p1NOTES

February-25-13
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Lines&Plane
sNotesNEW
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TO DO

☐ more space #5 pg-6

☐ p.8 #2 more space

☐ p.13 add intercept equation

↓ see below

Lines & Planes Unit - Notes

Tentative TEST date Tue. Mar. 26

Big idea/Learning Goals

In this unit, you will work with vector concepts you learned in the preceding units and use them to develop equations for lines and planes. We begin with lines in R^2 and then move to R^3. The determination of equations for lines and planes helps to provide the basis for an understanding of geometry in R^3. All of these concepts provide the foundation for the solution of systems of linear equations that result from intersections of lines and planes, which are considered in the next unit.

Corrections for the textbook answers: I would like to try working with

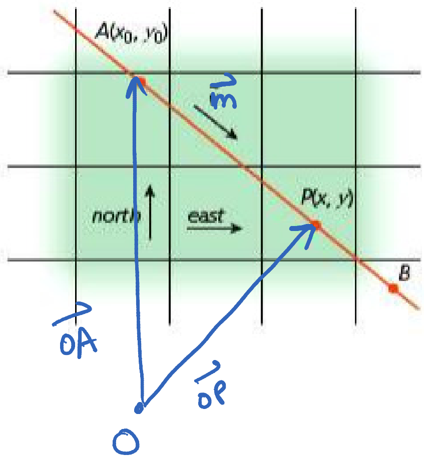
Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
Mar5	2-4	Vector & Parametric Equations of Lines in R^2 8.1	
	5-7	Cartesian Equation of a Line in R^2 8.2	
	8-10	Vector & Parametric & Symmetric Equations of Lines in R^3 8.3	
	11-12	Vector & Parametric Equations of Planes in R^2 8.4	
	13-15	Cartesian Equation of a Plane in R^3 8.5	
	16-19	Sketching & review 8.6	
		Review	

Reflect – previous TEST mark __________, Overall mark now__________.

Vector & Parametric Equations of a Line in R^2



Direction Vector of a Line

$\vec{m} = (a, b)$

gr. 9

$m = \frac{rise}{run} = \frac{b}{a}$

Vector Equation of a Line

$\vec{OP} = \vec{OA} + \vec{AP} = \vec{OA} + k\vec{m}$

$(x, y) = (x_0, y_0) + k(a, b)$

stretch factor = parameter, $k \in R$

Parametric Equations of a Line

textbook uses t and s

$x = x_0 + ka$

$y = y_0 + kb$

1. Highway 33 from Regina to Stoughton, Saskatchewan, is an almost straight line. Suppose you travel on this highway with a constant velocity (expressed in component form, where east and north are positive) $\vec{v} = (85, -65)$ km/h. How far south of Regina are you when you are at a position 102 km east of Regina?

gr. 9

$m = \frac{-65}{85}$ km/h

$b = 0$ Regina where you start.

$y = \frac{-65}{85}x + 0$

$y = \frac{-13}{17}x + 0$

Vectors course

slope or rate of change or direction vector

$\vec{m} = (85, -65)$

start at Regina ie. place x + y axes on Regina

$(x, y) = (0, 0) + k(85, -65)$

different k's.

or

$(x, y) = (0, 0) + k(17, -13)$

CAN reduce ONLY the direction vector.

sub in.

position given is $(102, y)$

$(102, y) = (0, 0) + k(17, -13)$

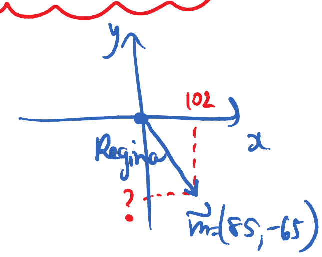
$102 = 0 + 17k$

$6 = k$

$y = 0 + -13k$

$y = 0 + -13(6)$

$y = -78$



2. State a direction vector for

a. the line that passes through the points $C(3, 4)$ and $D(7, 2)$

b. a line that has slope $-\frac{5}{3}$

c. a vertical line passing through the point ~~(−6, 5)~~ not needed unless ask for equation

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{7-3} = \frac{-2}{4} = -\frac{1}{2}$

(a) $\vec{CD} = (7-3, 2-4) = (4, -2)$ $\therefore \vec{m} = (2,-1)$ or $(-1, 2)$

(b) $\vec{m} = (3, -5)$ or $(-3, 5)$ or $(-9, 15)$ ....

