# nNOTESfixed2012

newExpLog
UnitNOTES

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# Exponentials & Logarithms Unit 8

Tentative TEST date______________

## Big idea/Learning Goals

This unit begins with the review of exponent laws, solving exponential equations (by matching bases method and trial & error method) and problems solving with exponential functions. You will then learn about the inverse of exponential functions – which is a logarithmic function of the same base. Just like there are several exponent laws there will be several logarithmic laws that you will have to know. These laws will help you in solving more complicated exponential equations where previous methods don't work or to avoid the use of trial & error method. The laws will also help you solve logarithmic equations. You will learn how to graph logarithmic functions by the use of transformations or by the use of key characteristics and finally you will study some real life situations that involve logarithms.

<u>Corrections for the textbook answers:</u>

| | | |
|---|---|---|
| Sec 8.1 | #9c) 3. | |
| Sec 8.2 | #4d) iii) domain x>0, iv) left by 2 | #6a) (25, -1) |
| Sec 8.3 | #4d) 1.40 #14a) 223 mph | |
| Sec 8.4 | #10c) x=4 | |
| Sec 8.5 | #6a) 9.01 | |
| Sec 8.6 | #9b) $2.5X10^{-4}$ | #10 x=2 the only solution |
| Sec 8.8 | #7 table is wrong for it to be exponential with same growth rate | |
| Review | #7d) log144 | |

## Success Criteria

- ☐ I am <u>ready for this unit</u> if I am confident in the following review topics
  - ☐ Exponent laws
  - ☐ Solve for the exponent by matching bases & by trial and error
  - ☐ Exponential word problems
  - ☐ Transformations
  - ☐ Inverses
  - ☐ Domain & range
  - ☐ Rates of change
- ☐ I <u>understand the new topics</u> for this unit if I can do the practice questions in the textbook/handouts

May 16

| Date | pg | Topics | # of quest. done? You may be asked to show them |
|---|---|---|---|
| May 9 | 2-4 | Review of Grade 11 – THREE Handouts<br>Exponent laws, Solve exponential equations by matching bases, Solve exponential word problems by trial & error | |
| | 5-6 | What is a Logarithmic Function?<br>Section 8.1 & 8.3 & THREE Handouts | |
| | 7-8 | Exponential & Logarithmic Functions<br>Section 8.2 & THREE Handouts | |
| | 9-10 | Laws of Logarithms<br>Section 8.4 & TWO Handouts | |
| May 14 | 11-12 | Solving Exponential Equations by using Logs<br>Section 8.5 & TWO Handouts | |
| | 13-14 | Solve Logarithmic Equations<br>Section 8.6 & TWO Handouts | |
| | 15-16 | Solve Problems<br>Section 8.7 & Handout | |
| | | EXTRA<br>• Assignment with a Catholic component | |

## Reflect – previous TEST mark __________, Overall mark now__________.

Looking back, what can you improve upon?

## Review of Grade 11

1. Summarize the laws you learned in grade 9-11 (multiplication, division, power of power, zero, negative, rational, distributive properties)

$a^m \cdot a^n = a^{m+n}$ ← Keep the base

$2^3 \cdot 2^4 = 4^7$ ✗ $2^7$ ✓

$\frac{a^m}{a^n} = a^{m-n}$ ← base is in numerator

$\frac{x^3}{x^7} = \frac{x^{-4}}{1} = \frac{1}{x^4}$

$a^{-n} = \frac{1}{a^n}$

$(a^n)^m = a^{nm}$

$a^0 = 1$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$\left(\frac{3ab^2}{c^3}\right)^5 = \frac{3^5a^5b^{10}}{c^{15}}$

$(3+ab^2)^5$ can't!!

2. Apply the laws to the following examples as you simplify the questions. Leave everything as exact numbers with positive exponent answers. Do power of power 1st, leave neg. exp. rule for last

a. $\frac{\left(-2xy^3\times 3x^{-3}y^{-2}\right)^3}{6x^0y^{-1}}$

$= \frac{(-1)^3(2)^3x^3y^9(3^3)x^{-9}y^{-6}}{6x^0y^{-1}}$

$9+-6--1$

$= -\frac{216^1x^{-6}y^4}{6^1}$

not proper to leave exponents as negatives.

$= -\frac{36y^4}{x^6}$

b. $\left(8x^6y^9\right)^{\frac{1}{3}}\left(27x^{-12}y^2\right)^{-\frac{1}{3}}$

$= 8^{\frac{1}{3}}x^2y^3\,27^{-\frac{1}{3}}x^4y^{-2/3}$

$= \frac{(2)x^6y^{7/3}}{27^{1/3}}$

$\frac{3}{3}3-\frac{2}{3}$

$\frac{9-2}{3}$

$= \frac{2x^6y^{7/3}}{3}$

or

$\frac{2}{3}x^6\sqrt[3]{y^7}$

c. $\left(\frac{64m^{15}}{343}\right)^{-\frac{2}{3}}$

$= \frac{49}{16m^{10}}$

d. $\frac{\left(-3c^4\right)^{-2}}{c^{-1}\times\left(3c^{-2}\right)^{-2}}$

$= \frac{1}{c^{11}}$

e. $\left(8x^{\frac{3}{4}}y^2\right)^{-\frac{1}{3}}$

$= \frac{1}{2x^{1/4}y^{2/3}}$

f. $\left(\frac{\left(-2a^{-2}\right)^3a^3}{4a^{-4}}\right)^{-3}$

$-1\times 2$

start from inner bracket!!

