

nNOTESfixed2012



newTrigIdU

nitNOTES

Inserted from: <<file:///C:/Users/MrsK/Desktop/12U%202011/7%20TrigId&Eqtn/newTrigIdUnitNOTES.docx>>

↓ See below

Trig Identities & Equations Unit 7

Tentative TEST date _____



Big idea/Learning Goals

In this unit you will learn how to find exact values of angles that are not directly related to the special triangles. This is only possible with the use of trig identities, so you will start with trig identity proofs where the use of proper notation is key to minimize errors as well as to get full marks. You will also learn how to solve more complicated trig equations that involve factoring not just the unit circle.

Corrections for the textbook answers:



Success Criteria

- I am ready for this unit if I am confident in the following review topics
 - Function notation
 - Fundamental trig identities from gr11
 - Unit circle
 - Special triangles
 - Factoring
 - Isolating a variable
 - Substituting values
- I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? <small>You may be asked to show them</small>
Apr. 30	2-3	Equivalent Trig Expressions = Identities Section 7.1 + Handout	
	4-6	Compound Angle Formula Section 7.2 & Handout	
May 2	7-9	Double Angle Formula (or Half Angle Formula) Section 7.3 & Handout	
	10-11	Proving Trig Identities Section 7.4 & Handout	(TWO)
May 7	12-13	Solving Trig Equations Section 7.5 & 7.6 & Handout	
		EXTRA - compound and double identities handout	



Reflect – previous TEST mark _____, Overall mark now _____.
Looking back, what can you improve upon?

Equivalent Trig Expressions = Identities

1. Clarify for yourself the difference between the following three words, use examples in your explanations.

Expression **SIMPLIFY** these
- no equals sign

$$\text{ex. } 2x^2 - 8$$

Equation **SOLVE** these
- has equals sign

$$\text{ex. } 2x^2 - 9 = -1$$

- only true for specific x values

Identity **PROVE** these
- has equals sign

$$\text{ex. } 2(x+2)(x-2) = 2x^2 - 8$$

$$2(x^2 - 4)$$

$$2x^2 - 8 = 2x^2 - 8$$

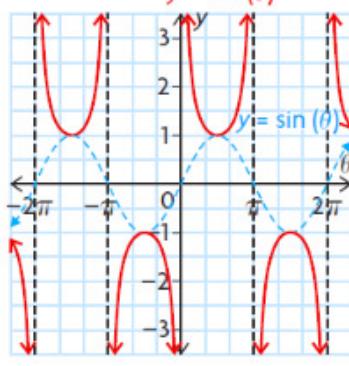
- true for all x values

2. Understand how to use all the identities (grouped by topic) ALL formulas are found on the last page

- a. Odd and even function properties

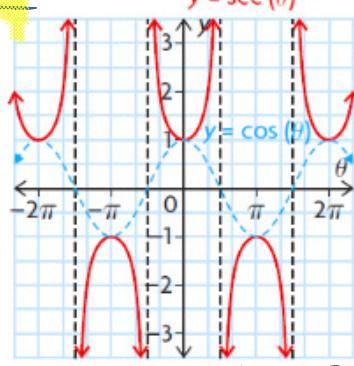


EVEN: $f(-x) = f(x)$



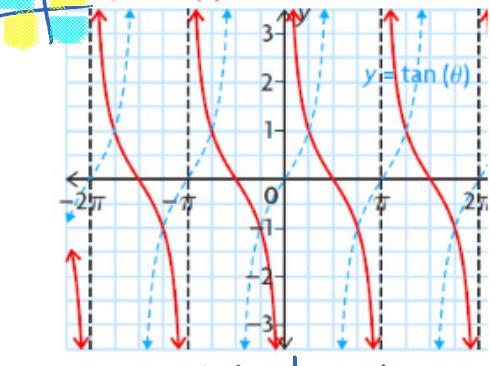
Sine + cosecant are ODD
 $\therefore \sin(-\theta) = -\sin\theta$
 $\csc(-\theta) = -\csc\theta$

ODD: $f(-x) = -f(x)$



Cosine + secant are EVEN
 $\therefore \cos(-\theta) = \cos\theta$
 $\sec(-\theta) = \sec\theta$

ODD: $y = \cot(\theta)$



Tangent + cotangent are ODD
 $\therefore \cot(-\theta) = -\cot\theta$
 $\tan(-\theta) = -\tan\theta$



- b. Practice writing the equivalent expressions for the following

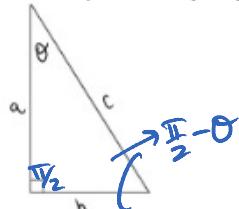
$$\text{i) } \sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}$$

$$\text{ii) } \sec\left(\frac{\pi}{2}\right) = \sec\left(-\frac{\pi}{2}\right)$$

$$\text{iii) } \cos(-A-B) = \cos(A+B)$$



- c. Complementary angles property = Cofunction identities



$$\sin\theta = \frac{b}{c} = \cos(\frac{\pi}{2} - \theta)$$

$$\sin(\frac{\pi}{2} - \theta) = \frac{a}{c} = \cos\theta$$

$$\cot\theta = \frac{a}{b} = \tan(\frac{\pi}{2} - \theta)$$

$$\therefore \sin\theta = \cos(\frac{\pi}{2} - \theta)$$

$$\therefore \cos\theta = \sin(\frac{\pi}{2} - \theta)$$

$$\therefore \cot\theta = \tan(\frac{\pi}{2} - \theta)$$

etc ...



