Lines & Planes Unit - Notes

Tentative TEST date_______________

Big idea/Learning Goals

In this unit, you will work with vector concepts you learned in the preceding units and use them to develop equations for lines and planes. We begin with lines in R^2 and then move to R^3. The determination of equations for lines and planes helps to provide the basis for an understanding of geometry in R^3. All of these concepts provide the foundation for the solution of systems of linear equations that result from intersections of lines and planes, which are considered in the next unit.

Corrections for the textbook answers:

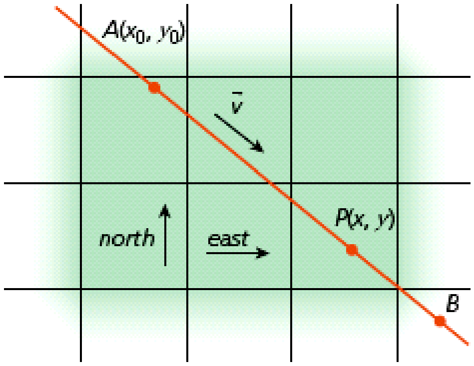
Success Criteria

- ☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	2-4	Vector & Parametric Equations of Lines in R^2 8.1	
	5-7	Cartesian Equation of a Line in R^2 8.2	
	8-10	Vector & Parametric & Symmetric Equations of Lines in R^3 8.3	
	11-12	Vector & Parametric Equations of Planes 8.4	
	13-15	Cartesian Equation of a Plane 8.5	
	16-19	Sketching & review 8.6	
		Review	

Reflect – previous TEST mark ___________, Overall mark now___________.

Vector & Parametric Equations of a Line in R^2



Direction Vector of a Line

Vector Equation of a Line

Parametric Equations of a Line

1. Highway 33 from Regina to Stoughton, Saskatchewan, is an almost straight line. Suppose you travel on this highway with a constant velocity (expressed in component form, where east and north are positive) $\vec{v} = (85, -65)$ km/h. How far south of Regina are you when you are at a position 102 km east of Regina?

2. State a direction vector for

a. the line that passes through the points $C(3, 4)$ and $D(7, 2)$

b. a line that has slope $-\frac{5}{3}$

c. a vertical line passing through the point $(-6, 5)$

3. A line passes through the point $(5, -2)$ with direction vector $(2, 6)$.

a. State the parametric equations of this line.

b. What point on the line corresponds to the parameter value $t = 3$?

c. Does the point $(1, -8)$ lie on this line?

d. Find the y-intercept and the slope of the line. Then, write the equation of the line in the form $y = mx + b$.

4. State a vector equation of the line passing through the points $P(4, 1)$ and $Q(7, -5)$.

5. Are the lines represented by the following vector equations coincident? That is, do these equations represent the same straight line?

a. $\vec{r} = (3, 4) + s(2, -1)$

b. $\vec{r} = (-9, 10) + t(-6, 3)$

Cartesian Equation of a Line

1. Find a normal to the line

   a. $y = -2x + 5$

   b. $(x, y) = (2, -3) + t(2, 5), t \in R$

2. Find the equation of the straight line with normal (5, 2), which passes through the point (– 2, 1). Write the equations in all forms.

Cartesian Equation of a Line

3. Find the scalar equation of the straight line with normal $(-6, 4)$ that passes through the point $(-3, -7)$.

4. Find the distance from the point $Q(5, 8)$ to the line $7x + y - 23 = 0$.

Shortest Distance
from point $Q(x_1,y_1)$ to line $Ax+By+C=0$

5. a. Prove that two lines in a plane are parallel if and only if their normals are parallel.
b. Prove that two lines in a plane are perpendicular if and only if their normals are perpendicular.

6. The angle α, $0° \leq \alpha \leq 180°$, that a line makes with the positive x-axis is called the **angle of inclination** of the line.
a. Find the angle of inclination of $\vec{r} = (2, -6) + t(3, -4)$
b. Prove that the tangent of the angle of inclination is equal to the slope of the line.

