nNOTESfixed2012

newRationalsUnitNOTES

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see below ↓

Rational Functions Unit

Tentative TEST date_______________

Big idea/Learning Goals

This unit begins with the study of how to graph rational functions that are reciprocals of linear and quadratic functions. Once you understand the idea of how zeros and vertical asymptotes are related, you will learn how to sketch rational functions with a non-constant numerator – this will involve the study of the main characteristics of the rational function: the y-intercepts, zeros, holes, vertical asymptotes, horizontal asymptotes and oblique asymptotes. You will learn to solve rational equations and inequalities as well as word problems that are modeled by rational functions. At the end you will yet again revisit the rates of change but with rational functions.

Corrections for the textbook answers:
Sec 5.1 #9c) y-int=-12, d) range <=0.5
Sec 5.2 #2i) HA at y=2
Sec 5.3 #5c) y=(x+5)/(4x-1) and y-int (0, -5) #8 HA not VA at y=1/2
Sec 5.4 #13b) 1 min 32 sec or 1.04 min
Sec 5.5 #4a) -5<x<-4.5 #5e) t<0, f) (-inf, -2.), (0,3) #6c) (-4, -2] or (-1, 2]
#7 x<-6, -1<x<1/2, 2<x #9 there is a solution 0<t<0.31 #11. 1<x<5
Sec 5.6 #9b)-1.2 #10a)11.39, b)1.29

Success Criteria

☐ I am ready for this unit if I am confident in the following review topics
- ☐ Rational expressions
- ☐ Transformations
- ☐ Domain & range
- ☐ Asymptotes
- ☐ End behaviour
- ☐ Symmetry
- ☐ Average and instantaneous rates of change
- ☐ Family of functions
- ☐ Solving equations and inequalities

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
Apr4	2-4	Reciprocal Graphs Section 5.1 & Handout	
Apr2	5-9	INVESTIGATION of Other Rational Graphs - 2 days Section 5.2 & Handout	
Apr2	10-12	Graphing Rational Functions Section 5.3 & THREE Handouts	
Apr11	13-15	Solve Rational Inequalities Section 5.5 & TWO Handouts	
Apr4	16-17	Solve Rational Equations Section 5.4 & THREE Handouts	
Apr2	18-19	Problem Solve TWO Handouts	
Apr11	20-21	Rates of Change of Rationals Section 5.6	
		TWO EXTRA assignments - Rationals and polynomials - Problem Solve with Rationals	

Apr.11

Reflect – previous TEST mark __________, Overall mark now__________.

Looking back, what can you improve upon?

Reciprocal Graphs → Rational graphs with only ONE in numerator

ex. $\frac{1}{x+3}$ or $\frac{1}{x^2-4}$

1. Describe what is the relationship between characteristics of $f(x)$ graph and its reciprocal, $\frac{1}{f(x)}$, graph.

(Include the following characteristics: zeros, y-intercepts, vertical asymptotes, end behaviour, horizontal asymptotes, increase and decrease intervals, points where the two graphs will meet.)

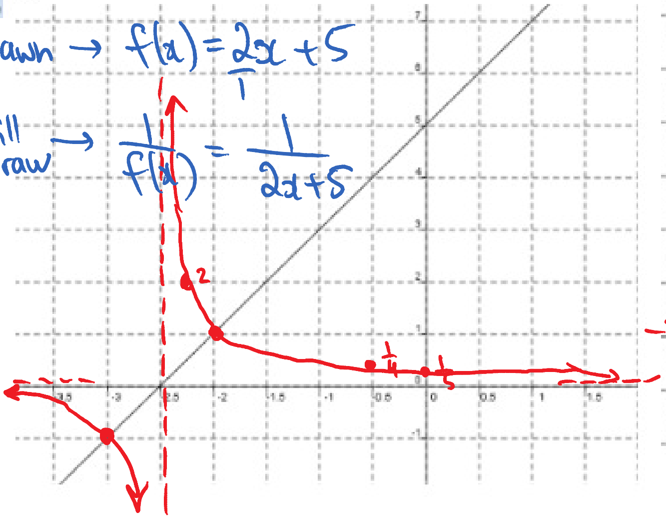
$\frac{\#}{0} = \text{undefined} = \infty$, $\frac{\#}{\infty} = 0$

VA. zeros

- zeros of f(x) will be vertical asymptotes of $\frac{1}{f(x)}$
- VA's of f(x) will be zeros of $\frac{1}{f(x)}$
- if end. behaviour of f(x) is $y \to \pm\infty$ then end behaviour of $\frac{1}{f(x)}$ is $y \to 0$
- all the y values of f(x) become $\frac{1}{y}$ values in $\frac{1}{f(x)}$
  ie. y-int (0,5) then it will be $(0, \frac{1}{5})$
- If graph of f(x) increases then graph of $\frac{1}{f(x)}$ will decrease
- f(x) will meet $\frac{1}{f(x)}$ at $y = \pm 1$

2. Sketch the reciprocal graphs for the following

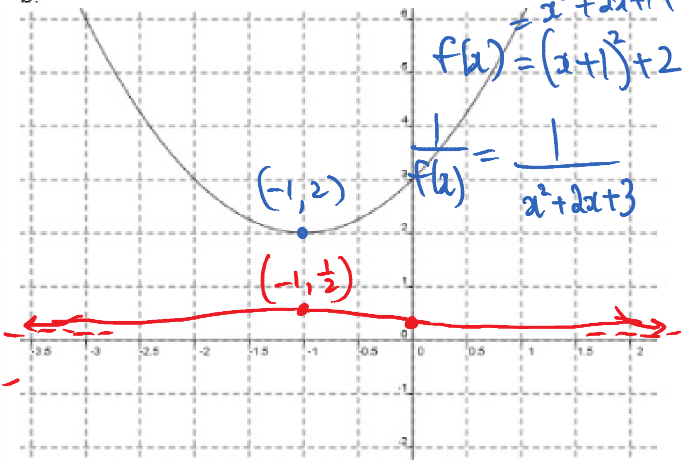
a.



drawn → $f(x) = \frac{2x+5}{1}$

will draw → $\frac{1}{f(x)} = \frac{1}{2x+5}$

b.



$f(x) = (x+1)^2+2 = x^2+2x+1+2$

$\frac{1}{f(x)} = \frac{1}{x^2+2x+3}$

Sketch reciprocals

$f(x)=$ c. $y=2x^2+x+1$

$\frac{1}{f(x)}=\frac{1}{2x^2+x+1}$

Complete sq.

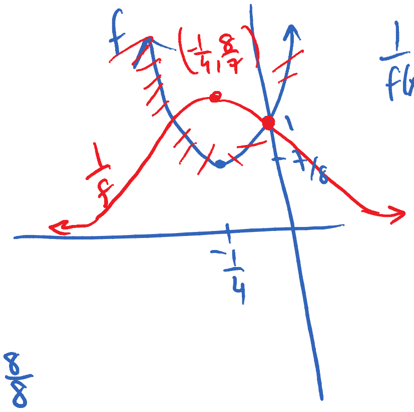
$2(x^2+\frac{1}{2}x+\frac{1}{16}-\frac{1}{16})+1$

$\left(\frac{\frac{1}{2}}{2}\right)^2$

$2(x^2+\frac{1}{2}x+\frac{1}{16})-\frac{1}{8}+\frac{8}{8}$

$2(x+\frac{1}{4})^2+\frac{7}{8}$

vertex $\left(-\frac{1}{4},\frac{7}{8}\right)$



$f(x)$
d. $y=9x^2+8x-x^3$

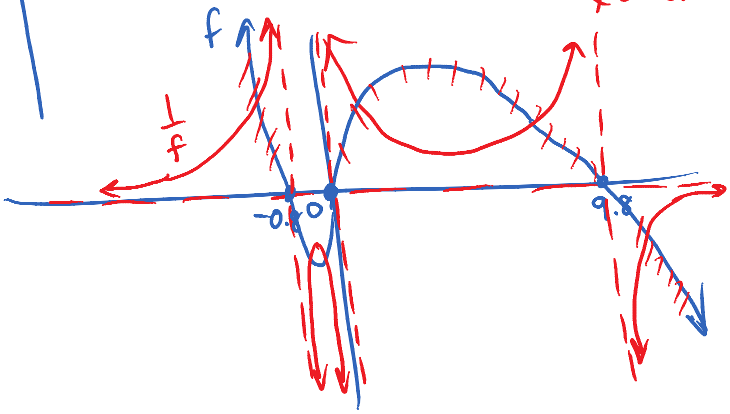
$\frac{1}{f(x)}=\frac{1}{-x^3+9x^2+8x}=\frac{1}{-x(x^2-9x-8)}$