- d. Practice writing the equivalent expressions for the following

$$\text{i) } \sin\left(\frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{3}\right)$$

$$\text{ii) } \sec\left(\frac{\pi}{8}\right)$$

$$= \csc\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$

$$= \csc\left(\frac{3\pi}{8}\right)$$

$$\text{iii) } \cot\left(2x + \frac{\pi}{4}\right)$$

$$= \tan\left(\frac{\pi}{2} - 2x - \frac{\pi}{4}\right)$$

$$= \tan\left(\frac{1}{4}\pi - 2x\right)$$

c. Horizontal translations of period multiples

Adding the period within the input, gives the same output

period = 2π → sine, cosine, cosecant, secant

period = π → tangent, cotangent

ex. $\tan(\theta) = \tan(\theta \pm \pi)$ etc...



Practice writing the equivalent expressions for the following

$$\text{i)} \cos\left(\frac{\pi}{3}\right) \xrightarrow{+2\pi} \cos\left(\frac{7\pi}{3}\right)$$

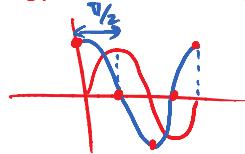
$$\text{ii)} \csc\left(\frac{3\pi}{8}\right) \xrightarrow{-2\pi \times 8} \csc\left(\frac{-13\pi}{8}\right)$$

$$\text{iii)} \tan(x - 4\pi) \xrightarrow{+4\pi} \tan(x - 3\pi) \xrightarrow{+2\pi} \tan(x - \pi) \xrightarrow{+2\pi} \tan x$$



d. Similar shape of the graph just with a shift

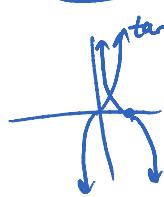
sine similar to cosine



$$\sin(\theta + \pi/2) = \cos(\theta)$$

$$\cos(\theta - \pi/2) = \sin(\theta)$$

} similar to
csc + sec



$$-\cot(\theta - \pi/2) = \tan(\theta)$$



Practice writing the equivalent expressions for the following

$$\text{i)} \sin\left(\frac{7\pi}{6}\right) \xrightarrow{= \cos\left(\frac{7\pi}{6} - \frac{\pi}{2} + 3\right)} \cos\left(\frac{4\pi}{6}\right) \xrightarrow{= \cos\left(\frac{2\pi}{3}\right)}$$

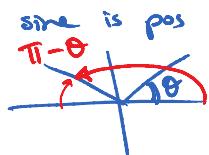
$$\text{ii)} \sec\left(\frac{4\pi}{5}\right)$$

$$\text{iii)} \cot(2x)$$

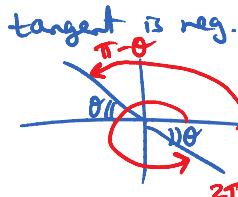


e. Principal and related acute angle characteristics

CAST



$$\therefore \sin(\theta) = \sin(\pi - \theta)$$



$$\tan(\pi - \theta) = \tan(2\pi - \theta)$$

etc ...



Practice writing the equivalent expressions for the following

$$\text{i)} \sin\left(\frac{11\pi}{6}\right) \xrightarrow{=} \sin\left(\frac{7\pi}{6}\right)$$

$$\text{ii)} \cos\left(\frac{5\pi}{7}\right) \xrightarrow{=} \cos\left(\frac{9\pi}{7}\right)$$

$$\text{iii)} \tan\left(-\frac{\pi}{4}\right)$$

3. Write down several equivalent expressions for the following, identify what property was used for each

Eg. a)

$$\sin\frac{5\pi}{6}$$

$$\begin{aligned} &= -\sin\left(-\frac{5\pi}{6}\right) \quad (\text{odd function}) \\ &= \cos\left(\frac{\pi}{2} - \frac{5\pi}{6}\right) \quad (\text{cofunction}) \\ &= \cos\left(-\frac{2\pi}{6}\right) \\ &= \cos\left(-\frac{\pi}{3}\right) \\ &= \sin\left(\frac{5\pi}{6}\right) \end{aligned}$$