$= \left[\frac{(-1)^3 2^3 a^{-6} a^3}{4a^{-4}}\right]^{-3}$

$= \frac{(-1)^{-9}2^{-9}a^{18}a^{-9}}{4^{-3}a^{12}}$

$= -\frac{4^3a^{-3}}{2^9} = \frac{-64}{512a^3} = \frac{-1}{8a^3}$

$e^x$ $e^1$

There are several useful constants that are used for math

$\pi$ is called __pi__ and $\pi \doteq$ __3.14159265 4....__ is used with anything circular

$e$ is called __Euler's constant__ and $e$ = __2.718281828...__ is used with exponential continuous growth/decay

Read up more on the number *e*:
http://betterexplained.com/articles/an-intuitive-guide-to-exponential-functions-e/\

When the variable is **on the base**, not in the exponent, to solve it you must isolate it by using __BEDMAS backward__

When the variable is **in the exponent**, not on the base, to solve it you must __Match bases (goal: have SAME single base on both sides of equation__

When the variable is **in the exponent** and bases **cannot** be matched you must use __Trial + error (or later can use logs.)__

3. Match the method to each given question. Then solve.

a. $2^{2x} = 8$

$2^{2x} = 2^3$

$\therefore 2x = 3$

$x = \frac{3}{2}$

b. $2x^{\frac{4}{5}} + 12 = 174$ $-12$

$\frac{2x^{4/5}}{2} = \frac{162}{2}$

$x^{4/5} = 81$ do power $\frac{5}{4}$

$\left(\sqrt[5]{x^4}\right)^5 = (81)^5$

$\sqrt[4]{x^4} = \sqrt[4]{81^5}$

$x = 81^{5/4} = 243$

c. $2 \cdot e^x + 5 = 22$

$2e^x = 17$

$e^x = 8.5$

$2.71...^x = 8.5$

$2.71^3$

$2.71^2$

$e^{2.15} \doteq 8.5$

$\therefore x \doteq 2.15$

4. Practice matching bases method:

a. $3^{2x-5} = 1$

$3^{2x-5} = 3^0$

$\therefore 2x - 5 = 0$

$x = \frac{5}{2}$

b. $\left(\frac{1}{5}\right)^{-3x} \cdot 25^{x-1} = \frac{1}{125}$

$x = 243$

c. $5^{x-2} + 5^{x-3} = 150$

$5^x \cdot 5^{-2} + 5^x 5^{-3} = 150$

let $a = 5^x$

$a\frac{1}{5^2} + a\frac{1}{5^3} = 150$

$\frac{5 \times a}{5 \times 25} + \frac{a}{125} = 150$

$\frac{6a}{125} = \frac{150}{1}$

$a = 3125$

$\therefore 5^x = 3125$

$5^x = 5^5$

$\therefore x = 5$

d. $4^x = \frac{1}{256}$

$x = \frac{5}{2}$

e. $2^1 \cdot \left(\frac{1}{64}\right)^x = \left(\frac{1}{2}\right)^{x+2.5}$

add

$2^1 \cdot 2^{-6(x)} = 2^{-1(x+2.5)}$

$2^{1-6x} = 2^{-x-2.5}$

$\therefore 1 - 6x = -x - 2.5$

$3.5 = 5x$

$0.7 = x$

f. $3^{x+3} - 3^x = 234$

$x = 2$

5. Here is the general equation for exponentials that will most often be used for exponential word problems that have horizontal asymptote of y = 0. Explain the significance of EACH letter in the context of a word problems and summarize how to find the 'b' in the equation.

$y = a(b)^{\frac{x}{p}}$

y = dependent variable / final amount
a = initial amount
b = growth / decay FACTOR

not the same as growth/decay
% = RATE = r

b = 1 + r if % increase
b = 1 − r if % decrease
b = 2, 3, ½ if double / triple / half

x = independent variable (usually time)
p = How long it takes to grow by factor b ?
ex. Half-life

6. Solve the following problems by using the trial & error method.
(you will later learn how to use logs to solve these without the use of trial and error)

a.
A drug's effectiveness (Mass) decreases as time passes. Each hour the 250mg drug loses 5% of its effectiveness. [How long will it take for the dose to reach the low level of 52mg?]

$b = 1 - r = 1 - 0.05 = 0.95$

$M(h) = a\,b^{\frac{h}{p}}$

$= 250(0.95)^{\frac{h}{1}}$

$\frac{52}{250} = \frac{250(0.95)^h}{250}$

$0.208 = (0.95)^h$

trial + error

$\therefore h \doteq 31$ hours

b.
Carbon-14 (Name) has a half life of 5730 years. $a = 100\%$ or $a = 1$
(If no initial amount is given, assume 100% is the initial amount)
Some pre-historic cave paintings were discovered in a cave in France. If the paint contained 48% of the orinial carbon-14, estimate the age of the painting.

$C(t) = 1\,(0.5)^{\frac{t}{5730}}$

$0.48 = (0.5)^{\frac{t}{5730}}$ let $a = \frac{t}{5730}$

$0.48 = (0.5)^a$

$a \doteq 1.06$

$\therefore 1.06 = \frac{t}{5730}$

$6074 \doteq t$ years

c.
A cottage is originally bought for $150 000. If the value of this cottage appreciates at the rate of 7% per year, when will the cottage be worth $200 000?

(4.3 years)

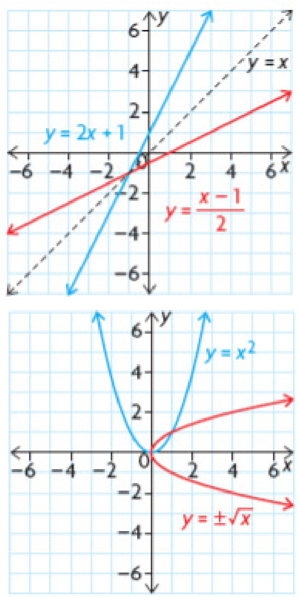
d.
The 200 fruit fly population doubles every 5 days. In how many days is the population up to 1000 flies?