$\doteq\frac{1}{-x(x-9.8)(x+0.8)}$

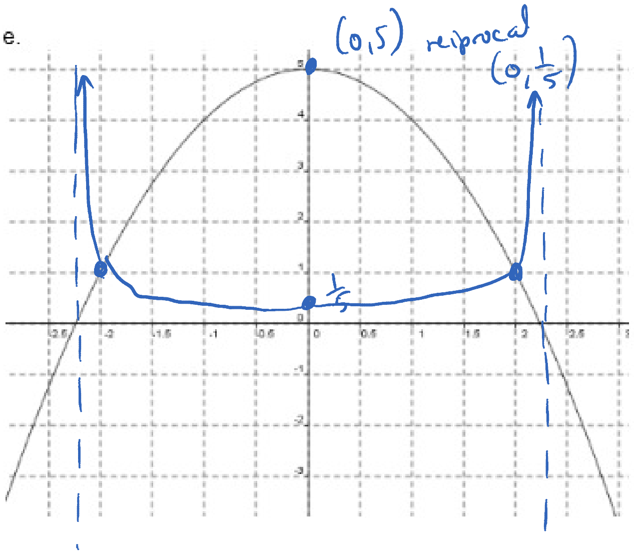
quad formula

$x=9.8$

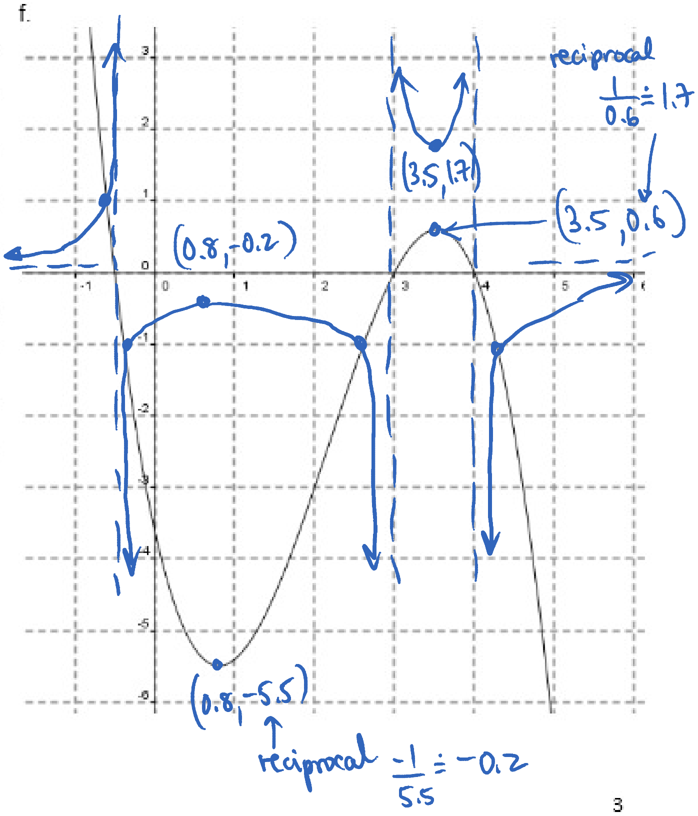
$x=-0.8$



e.

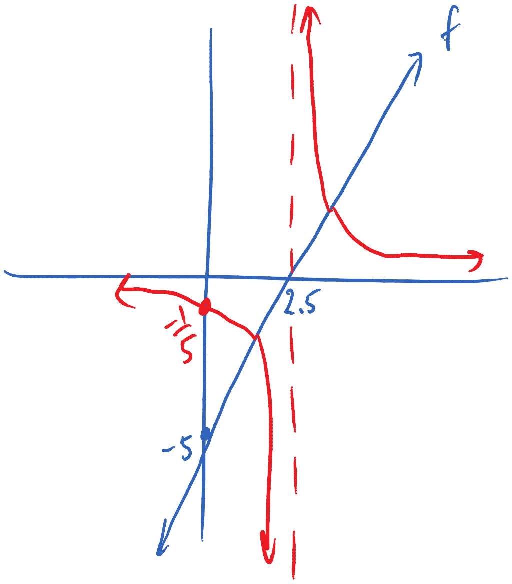


f.



g. $y = 2x - 5 = f(x)$

$$\frac{1}{f(x)} = \frac{1}{2x-5}$$

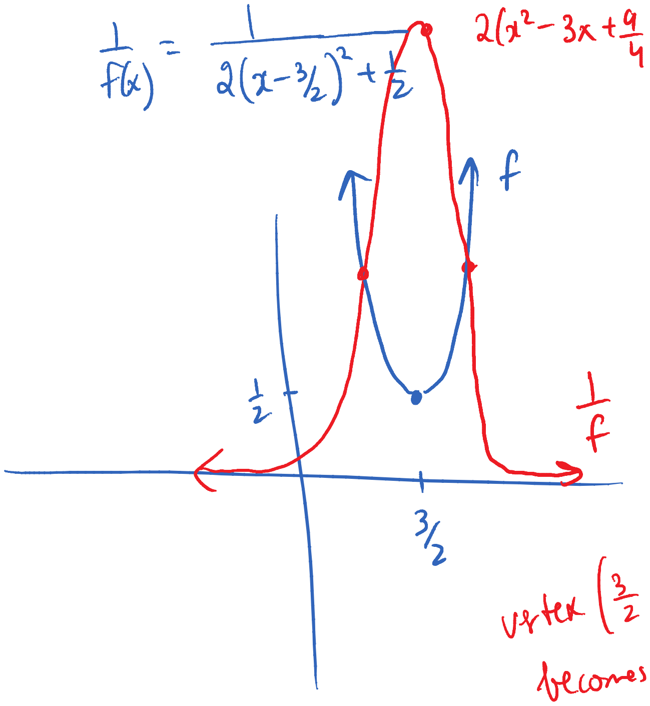


h. $y = 2x^2 - 6x + 5 = f(x)$

$2(x^2 - 3x \quad) + 5$

$2(x^2 - 3x + \frac{9}{4} - \frac{9}{5}) + 5$

$$\frac{1}{f(x)} = \frac{1}{2(x-3/2)^2 + \frac{1}{2}}$$

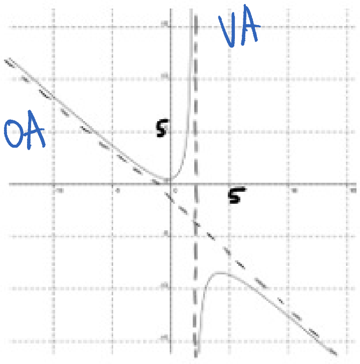


vertex $(\frac{3}{2}, \frac{1}{2})$

becomes

$(3/2, \frac{2}{1})$

3. You will learn how to sketch rational functions that have x's in the numerator. You will see that some functions will have an **oblique asymptote**. See the example graph →

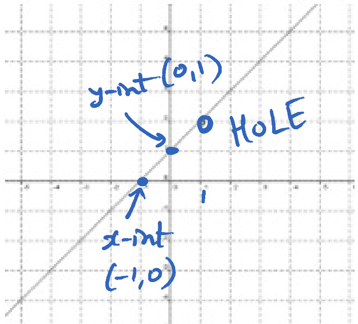


INVESTIGATION of Other Rational Graphs

1. Fill out the chart by looking at graphs to discover how to find the key characteristics from the equation without the use of graphs. Always try to factor both the numerator and the denominator to see if there are any cancellations. If things don't simplify and the numerator has a higher degree than the denominator do long division.