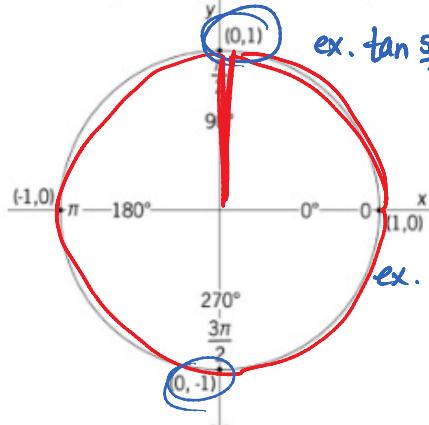
b)

$$\cos\frac{3\pi}{4}$$

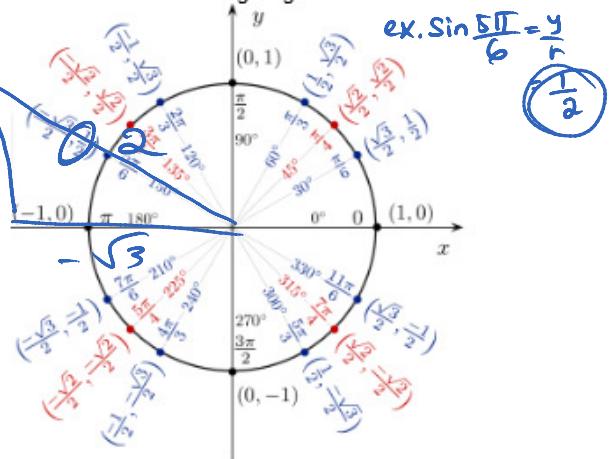
$$\begin{aligned} &= \cos\frac{\pi}{2} \quad (\text{principal}) \\ &= \sin\left(\frac{\pi}{2} - \frac{3\pi}{4}\right) \quad (\text{cofunction}) \\ &= \sin\left(-\frac{3\pi}{4}\right) \\ &= -\sin\left(\frac{3\pi}{4}\right) \quad (\text{odd}) \end{aligned}$$

Compound Angle Formula

Recall that you can find exact values for angles that fall on the x and y axes by using the four points on the unit circle.



Recall also that special triangles help you to find exact values for the following angles.



1. In this lesson you will learn how to find exact values of angles that are not directly related to the special triangles. For example, $\cos 105^\circ$. You may think that the following can be done but try it with the calculator to see that trig functions CANNOT be distributed.

$$\cos 105^\circ = \cos(135^\circ - 30^\circ) \neq \cos 135^\circ - \cos 30^\circ$$

~~-0.2588 ...~~ ~~= -1.5731 ...~~
not the same

You may recall that function notation uses brackets however it does NOT mean to multiply and so you cannot distribute it. This is especially important in this lesson.

Instead of distributing the following identity is true, check it with the use of calculator.

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos 105^\circ = \cos(135^\circ - 30^\circ) = \cos 135^\circ \cos 30^\circ + \sin 135^\circ \sin 30^\circ$$

-0.2588 ...

2. In universities you will be required to follow proofs to theorems that professors do in class. Read the following proof and make notes in the margins of what is going on to help you understand it.

Proof for

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$c^2 = 1^2 + 1^2 - 2(1)(1) \cos(A - B) \quad \text{cosine law on } \Delta$$

$$c^2 = 2 - 2 \cos(A - B) \quad \text{simplified}$$

$$(\sin A - \sin B)^2 + (\cos B - \cos A)^2 = c^2 \quad \text{pythagorean on } \Delta$$

$$\text{Substitute equation 1 into 2} \quad \text{equation 2 expanded}$$

$$\sin^2 A - 2 \sin A \sin B + \sin^2 B + \cos^2 B - 2 \cos A \cos B + \cos^2 A = 2 - 2 \cos(A - B)$$

$$1 + 1 - 2 \sin A \sin B - 2 \cos A \cos B = 2 - 2 \cos(A - B)$$

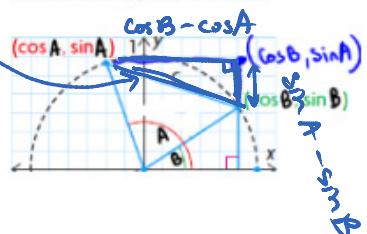
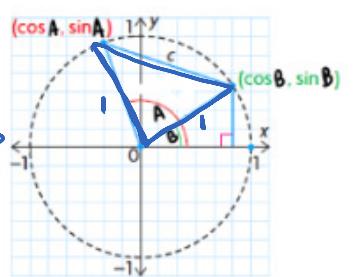
used pythag. id.

$$\sin A \sin B + \cos A \cos B = \cos(A - B)$$

$$\sin^2 A + \cos^2 A = 1$$

$$\sin^2 B + \cos^2 B = 1$$

and divide by -2



3. Identify the steps in the following completed proofs, if the proof is not there, prove it.

i) Identities used:

a $\cos(A - B) = \cos A \cos B + \sin A \sin B$

b $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$

c $\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Proof for

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned} \sin(A + B) &= \cos\left(\frac{\pi}{2} - (A + B)\right) \quad \text{use } b \\ &= \cos\left(\left(\frac{\pi}{2} - A\right) - B\right) \quad \text{multiply neg in regroup.} \\ &= \cos\left(\frac{\pi}{2} - A\right) \cos B + \sin\left(\frac{\pi}{2} - A\right) \sin B \quad \text{use } c \\ &= \sin A \cos B + \cos A \sin B \end{aligned}$$

ii) Identities used:

a $\cos(A - B) = \cos A \cos B + \sin A \sin B$

d $\cos(-\theta) = \cos \theta$

e $\sin(-\theta) = -\sin \theta$

Proof for

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned} \cos(A + B) &= \cos(A - (-B)) \quad \text{rewritten with negatives.} \\ &= \cos A \cos(-B) + \sin A \sin(-B) \quad \text{use } d \\ &= \cos A \cos B + \sin A(-\sin B) \quad \text{use } e \text{ and } c \\ &= \cos A \cos B - \sin A \sin B \quad \text{rewrite.} \end{aligned}$$

iii) Identities to use:

f $\sin(A - B) = \sin A \cos B + \cos A \sin B$

d $\cos(-\theta) = \cos \theta$

e $\sin(-\theta) = -\sin \theta$

Proof for

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\begin{aligned} \sin(A - B) &= \sin(A + (-B)) \quad \text{use } f \\ &= \sin(A) \cos(-B) + \cos(A) \sin(-B) \quad \text{use } d \\ &= \sin A \cos B + \cos A(-\sin B) \quad \text{use } e \\ &= \sin A \cos B - \cos A \sin B \end{aligned}$$

iv) Identities used:

f $\sin(A + B) = \sin A \cos B + \cos A \sin B$

g $\cos(A + B) = \cos A \cos B - \sin A \sin B$

h $\tan \theta = \frac{\sin \theta}{\cos \theta}$

Proof for

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} \quad \text{use } h$$

$$\begin{aligned} &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \quad \text{use } f \text{ and } g \\ &= \frac{\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\cos A} \sin B \right)}{\left(\frac{\cos A}{\cos A} \cos B - \frac{\sin A}{\cos A} \sin B \right)} \quad \text{divide all terms by } \cos A \cos B \\ &= \frac{\left(\frac{\sin A}{\cos A} \cos B + \frac{\cos A}{\cos A} \sin B \right)}{\left(\frac{\cos A}{\cos A} \cos B - \frac{\sin A}{\cos A} \sin B \right)} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{cancel + use } h \end{aligned}$$

v) Identities to use:

i $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

j $\tan(-\theta) = -\tan \theta$

Proof for

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\begin{aligned} \tan(A - B) &= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \quad \text{use } i \\ &= \frac{\tan A + (-\tan B)}{1 - \tan A(-\tan B)} \quad \text{use } j \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}$$

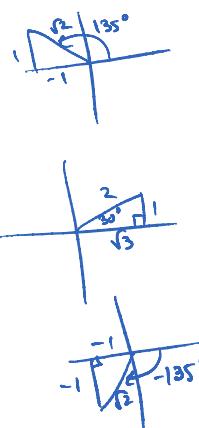
4. Find the exact values of the following

a. $\tan \frac{5\pi}{6}$

easier to see how angles are made with degrees → can use degrees as rough work

$= \tan 150^\circ$ ← already special angle

$$\begin{aligned} &= \frac{\sqrt{3}}{2} \\ &= \frac{1}{-\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3} \end{aligned}$$



b. $\cos \frac{-7\pi}{12}$

(show that you can use the sum or the difference formula to get the same result)

$= \cos(-105^\circ)$ not a special angle
rewrite -105° by using two special angles

$$\begin{aligned} &= \cos(30^\circ - 135^\circ) \\ &= \cos 30^\circ \cos 135^\circ + \sin 30^\circ \sin 135^\circ \\ &= \frac{\sqrt{3}}{2} \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= -\frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

or $= \cos(30^\circ + -135^\circ)$

$$\begin{aligned} &= \cos 30^\circ \cos(-135^\circ) - \sin 30^\circ \sin(-135^\circ) \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\sqrt{3}+1}{2\sqrt{2}} \end{aligned}$$

or $= \cos(255^\circ)$ if you add period

$= \cos(45^\circ + 210^\circ)$

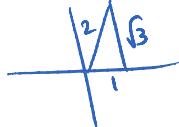
etc... also get the same result

c. $\sec \frac{7\pi}{3}$

$$\begin{aligned} &= \sec 420^\circ \\ &= \sec 60^\circ \\ &= \frac{1}{\cos 60^\circ} \\ &= \frac{1}{\frac{1}{2}} \text{ use special } \Delta \\ &= 2 \end{aligned}$$

*subtract period = 360°
inside input*

*rewrite as cosine
don't do $\cos 60^\circ$!!
reciprocal of ratios
not angles*



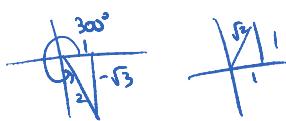
d. $\sin \frac{23\pi}{12}$

 $= \sin 345^\circ$ not special, rewrite using special angles

$= \sin(300^\circ + 45^\circ)$

many choices

$$\begin{aligned} &= \sin 300^\circ \cos 45^\circ + \cos 300^\circ \sin 45^\circ \\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\ &= -\frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$



@	⑥	⑦	⑧	⑨
$\sin(2A) = 2\sin A \cos A$	$\cos(2A) = \cos^2 A - \sin^2 A$	$\cos(2A) = 1 - 2\sin^2 A$	$\cos(2A) = 2\cos^2 A - 1$	$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

3. Which of the identities are useful in finding exact value of $\cos(157.5^\circ)$? Find it.

not special $A = 157.5^\circ$
 but $2A = 2(157.5) = 315^\circ$
 it's "special."
Cos A? use ⑦ since $\cos A$ is there