(11.5 days)

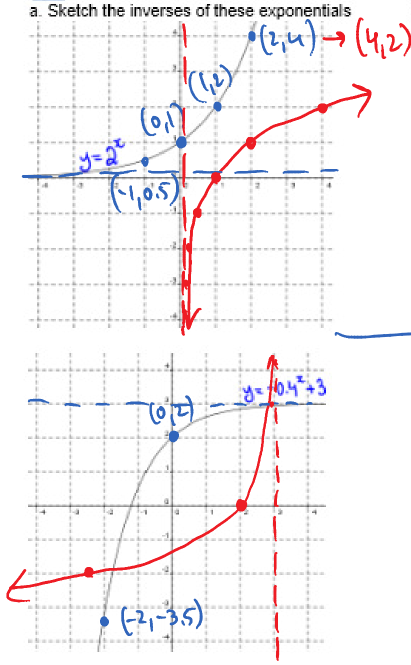
## What is a Logarithmic Function?

1. You've seen inverses for all the functions you've studied so far, except for exponentials. Recall that inverse graphs are just reflections in y=x lines and inverse equations have x=input and y=output switched.

recall inverses of lines and quadratics




a. Sketch the inverses of these exponentials




b. Find the inverse equations of these functions.

$y = 2^x$ original

$x = 2^y$ inverse

need to isolate y

switch form

$\log_2(x) = y$

$y = -1(0.4)^x + 3$ original

$x - 3 = -1(0.4)^y$ inverse

$-x + 3 = (0.4)^y$

switch

$\log_{0.4}(-x+3) = y$

2. The inverse equations you've found above don't have the output isolated and hence cannot be written in function notation. This is one of the reasons that logarithms were invented. Summarize the rule of switching exponential form to logarithmic form or vise versa, then write down the inverse functions using function notation for the above questions.

m ← input of exponential

$K = L^m \iff \log_L(K) = m$

input of log

output

output of exponential

Base of exponential

Base of logarithm

3. Practice switching the form

a. $r = \log_p q \longleftrightarrow p^r = q$ (switch form)

b. $a^b = c \longleftrightarrow \log_a(c) = b$

c. $\log_4 2 = \frac{1}{2} \longleftrightarrow 4^{1/2} = 2$

d. $\left(\frac{1}{5}\right)^2 = \frac{1}{25} \longleftrightarrow \log_{\frac{1}{5}}\left(\frac{1}{25}\right) = 2$

4. What meaning does $\log_a x$ have?

$\boxed{\text{a logarithm is an exponent!}}$ ie. $\log_a x = y \leftrightarrow a^y = x$

5. Evaluate the following

a. $\log_2 16$ – what exponent on base 2 gives the answer 16?
$= 4$ $\hookrightarrow = \log_2(2^4)$

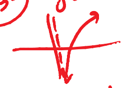
b. $\log_3 \sqrt[7]{9} = \log_3\left(3^{2/7}\right) = 2/7$

c. $\log_2 \frac{1}{64}$
$= \log_2\left(2^{-6}\right) = -6$

d. $\log_{\frac{1}{7}} 49^2 = \log_{\frac{1}{7}}\left(\frac{1}{7}\right)^{-4} = -4$
$(7^2)^2$
$7^4$

6. Find the inverse functions and state the domain and range of the given function. $\frac{1}{3}(4^x) - \frac{1}{3}$
(domain and range of the inverse, as you know, are just a switch between the domain and range of the original function)

a. $y = 2\log_3(x-1)$ original $= f$ — $\log_3 x$ parent — growth log



apply stretch + shift

$x = 2\log_3(y-1)$ inverse

$\frac{x}{2} = \log_3(y-1)$

$3^{\frac{x}{2}} = y - 1$

$3^{\frac{x}{2}} + 1 = y = f^{-1}$

or

$\left(\sqrt{3}\right)^x + 1 =$

can appear in different form

$D_f = \{x \in \mathbb{R}, x > 1\}$

$R_f = \{y \in \mathbb{R}\}$

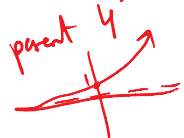
input of log must be pos

$x - 1 > 0$

$x > 1$

b. $y = \frac{4^x - 1}{3}$ original $= f$

parent $4^x$



$x = \frac{4^y - 1}{3}$ inverse

$3x = 4^y - 1$

$3x + 1 = 4^y$

$\log_4(3x+1) = y = f^{-1}$

$D_f = \{x \in \mathbb{R}\}$

$R_f = \{y \in \mathbb{R}, y > -\frac{1}{3}\}$

c. $y = \log_4 2x + 5$

$f^{-1} = y = \frac{4^{x-5}}{2} = \frac{(2^2)^{x-5}}{2}$

can appear in different form

or

$y = 2^{2x-11}$

logs do that too. will learn later

$D_f = \{x \in \mathbb{R}, x > 0\}$ "d"

$R_f = \{y \in \mathbb{R}\}$

d. $y = 2(3)^{x-9}$

$D_f = \{x \in \mathbb{R}\}$

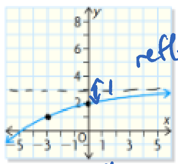
$R_f = \{y \in \mathbb{R}, y > 0\}$ "c"

$f^{-1} = y = \log_3\left(\frac{x}{2}\right) + 9$

## Exponential & Logarithmic Functions → Sketch

1. Review how to find the equation of the exponential function from a table or a graph

a.



reflected

Exponenial General equation

$y = a\,b^{k(x-d)} + c$

k and d can become part of base b to a

$\boxed{y = a\,b^{x} + c}$

distance between H.A. and y-intercept (can have reflection as well)

HA

$y = -1\,b^{x} + 3$

sub pt. $(-3, 1)$

$1 = -1(b)^{-3} + 3$

$\frac{-2}{-1} = \frac{-1}{-1} b^{-3}$

$2 = b^{-3}$

$\frac{2}{1} = \frac{1}{b^3}$ or flip both sides

$\sqrt[3]{\frac{1}{2}} = \sqrt[3]{b^3}$

$b \doteq 0.79$

$\therefore \boxed{y = -1(0.79)^x + 3}$

b.

