Trig Identities & Equations Unit 7

Tentative TEST date________________

Big idea/Learning Goals

In this unit you will learn how to find exact values of angles that are not directly related to the special triangles. This is only possible with the use of trig identities, so you will start with trig identity proofs where the use of proper notation is key to minimize errors as well as to get full marks. You will also learn how to solve more complicated trig equations that involve factoring not just the unit circle.

<u>Corrections for the textbook answers:</u>

Success Criteria

- ☐ I <u>understand the new topics</u> for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	2-3	Equivalent Trig Expressions = Identities Section 7.1 & Handout	
	4-6	Compound Angle Formula Section 7.2 & Handout	
	7-9	Double Angle Formula (or Half Angle Formula) Section 7.3 & Handout	
	10-11	Proving Trig Identities Section 7.4 & TWO Handouts	
	12-13	Solving Trig Equations Section 7.5 & 7.6 & Handout	
		REVIEW	

Reflect – previous TEST mark ___________, Overall mark now___________.

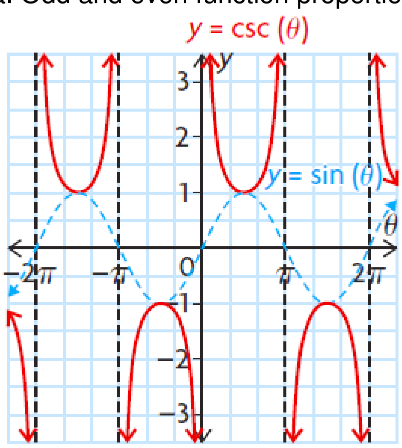
Equivalent Trig Expressions = Identities

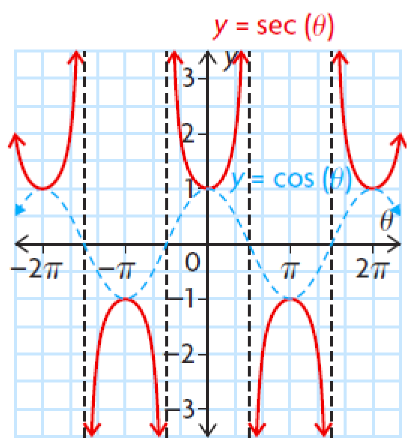
1. Clarify for yourself the difference between the following three words, use examples in your explanations.

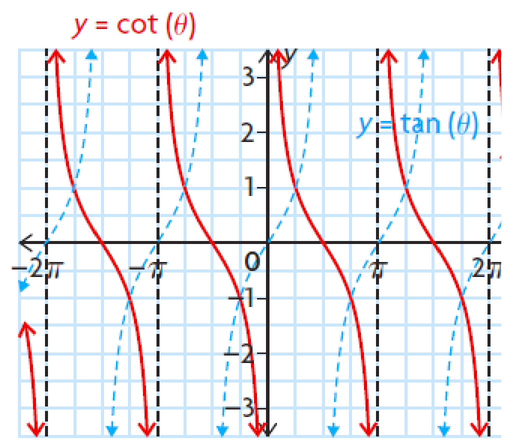
Expression Equation Identity

2. Understand how to use all the identities (gouped by topic) ALL formulas are found on the last page

a. Odd and even function properties



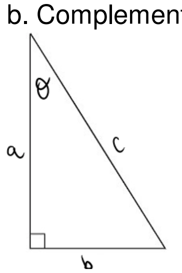




Practice writing the equivalent expressions for the following

i) $\sin\left(-\dfrac{\pi}{6}\right)$ ii) $\sec\left(\dfrac{\pi}{2}\right)$ iii) $\cos(-A-B)$

b. Complementary angles property = Cofunction identities



Practice writing the equivalent expressions for the following

i) $\sin\left(\dfrac{\pi}{6}\right)$ ii) $\sec\left(\dfrac{\pi}{8}\right)$ iii) $\cot\left(2x+\dfrac{\pi}{4}\right)$

c. Horizontal translations of period multiples

Practice writing the equivalent expressions for the following

i) $\cos\left(\frac{\pi}{3}\right)$ ii) $\csc\left(\frac{3\pi}{8}\right)$ iii) $\tan(x-4\pi)$

d. Similar shape of the graph just with a shift

Practice writing the equivalent expressions for the following

i) $\sin\left(\frac{7\pi}{6}\right)$ ii) $\sec\left(\frac{4\pi}{5}\right)$ iii) $\cot(2x)$

e. Principal and related acute angle characteristics

Practice writing the equivalent expressions for the following

i) $\sin\left(\frac{11\pi}{6}\right)$ ii) $\cos\left(\frac{5\pi}{7}\right)$ iii) $\tan\left(-\frac{\pi}{4}\right)$

3. Write down several equivalent expressions for the following, identify what property was used for each

a) $\sin\frac{5\pi}{6}$ b) $\cos\frac{3\pi}{4}$

Compound Angle Formula

1. Record the angles in both degrees and radians that are considered 'special' ie. Ratios can be found without calculator.