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 315^\circ = 2\cos^2 157.5^\circ - 1 \quad \text{isolate}$$

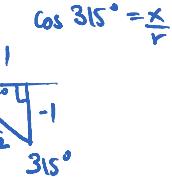
$$\frac{\cos 315^\circ + 1}{2} = \cos^2 157.5^\circ$$

$$\frac{\frac{1}{\sqrt{2}} + 1}{2} = \cos^2 157.5^\circ$$

- not proper form
 - radical in denominator
 - fraction within fraction.

$$\pm \sqrt{\frac{1}{2} \left(\frac{\sqrt{2}}{2} + 1 \right)} = \sqrt{\cos^2 157.5^\circ}$$

$$\text{in II } \textcircled{1} \sqrt{\frac{\sqrt{2}}{4} + \frac{1}{2}} = \cos 157.5^\circ \approx -0.9239\dots$$



4. Which one is useful for the exact value of $\sin 157.5^\circ$? Find it.

$$A = 157.5^\circ$$

$$2A = 315^\circ$$

⑨ $\cos(2A) = 1 - 2\sin^2 A$
 OR
 $\sin^2 A = \frac{1 - \cos 2A}{2}$ same id.
 just rearranged

$$\sin^2 157.5^\circ = \frac{1 - \cos 315^\circ}{2}$$

$$\sin^2 157.5^\circ = \frac{1 - \frac{1}{\sqrt{2}}}{2}$$

$$\sin^2 157.5^\circ = \frac{(1 - \frac{\sqrt{2}}{2})\frac{1}{2}}{2}$$

$$\sin 157.5^\circ = \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}}$$

in II

$$\approx 0.3827\dots$$

5. Find the exact values of the following by using the double angle identities. You may convert to degrees if that is easier for you. (follow instructions even if it is possible to find the answer in a faster way)

a. $\sin \frac{7\pi}{8} \cos \frac{7\pi}{8}$

$$= \sin 157.5^\circ \cos 157.5^\circ \quad \text{resembles ⑦}$$

$$= \frac{\sin(2 \times 157.5^\circ)}{2} \quad \text{since } \sin 2A = 2\sin A \cos A$$

$$= \frac{\sin 315^\circ}{2} \quad \text{since } \sin 2A = \sin A \cos A$$

$$= \frac{-\frac{1}{\sqrt{2}}}{2}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{2}$$

$$= \left(-\frac{\sqrt{2}}{2} \right) \frac{1}{2}$$

$$= -\frac{\sqrt{2}}{4}$$

$$\sin 157.5^\circ \cos 157.5^\circ \approx -0.3535\dots$$

b. $\tan \frac{2\pi}{3} = \tan 120^\circ$ it is special

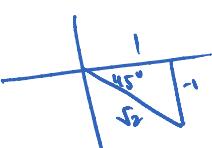
BUT we will practice using an identity.

$$= \frac{2\tan(60)}{1 - \tan^2(60)}$$

$$= \frac{2(\sqrt{3})}{1 - \frac{3}{4}}$$

$$= \frac{2(\sqrt{3})}{1 - \frac{9}{4}}$$

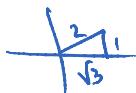
$$= \frac{2\sqrt{3}}{1 - 3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$



c. $\sin \frac{\pi}{3} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$$\begin{aligned} &= 2\sin 30^\circ \cos 30^\circ \\ &= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

practice using double $\sin 2A = 2\sin A \cos A$



d. $\cos \frac{3\pi}{2} = \cos 270^\circ = 0$

$$\begin{aligned} &= 1 - 2\sin^2 135^\circ \\ &= 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2 \\ &= 1 - 2\left(\frac{1}{2}\right) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

practice using double $\cos 2A = 1 - 2\sin^2 A$

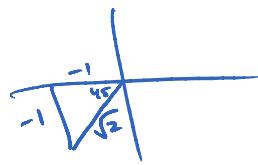


6. Find the exact values of the following by using the half angle identities.

a. $\cos \frac{5\pi}{8}$
 $= \cos 112.5^\circ$
 $= \sqrt{\frac{1 + \cos(225^\circ)}{2}}$
 $= \sqrt{\frac{1 + \frac{x}{r}}{2}}$
 $= \sqrt{\frac{1 + \frac{-1}{\sqrt{2}}}{2}}$
 $= \sqrt{\left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{2}} = \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}}$

$A = 112.5^\circ$
 $2A = 225^\circ$

$\cos^2 A = \frac{1 + \cos 2A}{2} \rightarrow \cos^2 \frac{\theta}{2} = \frac{1 + \cos(\theta)}{2}$



c. $\cos \frac{7\pi}{12} = \cos 105^\circ$
 $A = 105^\circ$
 $2A = 210^\circ$

$\cos^2 A = \frac{1 + \cos 2A}{2}$

$\cos^2 105^\circ = \frac{1 + \cos 210^\circ}{2}$

$\cos^2 105^\circ = \frac{1 + \frac{x}{r}}{2}$

$\cos^2 105^\circ = \left(1 + \frac{-\sqrt{3}}{2}\right)\frac{1}{2}$

$\cos 105^\circ = \text{in II} \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}$