| x | y |
|---|---|
| 2 | 14.75 |
| 4 | 113.19 |
| 6 | 728.42 |
| 8 | 4573.64 |

Horizontal asymptote at y=-4 c

$y = a b^x - 4$

pt. $(2, 14.75)$

$14.75 = ab^2 - 4$

$\boxed{18.75 = ab^2}$ ①

pt. $(4, 113.19)$

$113.19 = ab^4 - 4$

$\boxed{117.19 = ab^4}$ ②

② divided by ①

$\frac{117.19}{18.75} = \frac{\cancel{a}b^4}{\cancel{a}b^2}$

$18.75 = a(2.5)^2$

$3 = a$

$\pm\sqrt{6.25} \doteq \sqrt{b^2}$

$2.5 = b$

$\therefore \boxed{y = 3(2.5)^x - 4}$

2. Summarize the steps of sketching exponentials. $y = ab^{k(x-d)} + c$

1. Simplify the equation to look like $y = ab^x + c$
2. Sketch HA = c
3. sub x=0 to find y-int
4. Decide if growth/decay and if there are reflection ↳ draw below HA

Sketch the following functions

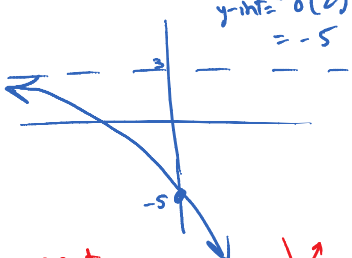
3. $y = -2(4)^{\frac{x}{2}+1} + 3$

$y = -2(4)^{\frac{x}{2}}(4)^1 + 3$

$y = -8(4^{\frac{1}{2}})^x + 3$

$y = -8(2)^x + 3$

y-int $= -8(2)^0 + 3$
$= -5$



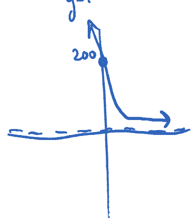

parent $y = 2^x$ growth but reflected in x-axis

4. $y = 100(0.5)^{2x-1}$

$y = 100(0.5^2)^x(0.5)^{-1}$

$y = 200(0.25)^x$

y-int = 200




5. $y = 5(3.5)^{\frac{8-x}{2}} - 10$

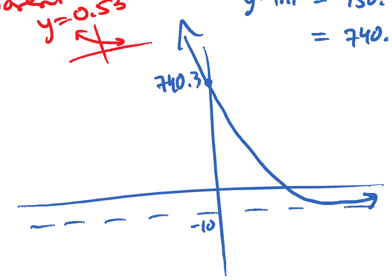
$y = 5(3.5)^{8/2}(3.5)^{-\frac{x}{2}} - 10$

$y = 5(3.5)^4(3.5^{-1/2})^x - 10$

$y \doteq 750.3(0.53)^x - 10$

parent $y = 0.53^x$

y-int $= 750.3(0.53)^0 - 10$
$= 740.3$



6. Summarize the steps of sketching logarithms: $y = a\log_b[k(x-d)]+c$

1. Factor out $k$ so that $d$ is visible, then VA $= d$
2. Find $x$-int sub $y=0$ + solving
3. Decide if it grows/decay and check if there are any reflections.

without reflections:


Sketch the following functions

7. $y = -\frac{1}{6}\log_{0.2} x - 3$

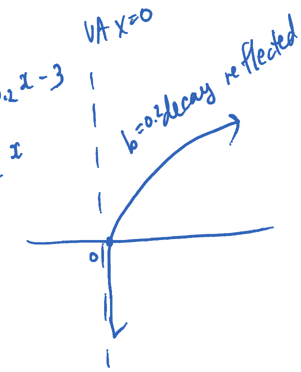
x-int

$0 = -\frac{1}{6}\log_{0.2} x - 3$

$3(-6) = \log_{0.2} x$

$0.2^{-18} = x$

tiny #.




8. $y = \log_{10}(6x+12)+0.5$

$y = \log_{10}(6(x+2)) + 0.5$

$\therefore$ VA $= -2$

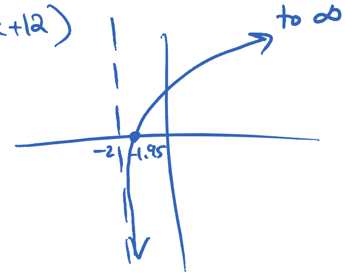
$0 = \log_{10}(6x+12)+0.5$

$-0.5 = \log_{10}(6x+12)$

$10^{-0.5} = 6x+12$

$\frac{(10^{-0.5}-12)}{6} = x$

$-1.95 \doteq x$




9. $y = 2\log_{0.5}(15-3x)+1$

$y = 2\log_{0.5}(-3(x-5)) + 1$

VA $= d = 5$

x-int:

$0 = 2\log_{0.5}(15-3x)+1$

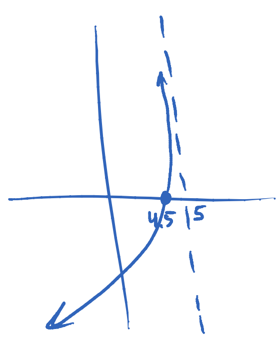
$-\frac{1}{2} = \log_{0.5}(15-3x)$

$0.5^{-1/2} = 15-3x$

$1.41 = 15-3x$

$\frac{-15}{-3}$

$4.5 \doteq x$




10. $y = 2+\log_3(4x+1)$

$y = \log_3(4(x+\frac{1}{4})) + 2$

VA $x = -\frac{1}{4}$

x-int

$0 = \log_3(4x+1)+2$

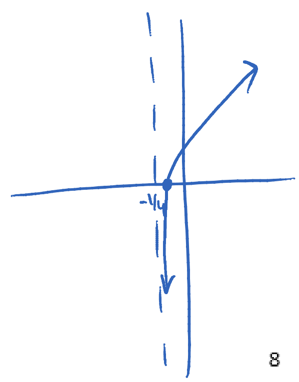
$-2 = \log_3(4x+1)$

$3^{-2} = 4x+1$

$\frac{\frac{1}{9}-1}{4} = x$

$-\frac{2}{9} = x$

$-0.\bar{2} \doteq x$



## Laws of Logarithms

Read and understand the following proofs to the new logarithm laws.