7. a. Show that the equation of a line that has an angle of inclination α can be expressed in the form $x \sin \alpha - y \cos \alpha + C = 0$.

b. Find the angle of inclination of $2x + 4y + 9 = 0$.

c. Find the scalar equation of the line through the point $(6, -4)$ with an angle of inclination of $120°$.

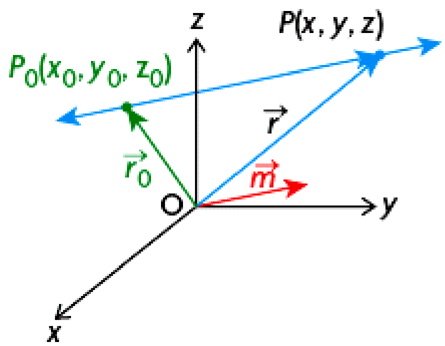
8. Find the angle between the two lines

$\vec{r}_1 = (5, 2) + t(-3, 6)$

$\vec{r}_2 = (5, 2) + u(11, 2)$

Angle between lines $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$

Vector, Parametric and Symmetric Equations of a Line in R^3



Vector Equation of a Line

Parametric Equations of a Line

Symmetric Equation of a Line

1. Write a vector equation for the line $-x = y + 2 = z$.

2. Find vector, parametric, and symmetric equations of
   a. the line that passes through the points $P(6, -4, 1)$ and $Q(2, -8, -5)$
   b. a line perpendicular to the xz-plane

3. Do the equations $\frac{x-5}{2} = \frac{y+4}{-5} = \frac{z+1}{3}$ and $\frac{x+1}{-4} = \frac{y-11}{10} = \frac{z+4}{-6}$ represent the same line?

4. Find the symmetric equations of the line that passes through the point $(-6, 4, 2)$ and is perpendicular to both of the lines $\frac{x}{-4} = \frac{y + 10}{-6} = \frac{z + 2}{3}$ and $\frac{x - 5}{3} = \frac{y - 5}{2} = \frac{z + 5}{4}$.

5. Find the distance from the point $Q(1, -2, -3)$ to the line $\vec{r} = (3, 1, 0) + t(1, 1, 2)$.

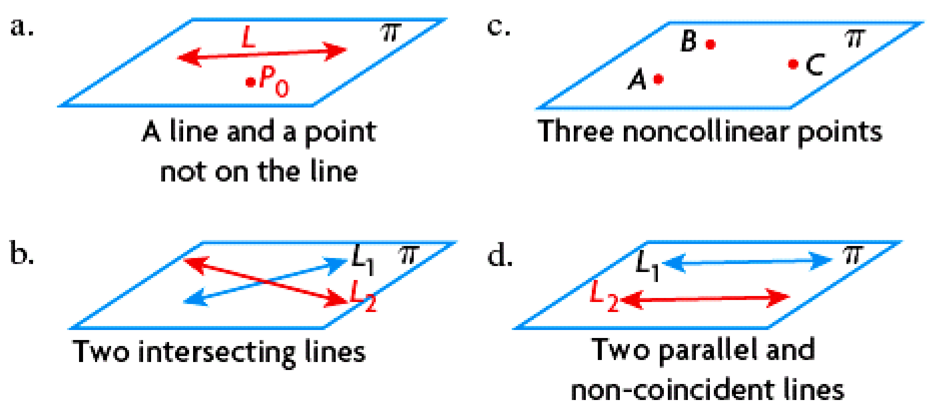
Shortest Distance
from point Q to line through P with direction vector $\vec{m}$

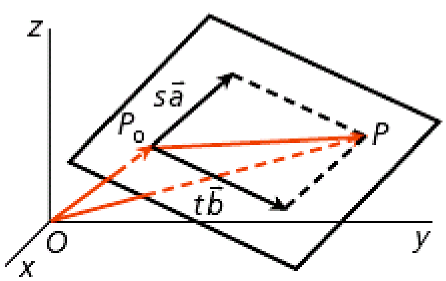
6. Find an equation of the line through the point (4, 5, 5) that meets the line $\frac{x-11}{3}=\frac{y+8}{-1}=\frac{z-4}{1}$ at right angles.