$\therefore \cos 105^\circ = -\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}$

$-\sqrt{\frac{2-\sqrt{3}}{4}}$

b. Find $\cos \frac{\theta}{2}$ if $270^\circ < \theta < 360^\circ$ and $\sin \theta = -\frac{7}{25}$ in IV

$$\cos^2 \frac{\theta}{2} = \frac{1 + \frac{x}{r}}{2}$$

$$=\frac{\frac{28}{25} + \frac{21}{25}}{2}$$

$$=\left(\frac{49}{25}\right)\frac{1}{2}$$

$$\cos^2 \frac{\theta}{2} = \frac{49}{50}$$

$\theta \text{ in IV}$
 $\therefore \frac{\theta}{2} \text{ in II}$

$$\therefore \cos \frac{\theta}{2} = \text{in II} \sqrt{\frac{49}{50}}$$

$$=-\frac{7}{\sqrt{50}} = -\frac{7\sqrt{50}}{50}$$

d. Find $\sin \frac{\theta}{2}$ if $180^\circ < \theta < 270^\circ$ and $\cos \theta = -\frac{15}{17}$

$\sin^2 A = \frac{1 - \cos 2A}{2}$

$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$

$\sin^2 \frac{\theta}{2} = \frac{1 - -\frac{15}{17}}{2}$

$\sin^2 \frac{\theta}{2} = \left(\frac{32}{17}\right)\frac{1}{2}$

$\sin^2 \frac{\theta}{2} = \frac{16}{17}$

$\sin \frac{\theta}{2} = \text{in II} \sqrt{\frac{16}{17}}$

$$= + \frac{4}{\sqrt{17}}$$

$$= \frac{4\sqrt{17}}{17}$$

$\theta \text{ in III}$
 $\text{means } \frac{\theta}{2} \text{ in II}$

Proving Trig Identities

1. The proofs to identities are not unique, hence it is important to write out what you do in each step. Here are some strategies to try when doing proofs.

- Convert everything to be in terms of sine and cosine (list the identities of use here)

$$\csc \theta = \frac{1}{\sin \theta} \quad \text{reciprocal}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad \text{quotient}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

- Use the pythagorean identity you learned in grade 11 (write all 3 ways it could appear)

Pythag: $\sin^2 \theta + \cos^2 \theta = 1$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

- Using the compound, double or other identities learned in this unit
- Factoring
- Expanding
- LCD – only if you have adding or subtracting of separate fractions.
- Distribute the denominator – only if you see a possible cancellation

see FORMULA page. usually move from complicated expression to a simpler one.

$$\frac{2+5}{4} = \frac{2}{4} + \frac{5}{4} \checkmark$$

$$\frac{4}{2+5} \neq \frac{4}{2} + \frac{4}{5} \times$$

Avoid the following mistakes

- Distributing the numerator – can't EVER do this (try with regular numbers to see this)
- Canceling wrong – remember "monomial cancels monomial", "binomial cancels binomial", etc...
- Forgetting to record brackets – especially if the question involves both multiplication and addition
- Forgetting to write the input beside the trig function – cos, sin, tan without θ are meaningless
- Incorrect placement of the exponent $\sin \theta^2 \times$ when you mean $\sin^2 \theta \checkmark (\sin \theta)^2 \checkmark$
- Forgetting to explain the steps you've done

2. Prove the following, explain all the steps.

g)

$$a. \frac{\csc^2 x - 1}{\csc^2 x} = 1 - \sin^2 x$$

$$LS = \frac{\csc^2 x}{\csc^2 x} - \frac{1}{\csc^2 x} \quad \text{distribute denom.}$$

$$b. \cos^4 x - \sin^4 x = \cos 2x$$

LS

$$\begin{aligned} &= 1 - \frac{1}{\csc^2 x} \\ &= 1 - 1 \div \frac{1}{\sin^2 x} \quad \text{replace with sine} \\ &= 1 - 1 \times \frac{\sin^2 x}{1} \quad \text{multiply by reciprocal} \\ &= 1 - \sin^2 x \\ &= RS \end{aligned}$$

$$\begin{aligned} &\text{diff. of sq. factor} \\ &= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) \\ &= 1 (\cos^2 x - \sin^2 x) \quad \text{pythag.} \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \quad \text{double id.} \\ &\therefore LS = RS \end{aligned}$$

g. $\sin 2x + \sin 2y = 2 \sin(x+y) \cos(x-y)$

$$RS = 2(\sin x \cos y + \cos x \sin y)(\cos x \cos y + \sin x \sin y) \rightarrow \text{compound.}$$

FOIL ↓

$$= 2 \left[\begin{matrix} \cancel{\sin x \cos y} & \cancel{\cos^2 y} \\ F & O \end{matrix} + \begin{matrix} \cancel{\sin^2 x} & \cancel{\sin y \cos y} \\ O & I \end{matrix} + \begin{matrix} \cancel{\cos^2 x} & \cancel{\sin y \cos y} \\ I & O \end{matrix} + \begin{matrix} \cancel{\sin x \cos y} & \cancel{\sin^2 y} \\ O & L \end{matrix} \right]$$