1. Proof for

$\boxed{\log_a a^x = x}$

let $f(x) = a^x$ then

$f^{-1}(x) = \log_a x$

you already know that

$f^{-1}(f(x)) = x$, therefore

$\log_a(a^x) = x$

2. Proof is similar for

$\boxed{a^{\log_a x} = x}$

if $f(x) = a^x$

then $f^{-1}(x) = \log_a x$ (inverse of exponential)

$LS = a^{\log_a x} = f(f^{-1}(x)) = x = RS$

inverses cancel each other

$\therefore LS = RS$

3. Proof for

$\boxed{\log_a(xy) = \log_a x + \log_a y}$

Use the exponent multiplication law:

take log of both sides

$\boxed{a^m \cdot a^n = a^{m+n}}$ product law for exponentials

$\log_a(a^m \cdot a^n) = \log_a a^{m+n}$

to get variables to match

let $x = a^m \Leftrightarrow \log_a x = m$ ★

and $y = a^n \Leftrightarrow \log_a y = n$ then

switch form

$LS = \log_a(x \cdot y) = \cancel{\log_a a}^{m+n}$ cancel off.

$\log_a(x \cdot y) = m + n$

$\log_a(x \cdot y) = \log_a x + \log_a y$ use ★

Notice bases have to match and coefficients as well.

$c\log_a(x \cdot y) = c\log_a x + c\log_a y$

4. Proof is similar for

$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$

$\frac{a^m}{a^n} = a^{m-n}$ use ★

exponent law

$LS = \log_a\left(\frac{x}{y}\right) = \log_a\left(\frac{a^m}{a^n}\right) = \cancel{\log_a}\left(\cancel{a}^{m-n}\right)$

cancel

$= m - n$

$= \log_a x - \log_a y$ use ★

$= RS$ $\therefore LS = RS$

5. Proof for

$\boxed{\log_a(x)^n = n\log_a x}$

Use the exponent power of power law

$\boxed{(a^m)^n = a^{mn}}$ exponent law

take log of both sides

$\log_a (a^m)^n = \log_a a^{mn}$

let $x = a^m \Leftrightarrow \log_a x = m$ then ★

$LS = \log_a(x)^n = \cancel{\log_a a}^{mn}$ cancel

$\log_a(x)^n = mn$

$\log_a(x)^n = n\log_a x$ ★

6. Proof for CHANGE of BASE

$\boxed{\log_b a = \frac{\log a}{\log b}}$ allows you to do any BASE on calculator

$b^x = a \Leftrightarrow \log_b a = x$

$b^x = a$

$\log b^x = \log a$

$x\log b = \log a$

$x = \frac{\log a}{\log b}$

$\log_b a = \frac{\log a}{\log b}$

ex. $\log_3 20$

$= \frac{\log 20}{\log 3}$

OR

$= \frac{\ln 20}{\ln 3}$

$= 2.7268\ldots$

7. Proof for

$\boxed{\log_a 1 = 0}$

Use the zero exponent law

$a^0 = 1$

$\log_a a^0 = \log_a 1$

$0 = \log_a 1$

$\log_2 32 = 5$

8. Expand the following by using laws of logs

a. $\log_a\left(\frac{xy^2}{z}\right)$

$= \log_a(xy^2) - \log_a z$

$= \log_a x + \log_a y^2 - \log_a z$

$= \log_a x + 2\log_a y - \log_a z$

b. $\log_a\left(y^3 z^{\frac{1}{3}}\right)$

$= \log_a y^3 + \log_a z^{1/3}$

$= 3\log_a y + \frac{1}{3}\log_a z$

c. $\log_a\sqrt{(yz)}$

$= \log_a (yz)^{1/2}$ $= \frac{1}{2}\log_a yz$

$= \log_a y^{1/2} z^{1/2}$ $= \frac{1}{2}(\log_a y + \log_a z)$

$= \frac{1}{2}\log_a y + \frac{1}{2}\log_a z$

d. $\log_a\left(\frac{x}{\sqrt{(xy)}}\right)^{\frac{1}{4}}$ $\frac{1}{2}$ $\frac{2\times 1}{2\times 4} - \frac{1}{8}$

$= \log_a\left(\frac{x^{1/4}}{x^{1/8}y^{1/8}}\right)$

$= \log_a\left(\frac{x^{1/8}}{y^{1/8}}\right)$

$= \log_a x^{1/8} - \log_a y^{1/8}$

$= \frac{1}{8}\log_a x - \frac{1}{8}\log_a y$

9. Condense the following into a single logarithm

a. $4\log_a\sqrt{x} - \log_a y$

$= \log_a \sqrt{x}^4 - \log_a y$

$= \log_a\left(\frac{x^2}{y}\right)$

before you combine terms remove coeffients (bring them up as exponents.)

b. $-5\log_a x + 2\log_a y + \frac{1}{3}\log_a z$

$= -\log_a x^5 + \log_a y^2 + \log_a z^{1/3}$

$= \log_a\left(\frac{y^2 z^{1/3}}{x^5}\right)$

$\log_a x^{-5} + \log_a y^2 + \log_a z^{1/3}$

$= \log_a\left(x^{-5}y^2z^{1/3}\right)$

c. $\frac{3}{2}\log_a z + \log_a y$

$\log_a\left(z^{3/2}y\right)$

d. $-2\log_a x^2 - \frac{1}{2}\log_a y + \frac{1}{5}\log_a z$

$= -\log_a x^4 - \log_a y^{1/2} + \log_a z^{1/5}$

$\log_a\left(\frac{z^{1/5}}{x^4 y^{1/2}}\right)$

10. Evaluate

a. $\log_{\frac{1}{3}} 9 + \log_5 625$

(possible without a calculator since bases can be matched)