Angles for using the **four points** on the unit circle:

Angles for using **special triangles**:

2. In this lesson you will learn how to find exact values of angles that are not directly related to the lists above. For example, $\cos 105^o$. You may think that the following can be done but try it with the calculator to see that trig functions CANNOT be distributed.

$$\cos 105^o = \cos(135^o - 30^o) \neq \cos 135^o - \cos 30^o$$

You may recall that function notation uses brackets however it does NOT mean to multiply and so you cannot distribute it.This is especially important in this lesson.

Instead of distributing, the following identity is true, check it with the use of calculator.

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

$$\cos 105^o = \cos(135^o - 30^o) = \cos 135^o \cos 30^o + \sin 135^o \sin 30^o$$

Now without using the calculator find the exact value for $\cos 105^o$

3. Here are the rest of the compound identities:

$$\boxed{\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B}$$

$$\boxed{\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B}$$

$$\boxed{\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}}$$

Proofs for these are located at the back for reference to those who plan to take more math after this course.

4. Find the exact value of $\tan 375^o$

5. Find the exact values of the following

a. $\tan\frac{5\pi}{6}$

b. $\cos\frac{-7\pi}{12}$

(show that you can use the sum or the difference formula to get the same result)

c. $\sec\frac{7\pi}{3}$

d. $\sin\frac{23\pi}{12}$

Double Angle Formulas (or Half Angle Formulas)

1. In this lesson you will learn how to find exact values of angles that are half the size of the ones you know how to calculate. For example $\cos 157.5^o$.
You may think that the following can be done but try it with the calculator to see that not only trig functions CANNOT be distributed, but they also CANNOT have constants pulled out of the input.

$$\cos 315^o = \cos(2\cdot 157.5^o) \neq \cos 2 \cdot \cos 157.5^o$$
$$\neq 2\cos 157.5^o$$

Instead of distributing or pulling out the two, the following identity is true, check it with the use of calculator.

$$\boxed{\cos(2A) = 2\cos^2 A - 1}$$

$$\cos 315^o = \cos(2(157.5^o)) = 2\cos^2 157.5^o - 1$$

Now without using the calculator find the exact value for $\cos 157.5^o$

There are three versions for cos2A, and two of them can be rearranged

$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$

$\boxed{\cos(2A) = 1 - 2\sin^2 A}$ → **same as** $\boxed{\sin^2 A = \frac{1-\cos(2A)}{2}}$

$\boxed{\cos(2A) = 2\cos^2 A - 1}$ → **same as** $\boxed{\cos^2 A = \frac{1+\cos(2A)}{2}}$

$\boxed{\sin(2A) = 2\sin A \cos A}$

$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$

Proofs for these are located at the back for reference to those who plan to take more math after this course.

2. Practice replacing the input with different things so you understand how the formulas work:

$\sin x =$

$\sin \frac{x}{4} =$

$\tan 3x =$

$\cos 4x =$

$\cos \frac{4\pi}{3} =$

Same

$\boxed{\cos(2A)=1-2\sin^2 A}$ a

$\boxed{\sin^2 A=\frac{1-\cos(2A)}{2}}$

$\boxed{\cos(2A)=2\cos^2 A-1}$ b

$\boxed{\cos^2 A=\frac{1+\cos(2A)}{2}}$

$\boxed{\cos(2A)=\cos^2 A-\sin^2 A}$ c

$\boxed{\sin(2A)=2\sin A\cos A}$ d

$\boxed{\tan 2A=\frac{2\tan A}{1-\tan^2 A}}$ e

3. Find the exact value of $\sin 157.5^o$

4. Find the exact value of $\cos 112.5$

5. Find the exact values of the following. You may convert to degrees if that is easier for you.

a. $\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}$

b. $\tan\frac{2\pi}{3}$

c. $\sin\frac{\pi}{3}$

d. $\cos\frac{3\pi}{2}$

e. $\cos\frac{5\pi}{8}$

f. Find $\cos\frac{\theta}{2}$ if $270^o < \theta < 360^o$ and $\sin\theta = -\frac{7}{25}$

g. $\cos\frac{7\pi}{12}$

h. Find $\sin\frac{\theta}{2}$ if $180^o < \theta < 270^o$ and $\cos\theta = -\frac{15}{17}$

