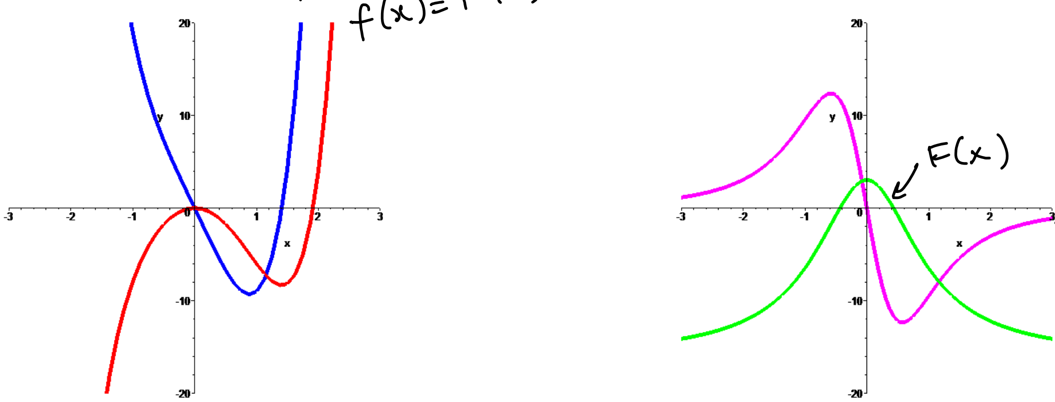
$f'(4) = \sin 4\,(4+1) \doteq -3.78$ slope

etc ...

Example: In each of the graphs below, determine which curve is $f(x)$, and which curve is the antiderivative of $f(x)$.

$F(x) =$ antideriv. of $f(x)$

$f(x) = F'(x) =$ slope of $F(x)$



Application: Suppose the rate of change of concentration of a vitamin in the bloodstream at time *t* is given by

$$\frac{dc}{dt} = -0.1e^{-0.3t}$$

$t=0$ $c=1$

If there is initially 1mg of the vitamin in the bloodstream, then what is the concentration as a function of time?
[Source: Modified from "Calculus for Biology and Medicine" by Claudia Neuhauser, 2nd ed., Prentice Hall, 2004]

$C = \frac{-0.1e^{-0.3t}}{-0.3} + D$

find constant D if $t=0$, $c=1$

$1 = \frac{0.1e^{-0.3(0)}}{0.3} + D$

$1 = \frac{1}{3} + D$

$\frac{2}{3} = D$

$\therefore C(t) = \frac{1}{3}e^{-0.3t} + \frac{2}{3}$

Application: A ball is thrown upward with a speed of 10 m/s from a building that is 30m tall. Find a formula describing the height of the ball above the ground *t* seconds later.

Choose up as positive

We know acceleration due to gravity

$a = -9.8$

$\boxed{a = v'(t)}$

$\therefore v(t) = -9.8t + c$

sub $v(0) = 10$

$10 = -9.8(0) + c$ $\therefore c = 10$

$\therefore v(t) = -9.8t + 10$

$\boxed{v(t) = s'(t)}$

$\therefore s(t) = -\frac{9.8}{2}t^2 + 10t + D$

sub $s(0) = 30$

$30 = -4.9(0)^2 + 10(0) + D$

$30 = D$

$\boxed{\therefore s(t) = -4.9t^2 + 10t + 30}$

areas under
s – position
v – velocity
a – acceleration
slopes

Note:
not all functions are exactly set up to be recognizable antiderivative
$\therefore$ need proper definition of the integral using areas under the curve (Ch. 5)

cool to be able to find position function with only knowing acceleration, speed + position at the beginning! :)