Graph | **Factor & simplify or do long division** | **State all the key characteristics:** y-intercept, zeros, holes, VA, HA, OA

a



$$f(x) = \frac{x^2-1}{x-1}$$

$$= \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$= x+1$ ← line

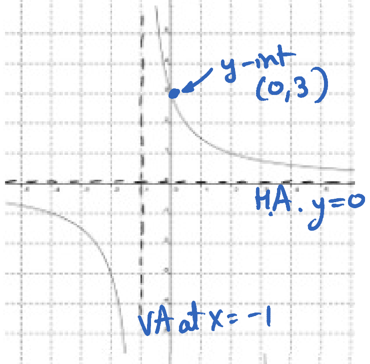
no x's in the denominator

Hole at x=1
the zero of the factor that cancels

y-int sub x=0 + solve for y

x-int the zero of the factor left in the numerator

b

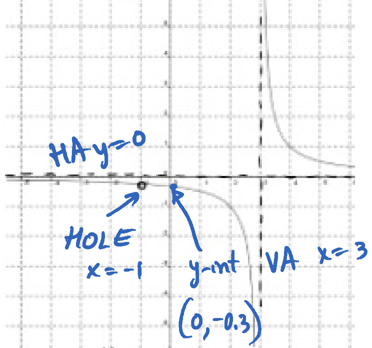


$$f(x) = \frac{3}{(x+1)}$$

VA the zero of the denominator that doesn't cancel.

HA ??

c



$$f(x) = \frac{x+1}{x^2-2x-3}$$

$$= \frac{\cancel{(x+1)}}{(x-3)\cancel{(x+1)}}$$

$$= \frac{1}{x-3}$$

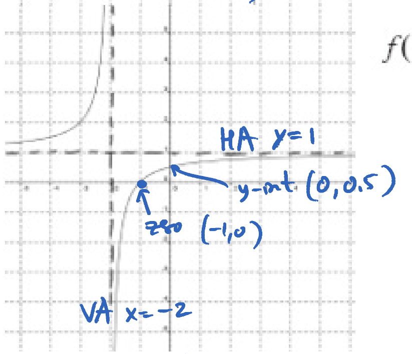
Hole at x=-1

y-int $y = \frac{0+1}{0^2-2(0)-3} = -\frac{1}{3}$

VA x-3=0
x=3

HA ??

d



$$f(x) = \frac{x+1}{x+2}$$

← zeros x+1=0
x=-1

← VA
x+2=0
x=-2

y-int $y = \frac{0+1}{0+2} = \frac{1}{2}$

long division:

$$x+2\overline{)\,x+1\,}^{\;1}$$
$$\underline{-\;x+2}$$
$$0\;\;-1$$

$\therefore \frac{x+1}{x+2} = \underbrace{1}_{\text{quot}} + \frac{-1}{x+2}$ Rem / Divisor

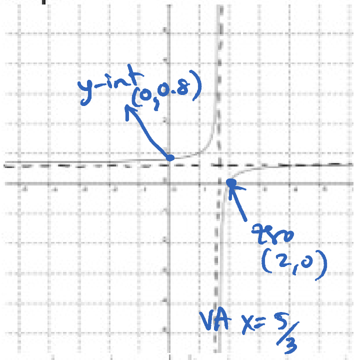
is this HA ??

Dividend = (Quotient)(Divisor) + Rem.

$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Rem}}{\text{Divisor}}$

Graph | **Factor & simplify or do long division** | **State all the key characteristics:** y-intercept, zeros, holes, VA, HA, OA

e



$$f(x)=\frac{2x-4}{3x-5}$$

y-int $y=\frac{2(0)-4}{3(0)-5}=\frac{-4}{-5}=0.8$

zeros $2x-4=0$, $2x=4$, $x=2$

VA $3x-5=0$, $x=\frac{5}{3}$

no Holes nothing cancels

$$\begin{array}{r} \frac{2}{3} \\ 3x-5 \overline{)\,2x-4} \\ \underline{2x-\frac{10}{3}} \\ -\frac{2}{3} \end{array}$$

$$\therefore \frac{2x-4}{3x-5}=\frac{2}{3}+\frac{-\frac{2}{3}}{3x-5}$$

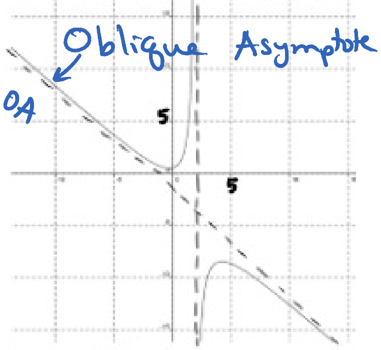
OA/HA = quotient when you long divide

HA $y=\frac{2}{3}$

quotient is with an x

quotient only a constant.

f



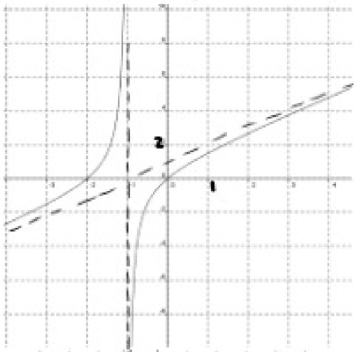
$$f(x)=\frac{1+x^2}{2-x}$$

$$\begin{array}{r} -x-2 \\ -x+2 \overline{)\,x^2+0x+1} \\ \underline{x^2-2x} \\ 2x+1 \\ \underline{2x-4} \\ 5 \end{array}$$

$$\therefore \frac{1+x^2}{2-x}=\boxed{-x-2}+\frac{5}{2-x}$$

OA $y=-x-2$

g



$$f(x)=\frac{x^2+2x}{x+1}$$

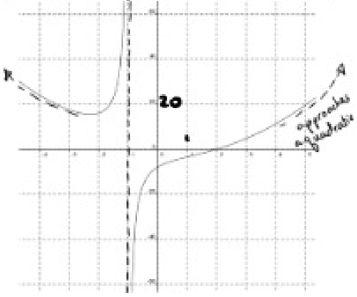
OA = ?

$$\begin{array}{r} x+1 \\ x+1 \overline{)\,x^2+2x+0} \\ \underline{-\;x^2+x} \\ x+0 \\ \underline{x+1} \\ -1 \end{array}$$

$$\therefore \frac{x^2+2x}{x+1}=x+1+\frac{-1}{x+1}$$

$\therefore$ OA $y=x+1$

h



$$f(x)=\frac{x^3-8}{x+1}$$

$$\begin{array}{r} x^2-x+1 \\ x+1 \overline{)\,x^3+0x^2+0x-8} \\ \underline{x^3+x^2} \\ -1x^2+0x \\ \underline{-1x^2-x} \\ x-8 \\ \underline{x+1} \\ -9 \end{array}$$

$$\therefore f(x)=(x^2-x+1)+\frac{-9}{x+1}$$

not OA since not linear.

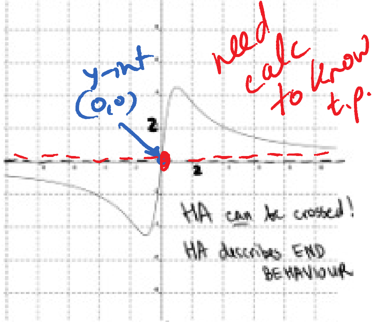
6

Graph

Factor & simplify or do long division

State all the key characteristics: y-intercept, zeros, holes, VA, HA, OA

i



$f(x) = \frac{9x}{1+x^2}$

y-int $y = \frac{9(0)}{1+0} = 0$

zeros $9x = 0$
$x = 0$

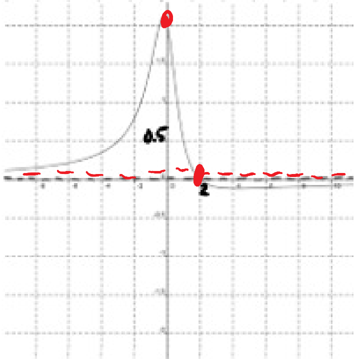
VA $1+x^2 = 0$
$x^2 = -1$
$x = \sqrt{-1}$ can't

$x^2+0x+1 \overline{)\,0x^2+9x+0}$ quotient 0 no holes

↑ quad ↑ linear doesn't divide in

∴ HA $y=0$

j



$f(x) = \frac{2-x}{1+x^2}$

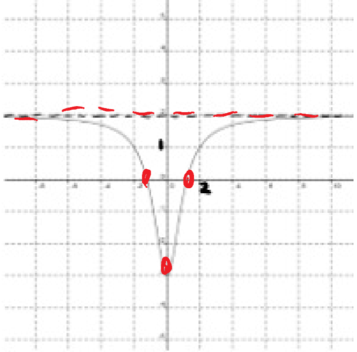
y-int $= \frac{2-0}{1+0} = 2$

zeros $2-x=0$
$2=x$

VA $1+x^2 \neq 0$ never.