↓ Regroup + common factor

$$= 2 \left[\cancel{\sin x \cos y} (\cos^2 y + \sin^2 y) + \cancel{\sin y \cos y} (\sin^2 x + \cos^2 x) \right] \downarrow \begin{matrix} \text{pythag} \\ = 1 \end{matrix}$$

↓ pythag

↓ pythag
= 1

$$= 2(\sin x \cos x + \sin y \cos y) \rightarrow \text{distribute}$$

$$= 2 \sin x \cos x + 2 \sin y \cos y \rightarrow \text{double}$$

$$= \sin 2x + \sin 2y = LS \quad \therefore LS = RS.$$



d. $\frac{\sin 2x}{2} = \sin x$

$$LS = \cancel{\frac{2 \sin x \cos x}{2}} = \cancel{\frac{1}{2} \cancel{2}}$$

double reciprocal of secant

cancel

$$= \sin x = RS$$

$$\therefore LS = RS$$

e. $\frac{1}{\cos^2 x \sin x} - \frac{\sin x}{\cos^2 x} = \csc x$

$$LS = \frac{(1 - \sin^2 x)}{\cos^2 x \sin x} \rightarrow \text{L.C.D.}$$

pythag.

$$= \frac{\cos^2 x}{\cos^2 x \sin x}$$

$$= \frac{1}{\sin x} \rightarrow \text{cancel}$$

reciprocal of sine

Solving Trig Equations – no factoring needed

Solve each of the following for x in $0 \leq x \leq 2\pi$.

Ex. 1. $\frac{2(5\sqrt{3}-3\sec x)}{2} = 6\frac{\sqrt{3}}{2}$

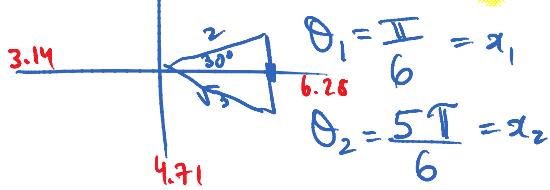
$$5\sqrt{3} - 3\sec x = 3\sqrt{3} - 5\sqrt{3}$$

$$-3\sec x = -2\sqrt{3}$$

$$\sec x = \frac{2\sqrt{3}}{3}$$

$$\therefore \cos x = \frac{3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$1.57 \quad \cos \theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} = \frac{x}{r}$$



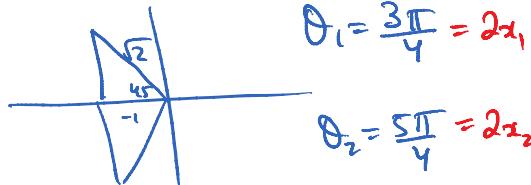
3. $2\cos 2x + \sqrt{2} = 0$

$$\theta = 2x$$

$$2\cos \theta + \sqrt{2} = 0$$

$$\cos \theta = -\frac{\sqrt{2}}{2}$$

$$\cos \theta = -\frac{1}{\sqrt{2}} = \frac{x}{r}$$



$$\therefore x_1 = \frac{3\pi}{8} = 2x_1$$

$$\theta_2 = \frac{5\pi}{4} = 2x_2$$

$$\therefore x_1 = \frac{3\pi}{8}$$

$$x_3 = \frac{11\pi}{8}$$

$$x_2 = \frac{5\pi}{8}$$

$$x_4 = \frac{13\pi}{8}$$

$$\text{period} = \frac{2\pi}{K} = \frac{2\pi}{2} = \pi$$

* if you see "Complicated" input

i.e. not just x

2. $\tan(2x) + \sqrt{3} = 0$

let $\theta = 2x$

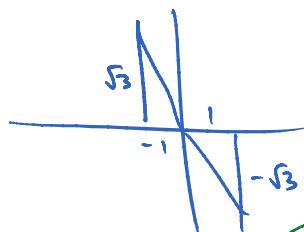
$$\tan \theta + \sqrt{3} = 0$$

Replace $\theta = \text{input}$

solve for θ_1, θ_2

then find x_1, x_2, x_3, \dots

$$\tan \theta = -\frac{\sqrt{3}}{1} = \frac{y}{x} = \frac{\sqrt{3}}{-1}$$



$$\theta_1 = \frac{2\pi}{3} = \frac{2x_1}{2}$$

$$\theta_2 = \frac{5\pi}{3} = \frac{2x_2}{2}$$

$$\text{period} = \frac{\pi}{K}$$

$$\therefore x_1 = \frac{\pi}{3}$$

$$x_2 = \frac{5\pi}{6}$$

add period until you get beyond $0 \leq x \leq 2\pi$

$$x_3 = \frac{4\pi}{3}$$

4. $\csc x + \cos x + 1 = \cot x \sin x$

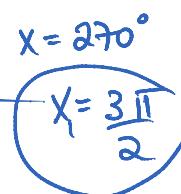
need to simplify to have one ratio, otherwise impossible to solve.

$$\frac{1}{\sin x} + \cos x + 1 = \frac{\cot x \sin x}{\sin x} - \cos x$$

$$\frac{1}{\sin x} = -1$$

$$y = -1 = \sin x$$

$$(0, 1)$$



$$x_5 = \frac{13\pi}{8} \text{ TOO BIG!}$$

Solving Trig Equations – with factoring

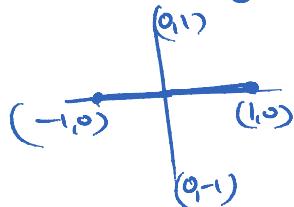
Common
Cross Cross Trinomial
Difference of Sq
factor so that each factor has only
one ratio

Solve each of the following for x in $0 \leq x \leq 2\pi$.