$= \cancel{\log_{\frac{1}{3}}}\left(\frac{1}{3}\right)^{-2}$

$= -2$

b. $\log_8 92$

(not possible without the change of base formula and calculator)

$= \frac{\ln(92)}{\ln(8)}$

$\doteq 2.17\ldots$

6.4% every 3 yrs $\left(1+\frac{0.064}{1}\right)^{\frac{t}{3}}$

6.4% monthly $\left(1+\frac{0.064}{12}\right)^{\frac{t}{\frac{1}{12}}}$ or 12t

## Solving Exponential Equations by using Logs

$y = ab^{\frac{x}{p}}$ ← how long it takes

initial

1+r
1−r
2 = double
$\frac{1}{2}$ = half-life

1. An investment of $500 is invested in an account that pays 6.4% compound annually. How long will it take for the original amount of the investment to triple?

$A(t) = 500(1+0.064)^{\frac{t}{1}}$

$A(t) = 500(1.064)^{t}$

$\frac{1500}{500} = \frac{500}{500}(1.064)^{t}$

$3 = 1.064^{t} \longleftrightarrow \log_{1.064}(3) = t$

$17.7 = \frac{\ln(3)}{\ln(1.064)} = t$

years

2. A culture of bacteria triples every 30 minutes. How long will it take a culture originally consisting of 40 bacteria to grow to a population of 200 000 bacteria?

$B(m) = 40(3)^{\frac{m}{30}}$ ← minutes

OR $B(h) = 40(3)^{\frac{h}{0.5}}$ ← hours

$\frac{200\,000}{40} = \frac{40}{40}(3)^{\frac{m}{30}}$

$5000 = 3^{\frac{m}{30}}$

$\log_3(5000) = \frac{m}{30}$

$30 \times \frac{\ln 5000}{\ln 3} = m$

233 minutes $= m$

Here are a few strategies to try when solving exponential equations

- Match bases if possible (goal is to have a single base on one side of the equation and the same single base on the other side of the equation – no coefficients)
- Make a substitution to simplify the equation (is is helpful for questions like c.)
- Take log or ln of both sides (as a whole, not log of separate terms!) (Do this ONLY if other methods don't work)
  → OR switch form...

3. Solve the following equations.

a. $5^{2x} = \sqrt{\frac{1}{625}}$

$5^{2x} = (5^{-4})^{\frac{1}{2}}$

$5^{2x} = 5^{-2}$

$\therefore 2x = -2$

$x = -1$

b. $\left(\frac{1}{4}\right)^{x+1} + \left(\frac{1}{4}\right)^{x} = 20$

$\left(\frac{1}{4}\right)^{x}\left(\frac{1}{4}\right)^{1} + \left(\frac{1}{4}\right)^{x} = 20$

let $a = \left(\frac{1}{4}\right)^{x}$

$\frac{1}{4}a + 1a = 20$

$\cancel{\frac{4}{5}} \times \cancel{\frac{5}{4}} a = 20 \times \frac{4}{5}$

$a = 16$

$\therefore \left(\frac{1}{4}\right)^{x} = 16$

$4^{-x} = 4^{2}$

$\therefore -x = 2$

$x = -2$

No Rule for $\log(x+y)$ - leave as is

$\frac{\log(x)}{\log(y)} \neq \log\left(\frac{x}{y}\right)$

$= \log_y x$

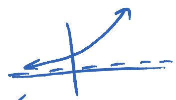
c. $4(3^x)+3^{2x}=-4$ $\quad (3^x)^2$

let $a=3^x$

$4a+a^2=-4$

$a^2+4a+4=0$

$(a+2)(a+2)=0$



$a+2=0$

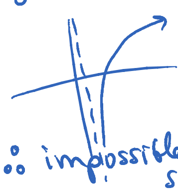
$a=-2$

$\therefore\ 3^x=-2$

$\log_3(-2)=x$

error $=\dfrac{\ln(-2)}{\ln(3)}=x$

graph



$\therefore$ impossible to solve $(LS\neq RS)$

d. $\left(\dfrac{5}{4}\right)^{10-x}=7^x$

*Don't switch form unless there is only ONE exponential

take ln of both sides.

only do this for MONOMIALS

$\ln\left(\dfrac{5}{4}\right)^{10-x}=\ln 7^x$

$(10-x)\ln(1.25)=x\ln 7$

$10\ln 1.25-x\ln 1.25=x\ln 7$

$10\ln 1.25=x\ln 7+x\ln 1.25$

common factor

$10\ln 1.25=x(\ln 7+\ln 1.25)$

$$\boxed{\dfrac{10\ln 1.25}{\ln 7+\ln 1.25}=x}$$

other versions!!

$\dfrac{10\ln 1.25}{\ln 8.75}$

e. $3^{x-4}=9^{x-6}$ similar to

match bases!!

$3^{x-4}=(3^2)^{x-6}$

$\therefore\ x-4=2(x-6)$

$x-4=2x-12$

$8=x$

$$\boxed{X=8}$$

f. $19^{\frac{x}{2}-4}=81$

switch. since there is ONE exponential!!

$\log_{19}(81)=\dfrac{x}{2}-4$

$+4$

$\times 2$

$X=\left(\log_{19}(81)+4\right)\times 2$

or other versions!!