$x^2+1 \overline{)\,-x+2}$ can't ∴ HA $y=0$

k



$f(x) = \frac{2x^2-3}{x^2+1}$

y-int $= \frac{0-3}{0+1} = -3$

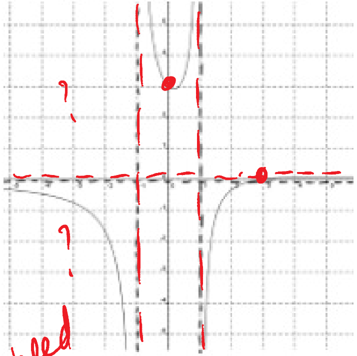
VA none

zeros $2(x^2-3/2)=0$
$2(x-\sqrt{\tfrac{3}{2}})(x+\sqrt{\tfrac{3}{2}})=0$
$x = \pm\sqrt{\tfrac{3}{2}}$

$x^2+0x+1 \overline{)\,2x^2+0x-3}$ quotient 2
$\underline{2x^2+0x+2}$
-5

∴ HA $y=2$

l



need +/- chart to know if above/below

$f(x) = \frac{x-3}{x^2-1}$

y-int $= \frac{0-3}{0-1} = 3$

zeros $x-3=0$
$x=3$

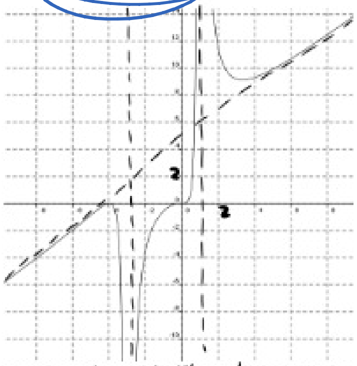
VA $x^2-1=0$
$(x+1)(x-1)=0$
$x = \pm 1$

can't divide ∴ HA $y=0$

~~8~~ 7

2. EXTENSION

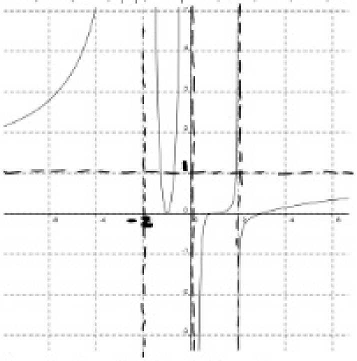
Note the orders of each factor and how it affects the behaviour near zeros and VAs



$$f(x)=\frac{x^3(x+4)^2}{(x-1)^2(x+3)^2}$$

zero x = 0 bend   x = -4 bounce

VA. x = 1 ~~bounce~~ same side   x = -3



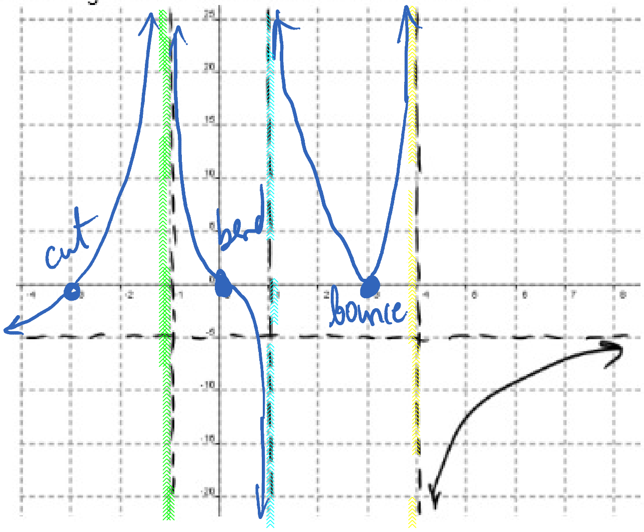
$$f(x)=\frac{(x+1)^2(x-3)(x-1)^3}{x^3(x-2)(x+2)^2}$$

zero x = -1 bounce   x = 3 cut   x = 1 bend

VA x = 0 ~~bend~~ opposite sides   x = 2 ~~cut~~ opposite sides   x = -2 ~~bounce~~ same side

3. Determine the behaviour near zeros and near VAs for the following and then finish off the started sketch.

$$f(x)=\frac{-5x^3(x-3)^2(x+3)}{(x-1)^3(x+1)^2(x-4)}$$



zeros x = 0 bend
x = 3 bounce
x = -3 cut

VA x = 1 opposite sides
x = -1 same sides
x = 4 opposite sides

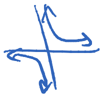
4 8

Graphing Rational Functions

Review how to graph the transformed parent $y = \frac{1}{x}$ graph.

1. Describe what each constant in $y = \frac{a}{k(x-d)} + c$ controls

If equation resembles transformed parent:
- VA $x = d$
- HA $y = c$
- use reflections to figure out shape



If it's reciprocal: $\frac{1}{f(x)}$:
- sketch $f(x)$
- HA $y=0$ always
- swap VA and Ø
- plot $\frac{1}{y}$ values → y-int → vertex → ±1

2. You can apply the usual transformation steps as you've done before or use the characteristics of rationals you've already learned to simplify the amount of work required. Describe the most effective method of sketching ~~the transformed parent~~ rational.

Regular steps for other rationals

Factor + cancel
- hole is zero of the cancelled factor
- y-int (sub x=0)
- zeros numerator = 0 OR $\boxed{\text{sub } y=0 \text{ + solve}}$
- VA denom = 0
- OA / HA find quotient using long division

3. Sketch the following

a. $f(x) = \frac{-2}{3x+6} \boxed{+4}$

- no holes
- y-int $y = \frac{-2}{2(0)+6} + 4 = \frac{-2}{6} + 4$
  $= -\frac{1}{3} + 4 = \boxed{\frac{11}{3} = y}$
- zeros $0 = \frac{-2}{3x+6} + 4$

$\frac{2}{3x+6} = \frac{4}{1}$

$2 = 4(3x+6)$

$2 = 12x + 24$

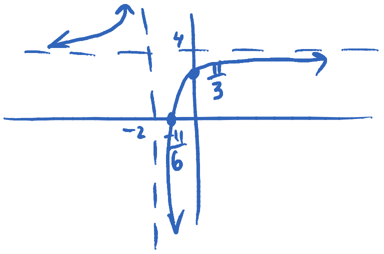
$-22 = 12x$

$\boxed{-\frac{11}{6} = x}$

VA. $3x+6=0$

$\boxed{x=-2}$

HA $\boxed{y=4}$



b. $f(x) = \frac{2+5x}{3-4x}$ no holes

y-int $y = \frac{2+5(0)}{3-4(0)} = \boxed{2/3 = y}$

zeros $\frac{0}{1} = \frac{2+5x}{3-4x}$

$0 = 2 + 5x$

$\boxed{-2/5 = x}$

VA $3 - 4x = 0$

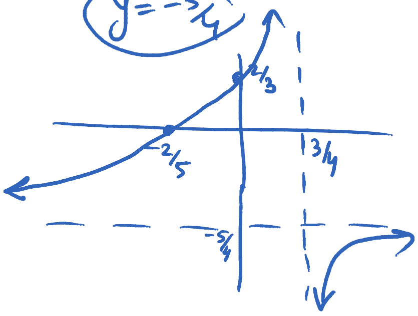
$3 = 4x$

$\boxed{\frac{3}{4} = x}$

$\boxed{\text{HA}}$ / ~~OA~~

$\boxed{y = -5/4}$

$-4x+3 \overline{)\,5x+2}$ quotient $-5/4$



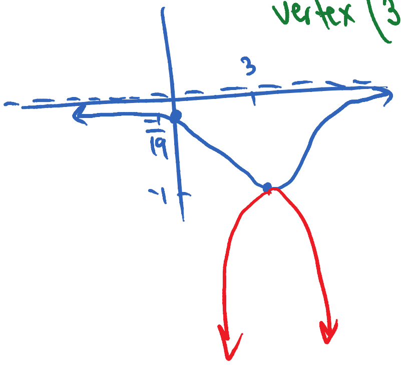
Reciprocal of Quadratic

c. $f(x)=\dfrac{1}{-2x^2+12x-19}$

Complete the sq.