(c) $\vec{m} = (0, 1)$ any #. except zero

equation of line: $(x, y) = (-6, 5) + k(0, 1)$

3. A line passes through the point A$(5, -2)$ with direction vector $(2, 6)$. $\vec{OA} = (5,-2)$, $\vec{m} =$ or $(1,3)$

a. State the parametric equations of this line.

b. What point on the line corresponds to the parameter value $k = 3$?

c. Does the point $(1, -8)$ lie on this line?

d. Find the y-intercept and the slope of the line. Then, write the equation of the line in the form $y = mx + b$.

(a) $(x, y) = (5, -2) + k(1, 3)$

$\therefore$ parametric: $x = 5 + k$
$y = -2 + 3k$

(b) $\begin{cases} x = 5 + 3 = 8 \\ y = -2 + 3(3) = 7 \end{cases}$ $\therefore$ pt. on the line is $(8, 7)$

(c) $\begin{cases} 1 = 5 + k \rightarrow k = -4 \\ -8 = -2 + 3k \end{cases}$

$-6 = 3k$
$-2 = k$

contradiction

$\therefore$ pt. $(1,-8)$ is not on the line.

(d) y-int sub $x = 0$

$\begin{cases} 0 = 5 + k \quad k = -5 \\ y = -2 + 3k \end{cases}$

$y = -2 + 3(-5)$
$y = -17$

$\therefore b = -17$
$m = \frac{3}{1}$

$\therefore y = 3x - 17$

4. State a vector equation of the line passing through the points $P(4, 1)$ and $Q(7, -5)$.

$\vec{m} = \vec{PQ} = (7-4, -5-1) = (3, -6) \quad \therefore \vec{m} = (1, -2)$

$(x, y) = (4, 1) + t(1, -2)$

or

$(x, y) = (7, -5) + t(1, -2)$

overlap, same

5. Are the lines represented by the following vector equations coincident? That is, do these equations represent the same straight line?

a. $\vec{r} = (3, 4) + k(2, -1)$ b. $\vec{r} = (-9, 10) + t(-6, 3)$

$\vec{m}_1 = (2, -1) \xleftrightarrow{\text{parallel}} \vec{m}_2 = (-6, 3)$

check if pt (position vector of the known pt.) of line (b) is on line (a)

$(-9, 10) = (3, 4) + k(2, -1)$

$\begin{cases} -9 = 3 + 2k \\ 10 = 4 - k \end{cases}$ $\quad -12 = 2k$, $\; -6 = k$

$6 = -k$

$-6 = k$

$\therefore$ the lines are the <u>same</u>

Cartesian Equation of a Line

1. Find a normal to the line (perpendicular vector.)

   a. $y = -2x + 5$ ($-\frac{2}{1}$)

   b. $(x, y) = (2, -3) + t(2, 5), t \in R$

ⓐ $\vec{m} = (1, -2)$ $\vec{m}_{\perp} = (2, 1)$ or $(-2, -1)$

run rise

x y

negative reciprocal $-\frac{2}{1} \to \frac{1}{2}$

ⓑ $\vec{m} = (2, 5)$ $\vec{m}_{\perp} = (-5, 2)$

or

$(5, -2)$

or

$(10, -4)$

or Scalar

2. Find the equation of the straight line with normal (5, 2), which passes through the point (– 2, 1). Write the equations in all forms.

$\vec{n} = (5, 2) \rightsquigarrow \vec{m} = (-2, 5)$

vector: $(x, y) = (-2, 1) + t(-2, 5)$

parametric: $\begin{cases} x = -2 - 2t \\ y = 1 + 5t \end{cases}$

Cartesian: $Ax + By + C = 0$

or Scalar

$5x + 2y + C = 0$ sub pt. $(-2, 1)$

$5(-2) + 2(1) + C = 0$

$C = 8$

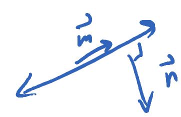
$\therefore 5x + 2y + 8 = 0$

Cartesian Equation of a Line

$Ax + By + C = 0$

$(A, B) = \vec{n}$

is the normal (perpendicular) to the line

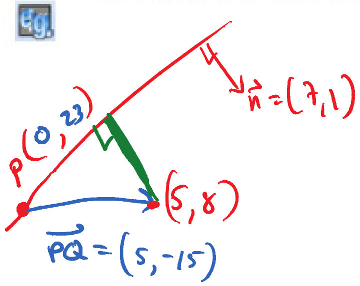


3. Find the scalar equation of the straight line with normal (−6, 4) that passes through the point (−3, −7).

$\therefore -6x + 4y + 10 = 0$

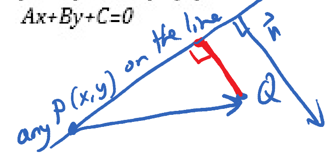
random pt. on line
sub $x=0$ or anything
$y=23$