Proving Trig Identities

1. The proofs to identities are not unique, hence it is important to write out what you do in each step. Here are some strategies to try when doing proofs.
   Gr12:
   - Using the compound, double or other identities (see back page)
   Gr.11:
   - Using quotient id, reciprocal id and pythagorean id (memorize these)

   - Factoring/Expanding
   - LCD – only if you have adding or subtracting of separate fractions.
   - Distribute the denominator – only if you see a possible cancellation ex. $\frac{\csc^2 x - 1}{\csc^2 x} = 1 - \sin^2 x$

Avoid the following mistakes
- Distributing the numerator –don't EVER do this (try with regular numbers to see this)
- Canceling wrong – remember "monomial cancels monomial", "binomial cancels binomial", etc…
- Forgetting to record brackets – especially if the question invoves both multiplication and addition
- Forgetting to write the input beside the trig function – $\cos, \sin, \tan$ without θ are meaningless
- Incorrect placement of the exponent $\sin\theta^2$ when you mean $\sin^2\theta$
- Forgetting to explain the steps you've done

2. Prove the following, explain all the steps.

a. $\cos^4 x - \sin^4 x = \cos 2x$

b. $\frac{1}{\cos^2 x \sin x} - \frac{\sin x}{\cos^2 x} = \csc x$

c. $\sin 2x + \sin 2y = 2\sin(x+y)\cos(x-y)$

d. $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$

e. $\sec 2x - \tan 2x = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

Solving Trig Equations – no factoring needed, what to do if you see ‘k’

Solve each of the following for x in $0 \le x \le 2\pi$.

1. $2(5\sqrt{3} - 3\sec x) = 6\sqrt{3}$

2. $\tan 2x + \sqrt{3} = 0$

3. $2\cos 2x + \sqrt{2} = 0$

4. $\csc x + \cos x + 1 = \cot x \sin x$

Solving Trig Equations – with factoring

Solve each of the following for x in $0 \le x \le 2\pi$.

5. $2\sqrt{3}\sin x \sec x - 4\sin x = 0$

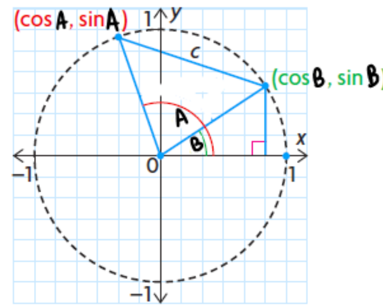
6. $10\cos 2x - 8\cos x + 1 = 0$

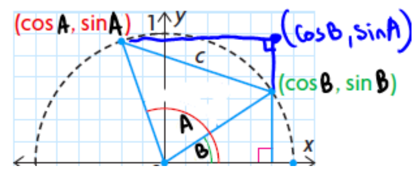
7. $\cos^2\theta = 2 - \cos\theta$

8. $20\cos^2 x + 10\cos x - 7 = 0$

Compound Angle PROOFS

In universities you will be required to follow proofs to theorems that professors do in class. Read the following proof and make notes in the margins of what is going on to help you understand it.





Proof for

$$\boxed{\cos(A-B)=\cos A\cos B+\sin A\sin B}$$

$$c^2=1^2+1^2-2(1)(1)\cos(A-B)$$
$$c^2=2-2\cos(A-B)$$
equation 1 from picture 1

$$(\sin A-\sin B)^2+(\cos B-\cos A)^2=c^2$$ equation 2 from picture 2

Substitute equation 1 into 2

$$\begin{aligned}\sin^2 A-2\sin A\sin B+\sin^2 B+\cos^2 B-2\cos A\cos B+\cos^2 A&=2-2\cos(A-B)\\ 1+1-2\sin A\sin B-2\cos A\cos B&=2-2\cos(A-B)\\ -2\sin A\sin B-2\cos A\cos B&=-2\cos(A-B)\\ \sin A\sin B+\cos A\cos B&=\cos(A-B)\end{aligned}$$