$\dfrac{1}{-2(x-3)^2-1}$

vertex $(3,-1)$



d. $f(x)=\dfrac{x^2-x+8}{x}$ ← check if factors

b^2-4ac

$(-1)^2-4(1)(8)$ neg!!

$=-31$ ∴ not factor

- no holes
- y-int $y=\dfrac{0^2-0+8}{0}=\dfrac{8}{0}$ ∴ ~~none~~ undefine ∴ no zeros.

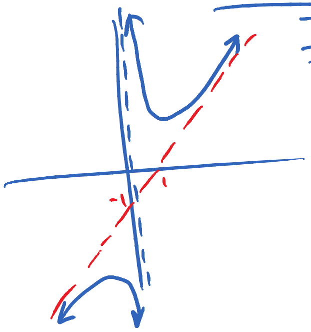
VA $\boxed{x=0}$

$$\begin{array}{r} x-1 \\ x+0\,\big)\overline{\,x^2-x+8\,} \\ \underline{x^2+0x} \\ -x+8 \\ \underline{-x+0} \\ 8 \end{array}$$

$\therefore f(x)=\boxed{x-1}+\dfrac{8}{x}$

OA $\boxed{y=x-1}$

y-int $(0,-1)$
x-int $(1,0)$



e. $f(x)=\dfrac{4}{1-x}-3$

Remainder
Quotient
Divisor

- holes? NONE
- zeros/x-int $\boxed{\text{sub } y=0}$, $\dfrac{4}{1-x}-3=0$
- y-int $\boxed{\text{sub } x=0}$

$\boxed{y=1}$

- VA $\boxed{\text{solve denom}=0}$

$1-x=0$

$\boxed{1=x}$

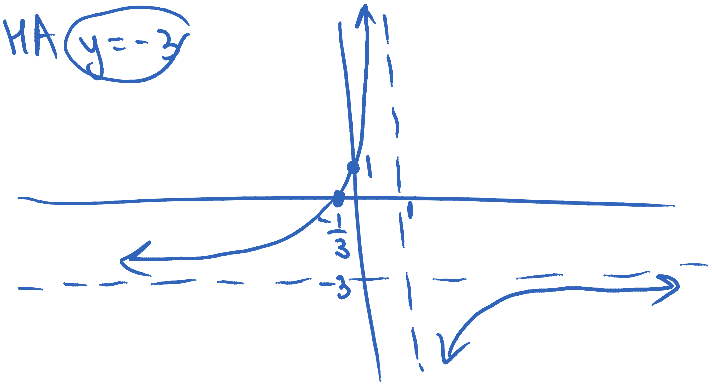
- HA $\boxed{y=-3}$

$\dfrac{4}{1-x}=\dfrac{3}{1}$

$4=3-3x$

$1=-3x$

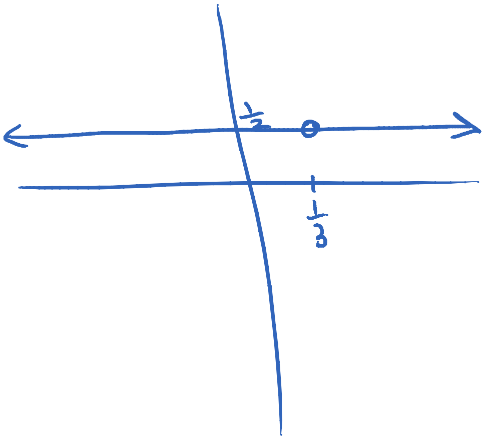
$\boxed{-\dfrac{1}{3}=x}$ zero



f. $f(x)=\dfrac{3x-1}{6x-2}=\dfrac{(\cancel{3x-1})}{2(\cancel{3x-1})}=\dfrac{1}{2}=y$

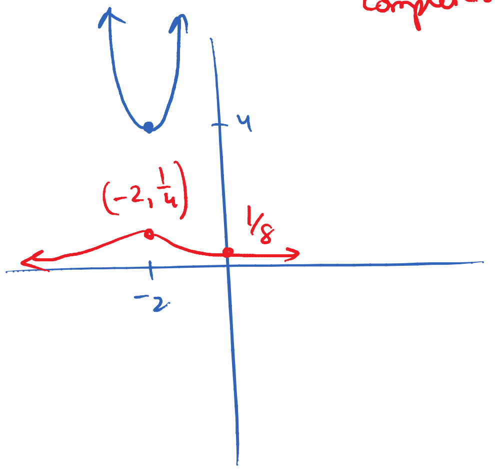
Horizontal line

- holes at $\boxed{x=\frac{1}{3}}$



g. $f(x)=\frac{1}{x^2+4x+8} = \frac{1}{(x+2)^2+4}$

complete sq.



h. $f(x)=\frac{2x-8}{3x}$

y-int $= \frac{0-8}{0} = N/A$

Zeros $2x-8=0$

$x=4$

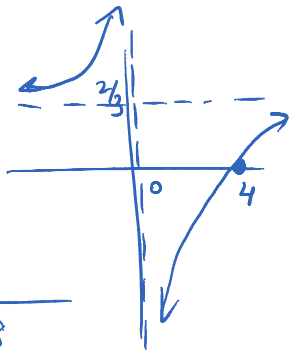
VA $3x=0$

$x=0$

$$\begin{array}{r} \frac{2}{3} \\ 3x+0\,\overline{)\,2x-8} \\ 2x+0 \\ \hline -8 \end{array}$$

HA.

$\therefore f(x)=\boxed{\frac{2}{3}} + \frac{-8}{3x}$



i. $f(x)=\frac{x^3-1x^2-2x}{x^2+4x-12}$

factor

$= \frac{x(x^2-x-2)}{(x+6)(x-2)}$

$= \frac{x\cancel{(x-2)}(x+1)}{(x+6)\cancel{(x-2)}}$

- hole at $\boxed{x=2}$
- zeros sub $y=0$

$\boxed{x=0}$ and $\boxed{x=-1}$

$\frac{0}{1} = \frac{x(x+1)}{(x+6)}$

$0 = x(x+1)$

- y-int sub $x=0$

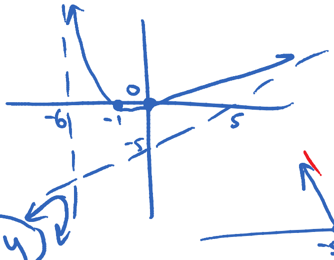
$\boxed{y=0}$

- VA $\boxed{x=-6}$
- HA/OA

$$\begin{array}{r} \boxed{x-5=y} \\ x^2+4x-12\,\overline{)\,x^3-1x^2-2x} \\ x^3+4x^2-12x \\ \hline -5x^2+10x \end{array}$$

y-int = -5

x-int = 5



ⓙ $f(x)=\frac{3x^2-75}{4-x} = \frac{3(x^2-25)}{4-x}$

$= \frac{3(x+5)(x-5)}{4-x}$

- no holes
- y-int $\boxed{y=-\frac{75}{4}}$
- zeros $\boxed{x=5}$ and $\boxed{x=-5}$
- VA $4-x=0$

$\boxed{4=x}$

OA/HA

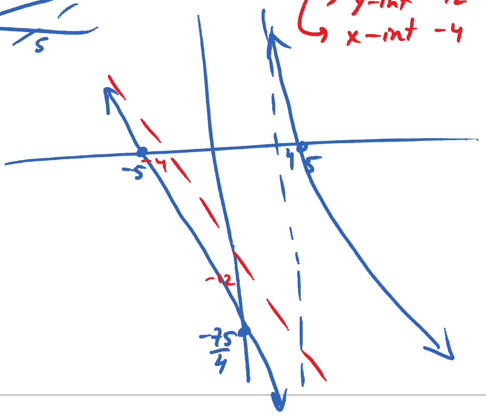
$$\begin{array}{r} -3x-12 \\ -x+4\,\overline{)\,3x^2+0x-75} \\ 3x^2-12x \\ \hline 12x-75 \end{array}$$

STOP!!