$x=\dfrac{2\ln 81+8\ln 19}{\ln 19}$

$x=\dfrac{\ln(81^2\cdot 19^8)}{\ln 19}$

$\doteq 10.98$

g. $4^x\cdot\dfrac{1}{16}=2^{3x+6}$

$(2^2)^x\cdot 2^{-4}=2^{3x+6}$

$2^{2x-4}=2^{3x+6}$

$\therefore\ 2x-4=3x+6$

$$\boxed{-10=x}$$

h. $\left(\dfrac{1}{2}\right)^{x+1}=\left(\dfrac{1}{3}\right)^x$

$2^{-x-1}=3^{-x}$

$\ln 2^{-x-1}=\ln 3^{-x}$

$(-x-1)\ln 2=-x\ln 3$

$-x\ln 2-\ln 2=-x\ln 3$

$-x\ln 2+x\ln 3=\ln 2$

$x(-\ln 2+\ln 3)=\ln 2$

$x=\dfrac{\ln 2}{\ln 3-\ln 2}\doteq 1.709\ldots$

or $\dfrac{-\ln 2}{-\ln 3+\ln 2}$

$x=\dfrac{\ln 2}{\ln(3/2)}$

## Solving Logarithmic Equations

When you are are solving logarithmic equations, keep in mind the domain of logarithms and discard any solutions that make $\log(zero) = undefined$ or $\log(negative) = undefined$

Also remember that you CANNOT distribute the log over separate terms, just like you CANNOT distribute exponents over several terms
Ex.

can't Don't ever distribute

$\log(3+x)$ not multiplication

can't

$(3+x)^2$

Here are a few strategies to try when solving logarithmic equations

- Change the form to an exponential $\log_b a = c \Leftrightarrow b^c = a$ one log only! and solve using exponential equation rules (can only do this when there is a single log on one side that is isolated)
- Equate the inputs of the logs (can do this when there is one log term on BOTH sides of the equation of the SAME base) Two logs
- Use laws of logs to condense the log terms to a single log for the whole equation or a one log for each side of equation, then attempt the other strategies above

Solve the following equations.

a. $\log_x \frac{16}{18} = 4$

$x^4 = \frac{16}{18}$

$x = \pm\sqrt[4]{\frac{16}{18}}$

$x = +\sqrt[4]{\frac{16}{18}}$ or $\cancel{x = -\sqrt[4]{\frac{16}{18}}}$ discard since base must be positive

$x \doteq 0.97$

b. $\log_{2.5}(6x) = \log_{2.5}(2x+36)$

$-2x$ $-6-x$

equate inputs

$\therefore 6x = 2x + 36$

$4x = 36$

$x = 9$

$-2x = -6 - x$

$-x = -6$

$x = 6$

But $\log_{2.5}(-2(6))$ undefined

$\therefore$ no sol.

c. $\log_4 18x - \log_4 2.5 = \log_4 39.6$

$\log_4\left(\frac{18x}{2.5}\right) = \log_4 39.6$

$\therefore$ equate inputs

$\frac{18x}{2.5} = 39.6 \times 2.5 \div 18$

$x = 5.5$

$a^b = c$

$\log_a c = b$

d. $\log_5(x+2\sqrt{6}) + \log_5(x-2\sqrt{6}) = 2$

$\log_5\left((x+2\sqrt{6})(x-2\sqrt{6})\right) = 2$

$\log_5\left(x^2 - \cancel{2\sqrt{6}x} + \cancel{2\sqrt{6}x} - 4(6)\right) = 2$

$\log_5(x^2 - 24) = 2$

$5^2 = x^2 - 24$

$25 = x^2 - 24$

$\pm\sqrt{49} = \sqrt{x^2}$

$x = 7$ or $x = -7$

discard since $\log(x+2\sqrt{6})$ $\log(-7+2\sqrt{6})$ log (negative!)

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e. $\log_x 8 = \frac{1}{4}$

$$\left(x^{1/4}\right)^4 = (8)^4$$

$$x = 8^4$$

$$x = 4096$$

f. $\log_9 \frac{9}{5}x = \log_9 \frac{63}{10} + \log_2 4^{-2}$

$$\log_9\left(\frac{9}{5}x\right) = \log_9\left(\frac{63}{10}\right) + \cancel{\log_2} 2^{-4}$$

$$\log_9\left(\frac{9}{5}x\right) = \log_9\left(\frac{63}{10}\right) + \log_9 9^{-4}$$

$$\log_9\left(\frac{9}{5}x\right) = \log_9\left(\frac{63 \times 9^{-4}}{10}\right)$$

$$\therefore \frac{9}{5}x = \frac{63}{65610}$$

$$x = \frac{7}{13122}$$

g. $\log_3 x^5 + \log_3 x = 12$

$$\log_3\left(x^6\right) = 12 \xleftrightarrow{\text{switch}} 3^{12} = x^6$$

$$\left(3^{12}\right)^{1/6} = x$$

$$3^2 = x$$

$$9 = x$$

OR

$$\frac{6\log_3 x}{6} = \frac{12}{6}$$

$$\log_3 x = 2$$

$$3^2 = x$$

$$9 = x$$

can't switch form if there is a coefficient

h. $\log_{10} x + \log_{10}(x-3) = \log_{10} 10$

$$\log_{10}\left(x(x-3)\right) = \log_{10} 10$$

$\therefore$ equate inputs

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

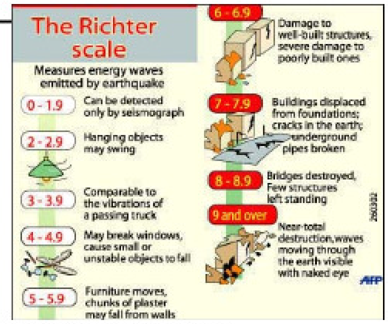
$$(x-5)(x+2) = 0$$

$\therefore$ ~~$x = -2$~~ or $x = 5$ ok

check: $\log x + \log(x-3)$

 How many times is 12 bigger than 2? 