Vector and Parametric Equations of a Plane

Plane –

Can define a unique plane from:





Direction Vectors of a Plane

Vector Equation of a Plane

Parametric Equations of a Plane

1. Find vector and parametric equations of the plane that contains the three points $A(1, 0, -3)$, $B(2, -3, 1)$, and $C(3, 5, -3)$.

2. Find the vector equation of the plane that contains the two parallel lines

$$l_1: \vec{r} = (2, 4, 1) + t(3, -1, 1)$$
$$l_2: \vec{r} = (1, 4, 4) + t(-6, 2, -2)$$

3. Find vector and parametric equations of the plane that contains the line $x = 7 - t$, $y = -2t$, $z = -7 + t$ and that does not intersect the z-axis.

4. a. Explain why the three points $(2, 3, -1)$, $(8, 5, -5)$, and $(-1, 2, 1)$ *do not* determine a plane.

b. Explain why the line $\vec{r} = (4, 9, -3) + t(1, -4, 2)$ and the point $(8, -7, 5)$ *do not* determine a plane.

Cartesian Equation of a Plane

1. a. Find a normal to the plane with vector equation $\vec{r} = (3, 0, 2) + s(2, 0, -1) + t(6, 2, 0)$.

   b. Show that the normal is perpendicular to every vector in the plane.

2. a. Find the scalar equation of the plane with vector equation $\vec{r} = (3, 0, 2) + p(2, 0, -1) + q(6, 2, 0)$.

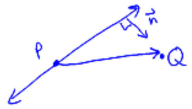
   b. Show that $\vec{r} = (-1, -2, 1) + s(5, 3, 2) + t(2, 4, 5)$ is another vector equation of the same plane.

Cartesian Equation of a Plane

3. Find the distance from the point $Q(1, 3, -2)$ to the plane $4x - y - z + 6 = 0$.

Shortest Distance
from point $Q(x_1,y_1)$ to line
$Ax+By+C=0$

$P(x,y)$ is pt on line



distance $= |\text{Proj}(\vec{PQ} \text{ onto } \vec{n})|$

$$= \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$$

$$= \frac{|(x_1-x, y_1-y)\cdot(A,B)|}{\sqrt{A^2+B^2}}$$

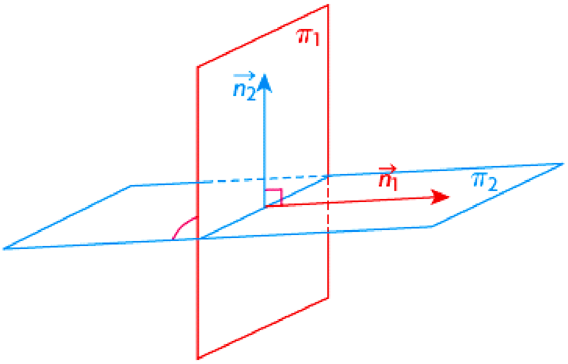
$$= \frac{|Ax_1+By_1-Ax-By|}{\sqrt{A^2+B^2}}$$

$$= \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

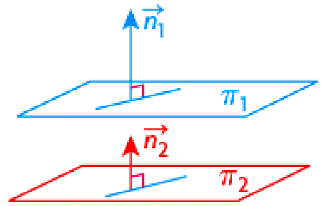
4. a. Find a vector equation for the plane with scalar equation $2x - y + 3z - 24 = 0$

b. Find an equation for the plane that passes through the origin and is perpendicular to plane in a.