Proof for

$$\boxed{\sin(A+B)=\sin A\cos B+\cos A\sin B}$$

$$\begin{aligned}\sin(A+B)&=\cos\left(\frac{\pi}{2}-(A+B)\right)\\ &=\cos\left(\left(\frac{\pi}{2}-A\right)-B\right)\\ &=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B\\ &=\sin A\cos B+\cos A\sin B\end{aligned}$$

Proof for

$$\boxed{\cos(A+B)=\cos A\cos B-\sin A\sin B}$$

$$\begin{aligned}\cos(A+B)&=\cos\big(A-(-B)\big)\\ &=\cos A\cos(-B)+\sin A\sin(-B)\\ &=\cos A\cos B+\sin A(-\sin B)\\ &=\cos A\cos B-\sin A\sin B\end{aligned}$$

Proof for:

$$\boxed{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

$$\begin{aligned}\tan(A+B)&=\frac{\sin(A+B)}{\cos(A+B)}\\ &=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}\\ &=\frac{\left(\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}\right)}{\left(\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}\right)}\\ &=\frac{\tan A+\tan B}{1-\tan A\tan B}\end{aligned}$$

Prove on your own :
(hint use compound sine with plus, odd and even properties)

$$\boxed{\sin(A-B)=\sin A\cos B-\cos A\sin B}$$

Prove on your own :
(hint use compound tangent with plus, odd and even properties)

$$\boxed{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

Double/Half Angle PROOFS

There are three versions for cos2A, but only one version for sin2A and tan2A

$$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$$

$$\boxed{\cos(2A) = 1 - 2\sin^2 A}$$

$$\boxed{\cos(2A) = 2\cos^2 A - 1}$$

$$\boxed{\sin(2A) = 2\sin A\cos A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

Proof for:

$$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$$

$$\begin{aligned}\cos(2A) &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

Proof for:

double angle: $\boxed{\cos(2A) = 1 - 2\sin^2 A}$ $\rightarrow$ when rearranged it is the same as half angle: $\boxed{\sin^2 A = \frac{1-\cos(2A)}{2}}$

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A\\ &= (1-\sin^2 A) - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\begin{aligned}\cos(2A) &= 1 - 2\sin^2 A\\ 2\sin^2 A &= 1 - \cos(2A)\\ \sin^2 A &= \frac{1-\cos(2A)}{2}\end{aligned}$$

Prove on your own :
(hint use first double formula from above, and Pythagorean)

double angle: $\boxed{\cos(2A) = 2\cos^2 A - 1}$ $\rightarrow$ when rearranged it is the same as half angle: $\boxed{\cos^2 A = \frac{1+\cos(2A)}{2}}$

Prove on your own :
(hint use compound formulas with A+A)

$$\boxed{\sin(2A) = 2\sin A\cos A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

FORMULA page

Reciprocal Identities

$\csc x = \dfrac{1}{\sin x}$

$\sec x = \dfrac{1}{\cos x}$

$\cot x = \dfrac{1}{\tan x}$

Quotient Identities

$\tan x = \dfrac{\sin x}{\cos x}$

$\cot x = \dfrac{\cos x}{\sin x}$

Co-function Identities

$\sin x = \cos(\dfrac{\pi}{2} - x)$

$\cos x = \sin(\dfrac{\pi}{2} - x)$

$\tan x = \cot(\dfrac{\pi}{2} - x)$

$\cot x = \tan(\dfrac{\pi}{2} - x)$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Horizontal Shift Identities

$\sin x = \sin(x \pm 2\pi)$

$\cos x = \cos(x \pm 2\pi)$

$\tan x = \tan(x \pm \pi)$

$\sin x = \cos(x - \dfrac{\pi}{2})$

$\cos x = \sin(x + \dfrac{\pi}{2})$

Compound Addition and Subtraction Formulas

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Double Angle Formulas

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

Principal&RelatedAcuteAngle Identities

$\sin x = \sin(\pi - x)$

$\sin(\pi + x) = \sin(2\pi - x) = \sin(-x) = -\sin x$

$\sin x = \sin(\pi - x)$

$\cos x = \cos(2\pi - x)$

$\cos(\pi - x) = \cos(\pi + x) = -\cos x$

$\tan x = \tan(\pi + x)$

$\tan(\pi - x) = \tan(2\pi - x) = \tan(-x) = -\tan x$

Even/odd Function Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Half-Angle Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\qquad$ $\cos^2 x = \dfrac{1 + \cos 2x}{2}$