$\therefore$ OA $\boxed{y=-3x-12}$

y-int -12

x-int -4



~~12~~ 11

Solve Rational Inequalities

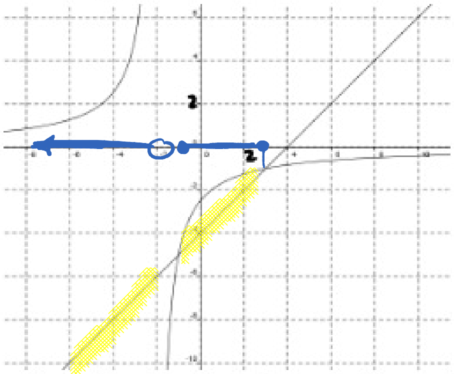
x can be neg!

1. Recall the rules of working with inequalities. Discuss why in addition to those rules you cannot cross multiply questions like the ones below.
   - must flip sign < to > when multiply or divide by a negative.
   - cannot sq. root inequalities, instead move all terms to one side factor and sketch or use +/- chart
   - cannot cross-multiply, instead move all terms to one side, factor +/- chart.

2. There are 3 ways to solve inequalities:
   A. Graph original 2 equations, and see where one is above/below the other (too long to sketch)
   B. Graph the related equation with ZERO on one side, and see where the graph is above/below x-axis
   C. Use +/- table on the equation with ZERO on one side, and pick positive/negative intervals

A: Graphing the original

$$x-4\le\frac{-5}{x+2}$$



a. How do you find the solution to this question?

for what x's is the line lower than rational? $x\in(-\infty,-2), [-1,3]$

Show the derivation of the related function that will be used in the next method:

$$x-4\le\frac{-5}{x+2}\xrightarrow{\text{show steps}}\frac{(x-3)(x+1)}{(x+2)}\le 0$$

b. Show steps

$$\frac{(x+2)}{(x+2)}\frac{(x-4)}{1}+\frac{5}{(x+2)}\le 0$$

$$\frac{x^2-4x+2x-8+5}{x+2}\le 0$$

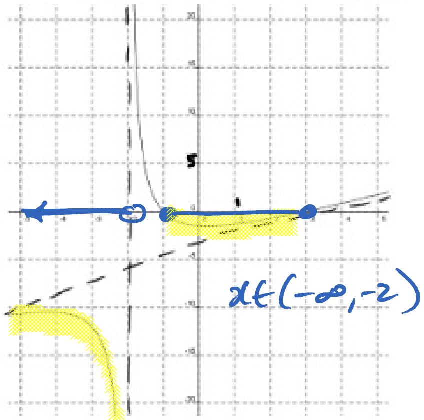
$$\frac{x^2-2x-3}{x+2}\le 0$$

$$\frac{(x-3)(x+1)}{(x+2)}\le 0$$

B: Graphing the related function (can move everything to either side)

choose to move terms to the left:

$$x-4+\frac{5}{x+2}\le 0\longrightarrow\frac{(x-3)(x+1)}{(x+2)}\le 0$$



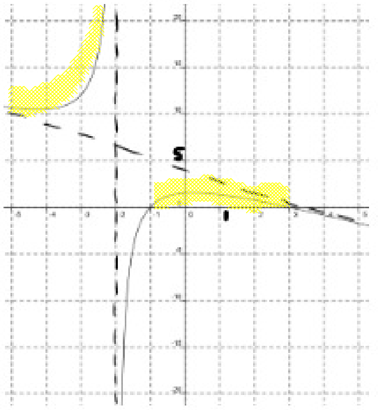
$x\in(-\infty,-2), [-1,3]$

look for x's where the graph is below x-axis since $f(x)\le 0$

OR

choose to move terms to the right:

$$0\le -x+4-\frac{5}{x+2}\longrightarrow 0\le\frac{-(x-3)(x+1)}{(x+2)}$$



look for x's where graph is above x-axis since $f(x)\ge 0$

C: Solving the related function with +/- table

choose to move terms to the left:

$$x-4+\frac{5}{x+2}\le 0 \longrightarrow \frac{(x-3)(x+1)}{(x+2)}\boxed{\le 0}$$

lower than zero

−∞ −2 −1 3 ∞

$x-3$	−	−	−	+
$x+1$	−	−	+	+
$x+2$	−	+	+	+
	−	+	−	+

OR

choose to move terms to the right: Higher than zero

$$0\le -x+4-\frac{5}{x+2} \longrightarrow 0\le \frac{-(x-3)(x+1)}{(x+2)}$$

−∞ −2 −1 3 ∞

-1	−	−	−	−
$x-3$	−	−	−	+
$x+1$	−	−	+	+
$x+2$	−	+	+	+
	+	−	+	−

3. Discuss what method, of the 3 shown above, is the best one to use on the following question, and then solve it.

$$\frac{2}{x+3}\le\frac{1}{x^2+1}\longrightarrow\frac{(x-1)(2x+1)}{(x+3)(x^2+1)}\le 0$$

not easy to sketch if doesn't factor ∴ METHOD C.

$$\frac{(x^2+1)\,2}{(x^2+1)(x+3)}-\frac{1\,(x+3)}{(x^2+1)(x+3)}\le 0$$

$$\frac{2(x^2+1)-1(x+3)}{(x+3)(x^2+1)}\le 0$$

* expand only the numerator

$$\frac{2x^2+2-x-3}{(x+3)(x^2+1)}\le 0$$

2 −1
1 −1

$$\frac{2x^2-x-1}{(x+3)(x^2+1)}\le 0$$

$$f(x)=\frac{(2x+1)(x-1)}{(x+3)(x^2+1)}\le 0$$

VA $x=-3$
zeros $x=1$ and $-\frac{1}{2}$

−∞ −3 −½ 1 ∞

$2x+1$	−	−	+	+
$x-1$	−	−	−	+
$x+3$	−	+	+	+
x^2+1	+	+	+	+
$f(x)$	−	+	−	+

$\therefore x\in(-\infty,-3)\,,\,[-\frac{1}{2},1]$

4. Summarize the steps of Method C: Using the +/- table for solving an inequality

① Move all terms to one side (Don't cross mult!!)
② LCD + Factor
③ Put VA and zeros onto the columns of +/− chart
④ if $f(x)\ge 0$ look for positive intervals
if $f(x)\le 0$ look for negative intervals

* always exclude VA values
* include zeros only if the question has $\le$, $\ge$
not include if $<$, $>$

~~14~~
13

5. Solve the following

a. $\dfrac{2x+1}{2x-3} \geq \dfrac{x+1}{x-5}$

$$\frac{(2x+1)}{(2x-3)} - \frac{(x+1)}{(x-5)} \geq 0$$

$$\frac{\left[(2x+1)(x-5) - (x+1)(2x-3)\right]}{(2x-3)(x-5)} \geq 0$$

*don't cancel! expand only numerator

$$\frac{2x^2 - 10x + x - 5 - (2x^2 - 3x + 2x - 3)}{(2x-3)(x-5)} \geq 0$$

$$\frac{-8x-2}{(2x-3)(x-5)} \geq 0$$

$$f(x) = \frac{-2(4x+1)}{(2x-3)(x-5)} \geq 0$$

zeros : $-1/4$

VA : $3/2$ and 5

	$-\infty$		$-1/4$		$3/2$		5		∞
-2		$-$		$-$		$-$		$-$	
$4x+1$		$-$	•	$+$		$+$		$+$	
$2x-3$		$-$		$-$	∞	$+$		$+$	
$x-5$		$-$		$-$		$-$	∞	$+$	
$f(x)$		$+$		$-$		$+$		$-$	