5. $2\sin x \sec x - 2\sqrt{3} \sin x = 0$

$$(2\sin x)(\sec x - \sqrt{3}) = 0$$

$$\frac{2\sin x}{2\sin x} = 0$$



$$\begin{aligned} x_1 &= 0 \\ x_2 &= \pi \\ x_3 &= 2\pi \end{aligned}$$

7. $\cos^2 \theta = 2 - \cos \theta$

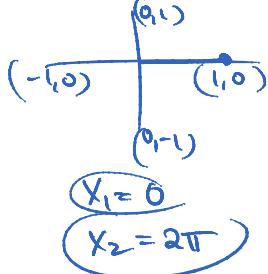
$$\cos^2 \theta + \cos \theta - 2 = 0$$

$$1 \quad 2$$

$$(\cos \theta + 2)(\cos \theta - 1) = 0$$

$$\cos \theta = -2 \quad \text{or} \quad \cos \theta = 1$$

NEVER
cosine
between
1 and -1



$$\begin{aligned} x_1 &= 0 \\ x_2 &= 2\pi \end{aligned}$$

6. $10\cos(2x) - 8\cos x + 1 = 0$

(double)

$$10(2\cos^2 x - 1) - 8\cos x + 1 = 0$$

$$20\cos^2 x - 10 - 8\cos x + 1 = 0$$

$$20\cos^2 x - 8\cos x - 9 = 0$$

$$\left(\frac{10}{2}\right)_2 20$$

$$\begin{array}{|c|c|c|} \hline -9 & 1 & 3 \\ \hline 1 & 9 & 3 \\ \hline \end{array}$$

one neg.

$$(10\cos x - 9)(2\cos x + 1) = 0$$

$$\cos x = \frac{9}{10} \quad \text{OR} \quad 2\cos x = -\frac{1}{2} = \frac{x}{r}$$

$$\begin{array}{|c|c|c|} \hline 1.57 & 6.28 & 2\pi \\ \hline 3.14 & 4.71 & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 5.37 & 2\pi & \\ \hline -\sqrt{3} & -1 & \\ \hline \end{array}$$

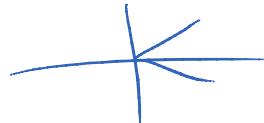
$$\begin{aligned} x_1 &= 0.45 \\ x_2 &= 6.28 - 0.45 \\ x_3 &= 5.37 \\ x_4 &= 4\pi \end{aligned}$$

Cross cross won't work
Quad.

$$\cos x = \frac{-10 \pm \sqrt{10^2 - 4(20)(-7)}}{2(20)}$$

$$\cos x = \frac{-10 \pm \sqrt{660}}{40}$$

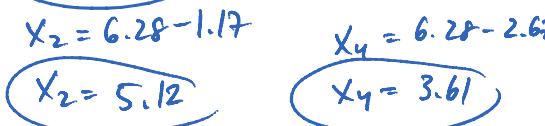
$$\cos x = 0.3923 \quad \text{or} \quad \cos x = -0.8923$$



$$x_1 = 1.17$$

$$x_2 = 6.28 - 1.17$$

$$x_3 = 2.67$$



$$x_4 = 6.28 - 2.67$$

$$x_4 = 3.61$$

FORMULA page**Reciprocal Identities**

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Co-function Identities

$$\sin x = \cos(\frac{\pi}{2} - x)$$

$$\cos x = \sin(\frac{\pi}{2} - x)$$

$$\tan x = \cot(\frac{\pi}{2} - x)$$

$$\cot x = \tan(\frac{\pi}{2} - x)$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

Horizontal Shift Identities

$$\sin x = \sin(x \pm 2\pi)$$

$$\cos x = \cos(x \pm 2\pi)$$

$$\tan x = \tan(x \pm \pi)$$

$$\sin x = \cos(x - \frac{\pi}{2})$$

$$\cos x = \sin(x + \frac{\pi}{2})$$

Compound
Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Principal & Related Acute Angle Identities

$$\sin x = \sin(\pi - x)$$

$$\sin(\pi + x) = \sin(2\pi - x) = \sin(-x) = -\sin x$$

$$\sin x = \sin(\pi - x)$$

$$\cos x = \cos(2\pi - x)$$

$$\cos(\pi - x) = \cos(\pi + x) = -\cos x$$

$$\tan x = \tan(\pi + x)$$

$$\tan(\pi - x) = \tan(2\pi - x) = \tan(-x) = -\tan x$$

Even/Odd Function Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$