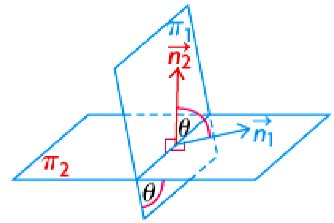
Perpendicular Planes



Parallel Planes



Angle between Intersecting Planes



5. Determine whether the following pairs of planes are coincident, parallel and distinct, or neither. If neither find the angle between the planes.

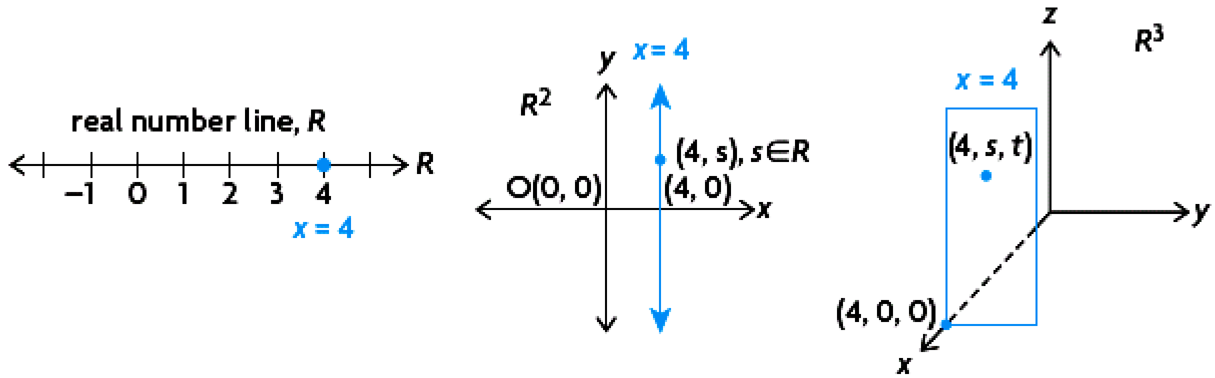
a. $x + 3y - z - 2 = 0$ and $2x + 6y - 2z - 8 = 0$

b. $2x + y + z - 3 = 0$ and $6x + 2y + 2z - 9 = 0$

c. $3x - 3y + z - 2 = 0$ and $6x - 6y + 2z - 4 = 0$

Sketching & Review

Different interpretations of the graph with equation $x = 4$



1. Sketch $3x-4y-12=0$

2. Sketch $-3x+2y+z-6=0$

3. Sketch

   a. $3xy+y=0$

   b. $xy+2y-3x-6=0$

   c. $z^2-4=0$

4. Identify what each model represents, then give another version for the model.

a. $1-x=z,\ y=5$

b. $(x, y, z)=(2, 3, 5) + t(7, 8, 9) + r(-3, 2, 1)$

c. $x=2-t,\ y=7+2t$

d. $2x+10y-z+7=0$

e. $(x, y, z)=(3, 6, 5) + t(1, 2, 3) + r(2, 4, 6)$

f. $x=2,\ y=7+t,\ z=-4+5r$

g. $(x, y, z)=(1, 1, 1) +t(0, 0, 1)$

h. $x=7-y=z+1$

i. $(x, y)=(2, 0) +t(7, 5)$

j. $2x-4y-10=0$

5. A line goes through the points (9, 2) and (3, 4). Determine

a. its vector equation

b. its parametric equations

c. its symmetric equation

d. its scalar equation

6. Find the scalar equation of the line which is perpendicular to the line $2x - 3y + 18 = 0$ and has the same y-intercept as the line $(x, y) = (0, 1) + t(-3, 4)$.

7. Find the distance from the point $(1, -2, -3)$ to the line $x = y = z - 2$.

8. A line through the origin has direction angles $\beta = 120^\circ$ and $\gamma = 45^\circ$. Find a vector equation for the line.

9. Find the scalar equation of the plane containing the line $x = y$, $z = 0$ and the point $(2, -5, -4)$.