$\therefore x \in (-\infty, -\frac{1}{4}], (3/2, 5)$

b. $-\dfrac{2}{x} < x+1$

$$0 < x + 1 + \frac{2}{x}$$

$$0 < \frac{x^2 + x + 2}{x}$$

$b^2 - 4ac$ is neg. $\therefore$ parabola no zeros

opens up $\therefore$ always positive

	$-\infty$		0		∞
x^2+x+2		$+$		$+$	
x		$-$		$+$	
		$-$		$+$	

$\therefore x \in (0, \infty)$

~~16~~ 14

Solve Rational Equations

1. Unlike inequalities, equations can be cross multiplied. The only thing you must watch out for is whether your final solution is part of restriction or not. Solve the following questions and check for extraneous solutions.

extra answer that is not a solution

a. $\frac{2x}{5} = \frac{x^2-5x}{5x}$

$10x^2 = 5x^2 - 25x$

$0 = -5x^2 - 25x$

$0 = -5x(x+5)$

$\therefore \cancel{x=0}$ restriction OR $\boxed{x=-5}$

$\frac{2(0)}{5} = \frac{0^2-5(0)}{5(0)}$?

b. $\frac{(2+x)}{3x} = \frac{(x+1)}{(x+2)}$

$(2+x)(x+2) = 3x(x+1)$

$x^2+4x+4 = 3x^2+3x$

$0 = 2x^2 - 1x - 4$

2 1   4 1 2   1 4 2

b^2-4ac

$(-1)^2-4(2)(-4)$

$1+32$

$x = \frac{+1 \pm \sqrt{33}}{2(2)}$ $\boxed{x \doteq 1.7}$ $\boxed{x \doteq -1.2}$

restrictions

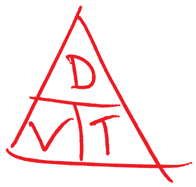
$x \neq 0$

$x \neq -2$

Next topic is to solve rational as well as other type word problems. Use the following already set up word problems to learn how to come up with equations then use the given equations to find the solution.

2. Dan and Sue set off at the same time on a 42 km go-cart race. Dan, drives 0.4 km/h faster than Sue, but has to stop en route and fix his go-cart for one-half hour. This stop costs Dan to arrive 15 min after Sue. How fast was each person driving?

	Distance, D	Speed, V	Driving only. Time $T=\frac{D}{V}$
Dan	42	X+0.4	$\frac{42}{x+0.4}$
Sue	42	X	$\frac{42}{x}$



Relate their times to make them equal: Dan's driving time + Dan stops to fix 30min(0.5h) = Sue's driving time + Sue stops at the end to wait for Dan 15min(0.25h)

Solve: $\frac{42}{x+0.4} + 0.5 - 0.25 = \frac{42}{x} + 0.25$

Cross multiply only if one fraction exists on each side!

$\frac{42}{x+0.4} + \frac{0.25(x+0.4)}{1(x+0.4)} = \frac{42}{x}$

$\frac{42+0.25(x+0.4)}{x+0.4} = \frac{42}{x}$

$\cancel{42x} + 0.25x^2 + 0.1x = \cancel{42x} + 16.8$

$0.25x^2 + 0.1x - 16.8 = \boxed{0}$ ☺

$x = \frac{-0.1 \pm \sqrt{0.1^2 - 4(0.25)(-16.8)}}{2(0.25)}$

~~$x = 8.4$~~

$x = 8$

$\therefore$ Sue's speed was 8km/h

Dan's speed was 8+0.4

8.4 Km/h

3. Connie can type 600 words in 5 minutes less than it takes Katie to type 600 words. If Connie types at a rate of 20 words per minute faster than Katie types, find the typing rate of each woman.

	Work Done (words)	Rate (words/min)	Time (min)
Connie	600	R + 20	T – 5
Katie	600	R	T

Relate the quantities together in two equations and two unknowns, then use substitution method:

Work = Rate × Time

① $600 = RT$

② $600 = (R+20)(T-5)$

isolate the first equation for T

$$\frac{600}{R} = T$$

sub this into the second equation

$$600 = (R+20)(\frac{600}{R} - 5)$$

expand

$$\cancel{600} = \cancel{600} - 5R + \frac{12000}{R} - 100$$

Solve:

$$100 = \frac{-5R}{1}\,{}^{\times R}_{\times R} + \frac{12000}{R}$$

$$\frac{100}{1} = \frac{-5R^2 + 12000}{R}$$

$$100R = -5R^2 + 12000$$

$$0 = -5R^2 - 100R + 12000$$

$$R = \frac{+100 \pm \sqrt{(-100)^2 - 4(-5)(12000)}}{2(-5)}$$

$$R = \frac{100 \pm 500}{-10}$$

$$R = \frac{600}{-10} \quad \text{or} \quad R = \frac{-400}{-10}$$

$$\cancel{R = -60} \qquad R = 40$$

∴ Katies typing rate was 40 words/min

Connie 40 + 20 = 60 words/min.

4. Pure alcohol is being added to 50 gallons of coolant mixture that is 40% alcohol. Find the rule of the concentration function c(x) that expresses the percentage of alcohol in the resulting mixture as a function of x gallons of pure alcohol that are added. Determine algebraically the exact amount of pure alcohol that must be added to produce a mixture that is 70% alcohol.

Relate the quantities in the equation:

$$\text{Final \% as a decimal} = \frac{\text{Partial \% of Total}}{\text{Total}}$$

$$\frac{0.7}{1} = \boxed{\frac{0.4*50 + 1*x}{50+x} = c(x)}$$

Solve:

$$0.7(50+x) = 20 + x$$

$$35 + 0.7x = 20 + x$$

$$35 - 20 = x - 0.7x$$

$$\frac{15}{0.3} = \frac{0.3x}{0.3}$$

$$x = 50$$

∴ 50 more gallons of 100% alcohol must be added to get 100 gallons of 70%

#16

Problem Solving

1. Rod agreed to mow a vacant lot for $12. It took him an hour longer than what he had anticipated, so he earned $1 per hour less than he originally calculated. How long had he anticipated that it would take him to mow the lot?

	Work (Earn in total) $	Rate $/hr	Time hr
anticipated	12	R	t
actual	12	R − 1	t + 1

Work = Rate × Time

① $12 = Rt$ → $\frac{12}{R} = t$

② $12 = (R-1)(t+1)$

$12 = (R-1)\left(\frac{12}{R} + 1\right)$

expand

$12 = \cancel{12}_{-12} + R - \frac{12}{R} - 1$

$0 = \frac{R \times R}{R \times 1} - \frac{1 \times R}{1 \times R} - \frac{12}{R}$

$\frac{0}{1} = \frac{R^2 - R - 12}{R}$

$0 = R^2 - R - 12$

$0 = (R-4)(R+3)$

~~$R = \frac{+1 \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)}$~~

~~$R = 1 \pm$~~ faster to factor

~~$R = -3$~~ or $R = 4$

∴ $t = \frac{12}{R}$

$t = \frac{12}{4} = 3$ ∴ He anticipated to work 3 hrs.

2. Suppose your mark in the math class is 60%. What mark, on average, do you need to get on the remaining 3 tests out of the total 9 tests to get your mark to be 70%?

$$\text{Final \% as a decimal} = \frac{\text{Partial \% of Total}}{\text{Total}}$$

$\frac{0.70}{1} = \frac{6(0.6) + 3x}{6 + 3}$

$0.7(9) = 3.6 + 3x$

$6.3 = 3.6 + 3x$

$\frac{2.7}{3} = \frac{3x}{3}$

$0.9 = x$

∴ you need to get on average 90% on next 3 tests to raise the mark from 60% to 70%

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3. To travel 60 miles, it takes Sue, riding a moped, 2 hours less than it takes Doreen to travel 50 miles on a bicycle. Sue travels 10 miles per hour faster than Doreen. Find the times and rates of speed of both women.