## Solve Problems




The logarithms are used in several different real life applications

- Earthquake magnitudes (developed by Charles F. Richter)
- Intensity of sound waves (how loud things are)
- pH scale

$$\boxed{M = \log\left(\frac{I}{I_R}\right)}$$

$M$ = the magnitude of the earthquake on the Richter scale
$I_0$ = the intensity/amplitude of the reference earthquake
$I$ = the intensity/amplitude of the wave detected by the **seismograph** of the earthquake being measured

1. In October of 2005, Pakistan ($I_P$) experienced an earthquake of magnitude 7.6 resulting in the death of over 73 000 people. Later on that month, Owen Sound experienced an earthquake of 4.2 in magnitude. How many times more intense was the Pakistan earthquake to the quake in Owen Sound? $I_W$

$$\frac{I_P}{I_W} = ?$$

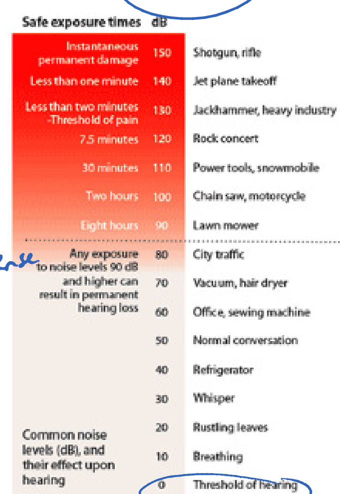
$$7.6 = \log\left(\frac{I_P}{I_0}\right) \qquad 4.2 = \log\left(\frac{I_W}{I_0}\right)$$

$$10^{7.6} = \frac{I_P}{I_0} \qquad 10^{4.2} = \frac{I_W}{I_0}$$

$$10^{7.6} I_0 = I_P \qquad 10^{4.2} I_0 = I_W$$

$$\therefore \frac{I_P}{I_W} = \frac{10^{7.6}\cancel{I_0}}{10^{4.2}\cancel{I_0}} = 10^{3.4} \doteq 2512$$

times more intense




$$\boxed{L = 10\log\left(\frac{I}{I_0}\right)}$$

$L$ = the loudness of sound in decibels
$I_0$ = the intensity of sound power per unit area (watts/m$^2$) of the threshold of hearing
$I$ = the intensity of sound power per unit area (watts/m$^2$) being measured

2. How many times more loud is a rock concert with a sound intensity of 123 dB than the threshold of sound?

$\div$ $\qquad \frac{I_R}{I_0} = ?$ $\qquad L=$ $\qquad$ 0 dB $= L$

$$\frac{123}{10} = \frac{10}{10}\log\left(\frac{I_R}{I_0}\right) \qquad 0 = \frac{10}{10}\log\left(\frac{I_0}{I_0}\right)$$

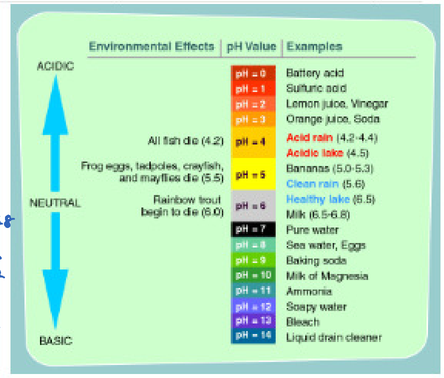
$$12.3 = \log\left(\frac{I_R}{I_0}\right) \qquad 0 = \log(1)$$

$$1.99\ldots \times 10^{12} = 10^{12.3} = \frac{I_R}{I_0}$$

DONE

more times intense.

$$\boxed{pH = -\log\left(H^{+}\right)}$$




pH = the acidity of the substance
$H^+$ = the concetration of hydrogen ions (mol/L)

3. A fish tank's water was recently changed with distilled water of pH 7. The day after it was changed, apple juice was spilled into it which caused the pH to drop to 5.8. By what factor has [$H^+$] changed?

$7 = -\log(H^+_{water})$
$-7 = \log(H^+_{water})$
$10^{-7} = H^+_{water}$

$5.8 = -\log(H^+_{app.})$
$10^{-5.8} = H^+_{app}$

$\div \quad \dfrac{H^+_{applewater}}{H^+_{water}} = \dfrac{10^{-5.8}}{10^{-7}} = 10^{1.2}$

$\doteq 15.8$ is the factor

4. A radioactive substance has a half-life of 3 days. Suppose you have 750 000 g of this substance now. In how many days will the mass be 2.3 x $10^{-2}$?

$y = ab^{x/p}$
$y = 750\,000(0.5)^{x/3}$

$2.3\times10^{-2} = 0.023 = 750\,000(0.5)^{x/3}$
$\dfrac{0.023}{750\,000}$
$3.1\times10^{-8} = 0.5^{x/3}$
$\log_{0.5}(3.1\times10^{-8}) = \dfrac{x}{3}$

$\dfrac{\ln(3.1\times10^{-8})}{\ln(0.5)} \times 3 = x$

75 days $\doteq x$

5. A boat sells at $16 000. Each year it depreciates (decreases in value) by 15%. In how many years will the boat's value be $10 000?

$1 - 0.15$

$y = ab^{x/p}$
$10\,000 = 16\,000(0.85)^{x/1}$

- - - - - - 2.9 years

6. Find the pH of a solution with hydronium ion concentration of 4.5x$10^{-5}$

$pH = -\log(H^+)$
$? = -\log(4.5\times10^{-5})$
$\doteq 4.3$ on pH scale

7. Find the decibel rating of a sound with intensity of 5000$I_o$

$L = 10\log\left(\dfrac{I}{I_o}\right)$
$? = 10\log\left(\dfrac{5000\,\cancel{I_o}}{\cancel{I_o}}\right) \doteq 37$ decibels

8. If a sound has a decibel rating of 85, how much more intense is it than the threshold sound?

$I_s \qquad \div \qquad I_o$

$\dfrac{I_s}{I_o} = ?$

$L = 10\log\left(\dfrac{I_s}{I_o}\right)$
$85 = 10\log\left(\dfrac{I_s}{I_o}\right)$
$10^{8.5} = \dfrac{I_s}{I_o} \doteq 3.2\times10^{8}$ times more intense.

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