Two options:

① If you use **ONE VARIABLE** for speed and $T=\frac{D}{V}$ in the last column, then relate the girls' times to make them equal

	Distance, D	Speed, V	Time
Sue	60	V +10	$\frac{60}{V+10}$
Doreen	50	V	$\frac{50}{V}$

Sue`s driving time + Sue waits = Doreen`s driving time

$$\frac{60}{V+10}+2=\frac{50}{V}$$

② If you use **TWO VARIABLES**, one for speed, one for time:

	Distance, D	Speed, V	Time
Sue	60	V +10	T – 2
Doreen	50	V	T

use $D=VT$ to relate the variables, then sub one equation into another.

$60=(V+10)(T-2)$

$50=VT \qquad T=\frac{50}{V}$

$$60=(V+10)\left(\frac{50}{V}-2\right)$$

$$60=50-2V+\frac{500}{V}-20$$

$$0=-\frac{2V}{1}\cdot\frac{\times V}{\times V}+\frac{500}{V}-\frac{30}{1}\cdot\frac{V}{V}$$

$$\frac{0}{1}=\frac{-2v^2-30v+500}{V}$$

$$0=-2v^2-30v+500$$

$$0=-2(v^2+15v-250) \qquad \begin{matrix}+25\\-10\end{matrix}$$

$$0=-2(v+25)(v-10)$$

$V=\cancel{-25}$ or $V=10$

Doreen's speed is 10 mph

∴ Sue's speed 20mph

4. A tank has a capacity of 10 gallons. When it is full, it contains 15% alcohol. How many gallons must be replaced by an 80% alcohol solution to give 10 gallons of 70% solution?

$$\text{Final \% as a decimal} = \frac{\text{Partial \% of Total}}{\text{Total}}$$

$$0.7=\frac{10(0.15)-x(0.15)+x(0.80)}{10-x+x}$$

$$0.7(10)=1.5+0.65x$$

$$7-1.5=0.65x$$

$$5.5=0.65x$$

$$8.5\doteq x$$

8.5 gallons

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5. Walt can mow a lawn in 1 hour, while his son, Malik, can mow the same lawn in 50 minutes. One day Malik started mowing the lawn by himself and worked for 30 minutes. Then Walt joined him and they finished the lawn. How long did it take them to finish mowing the lawn after Walt started to help?

		Work (lawn)	Rate (lawn/min)	Time (min)
Anticipated	Walt by himself	1	1/60	60
	Malik by himself	1	1/50	50
Actual	Malik starts	3/5 done	1/50	30
	Finish together	2/5 to do	(1/50 + 1/60)	t

Work = Rate × Time

$$\frac{2}{5} = \left(\frac{1}{50} + \frac{1}{60}\right)t$$

$$\frac{2}{5} = \frac{11t}{300}$$

$$600 = 55t$$

$t = 10.9$ minutes to finish

6. A rock is dropped into a well, and 3 seconds later the sound of its splash is heard. How deep is the well? Assume the sound travels at 1100 feet per second and that an object falls a distance of $16t^2$ feet in t seconds. (t represents the time for the rock to reach the water)

	Distance, D	Speed, V	Time $T = \frac{D}{V}$
Rock	$16t^2$		T
Sound	$16t^2$	1100	$\frac{16t^2}{1000}$

rock`s time + sound`s time = total time

$$t + \frac{16t^2}{1100} = 3$$

$$\frac{16t^2}{1100} = \frac{3-t}{1}$$

$$16t^2 = 3300 - 1100t$$

$$16t^2 + 1100t - 3300 = 0$$

$t \doteq 2.9$ or $t = \cancel{-71.6}$

∴ Distance $16t^2$ $\doteq 135$ feet

7. A homemade loaf of bread turns out to be a perfect cube. Five slices of the bread, each 0.6 inch thick, are cut from one end of the loaf. The remainder of the loaf now has a volume of 700 cubic inches. What were the dimensions of the original loaf?

	Length	Width	Height	Volume
Original	x	x	x	x^3
Remove	5×0.6	x	x	$3x^2$
Result	n/a	n/a	n/a	700

original - remove = result

$$x^3 - 3x^2 = 700$$

$$f(x) = x^3 - 3x^2 - 700 = 0$$

$f(10) = 0$ ∴ $(x-10)$ is a factor

$$\begin{array}{r|rrrr} 10 & 1 & -3 & 0 & -700 \\ & \downarrow & 10 & 70 & 700 \\ \hline & 1 & 7 & 70 & 0 \end{array}$$

$$\therefore f(x) = (x-10)(x^2+7x+70)$$

$b^2 - 4ac$ is neg

∴ no more solutions

ONLY $x = 10$

∴ original loaf was $10 \times 10 \times 10$ inches^3

$$a.r.o.c = \frac{f(x_2)-f(x_1)}{x_2-x_1} \qquad a.r.o.c = \frac{f(x+h)-f(x)}{h} \xrightarrow[h\to 0]{} = i.r.o.c.$$

Rates of Change of Rationals

1. After you eat something that contains sugar, the pH of acid level in your mouth changes. This can be modeled by the function below where *L* is the pH level and *m* is the number of minutes that have elapsed since eating. Find the average rate of change $\boxed{\text{in the first 5 minutes}}$ $m \in [\underset{x_1}{0}, \underset{x_2}{5}]$

$$L(m) = \frac{-20.4m}{m^2+36} + 6.5$$

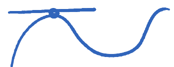
$$a.r.o.c = \frac{L(5)-L(0)}{5-0} = \frac{\left(\frac{-102}{61} \cancel{+6.5}\right) - \left(\cancel{\frac{0}{36}} \cancel{+6.5}\right)}{5}$$

$$\doteq -0.3344 \quad \text{ph/min}$$

$$\uparrow \frac{\text{units of } y}{\text{units of } x}$$

2. Find the turning points of $f(x) = \dfrac{x^2+1}{2-x}$

slope of tangent at t.p is zero



i.r.o.c. = mtan = 0

$$a.r.o.c = \frac{f(x+h)-f(x)}{h}$$

$$= \frac{\frac{(x+h)^2+1}{2-(x+h)} - \frac{x^2+1}{2-x}}{h}$$

$$= \frac{1}{h}\left[\frac{(x^2+2xh+h^2+1)}{(2-x-h)} - \frac{(x^2+1)}{(2-x)}\right]$$

$$= \frac{1}{h}\left[\frac{(x^2+2xh+h^2+1)(2-x) - (x^2+1)(2-x-h)}{(2-x-h)(2-x)}\right]$$

$$= \frac{1}{h}\left[\frac{\cancel{2x^2} \cancel{-x^3} + 4xh - 2x^2h + 2h^2 - xh^2 \cancel{+2} \cancel{-x} \;/\!/\!/ \left(\cancel{2x^2} \cancel{-x^3} - x^2h \cancel{+2} \cancel{-x} - h\right)}{(2-x-h)(2-x)}\right]$$

$$= \frac{1}{h}\left[\frac{4xh - 1x^2h + 2h^2 - xh^2 + h}{(2-x-h)(2-x)}\right]$$

$$= \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(4x - x^2 + 2h - xh + 1)}{(2-x-h)(2-x)}\right] \xrightarrow[h\to 0]{} i.r.o.c. = \frac{4x - x^2 + \cancel{2(0)} - \cancel{x(0)} + 1}{(2-x-\cancel{0})(2-x)}$$

solve for turning points

$$\frac{0}{1} = \frac{-x^2+4x+1}{(2-x)^2}$$

$$0 = -x^2+4x+1$$

$$x = \frac{-4 \pm \sqrt{20}}{-2}$$

$x = -0.24$
$x = 4.24$
} turng pts. $(-0.24, 0.47)$ $(4.24, -8.47)$