4. Find the distance from the point $Q(5, 8)$ to the line $7x + y - 23 = 0$.



$$D = \left|\text{proj}(\vec{PQ} \text{ on } \vec{n})\right|$$
$$= \frac{|(5,-15)\cdot(7,1)|}{\sqrt{7^2+1^2}}$$
$$= \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Shortest Distance
from point $Q(x_1,y_1)$ to line $Ax+By+C=0$



$$\text{Distance} = \left|\text{proj}(\vec{PQ} \text{ on } \vec{n})\right|$$
$$= \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right|$$
$$= \frac{|(x_1-x,\, y_1-y)\cdot(A,B)|}{\sqrt{A^2+B^2}}$$
$$= \frac{|Ax_1+By_1-Ax-By|}{\sqrt{A^2+B^2}}$$
$$= \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

pt. not on the line.

5. a. Prove that two lines in a plane are parallel if and only if their normals are parallel.
   b. Prove that two lines in a plane are perpendicular if and only if their normals are perpendicular.

(a) line with dir. vector $\vec{m}_1$ is parallel (collinear) to line with $\vec{m}_2$

iff $\vec{m}_1 = k\vec{m}_2$

$(a_1, b_1) = k(a_2, b_2)$

$\vec{n}_1 = (b_1, -a_1)$ is normal to line(1)

$\vec{n}_2 = (b_2, -a_2)$

$\therefore \vec{n}_1 = (b_1, -a_1) = (kb_2, -ka_2)$

$= k(b_2, -a_2) = kn_2$ DONE :)

$\therefore \vec{n}_1$ is parallel to $\vec{n}_2$

(b) $\vec{m}_1 \perp \vec{m}_2$

iff $(a_1, b_1)\cdot(a_2, b_2) = 0$

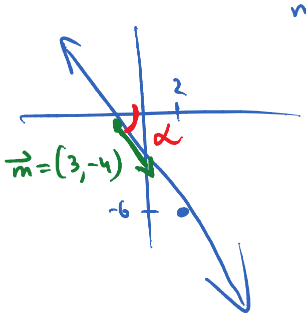
iff $a_1a_2 + b_1b_2 = 0$

iff $b_1b_2 + (-a_1)(-a_2) = 0$

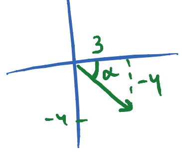
$(b_1, -a_1)\cdot(b_2, -a_2) = 0$

$\therefore \vec{n}_1 \perp \vec{n}_2$

6. The angle α, $0° \le \alpha \le 180°$, that a line makes with the positive x-axis is called the **angle of inclination** of the line.
   a. Find the angle of inclination of $\vec{r} = (2, -6) + t(3, -4)$
   b. Prove that the tangent of the angle of inclination is equal to the slope of the line.



move direction vector to origin



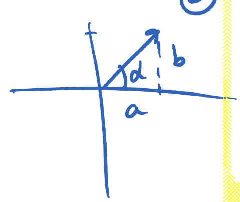
TOA

(a) $\alpha = \tan^{-1}\left(\frac{-4}{3}\right) = -53°$ not in

$\therefore \alpha = 127°$

$\vec{m} = (a, b)$

(b) $\tan\alpha \stackrel{?}{=} \text{slope}$



$\frac{\text{opp}}{\text{adj}} = \frac{b}{a} = \frac{\text{rise}}{\text{run}}$

Prove

$\sin\alpha$ $\quad -\cos\alpha$

7. a. Show that the equation of a line that has an angle of inclination α can be expressed in the form $x \sin \alpha - y \cos \alpha + C = 0$.

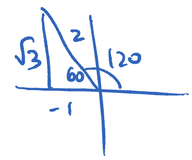
b. Find the angle of inclination of $2x + 4y + 9 = 0$.

c. Find the scalar equation of the line through the point $(6, -4)$ with an angle of inclination of 120°.

ⓐ $\tan\alpha = \frac{b}{a}$ $\quad \vec{m} = (a, b)$ $\quad \vec{n} = (b, -a) = (\sin\alpha, -\cos\alpha) = (A, B)$

$\frac{\sin\alpha}{\cos\alpha} = \frac{b}{a}$

sub in



ⓑ $\vec{n} = (2, 4)$

$\vec{m} = (4, -2)$

$\therefore \tan\alpha = \frac{-2}{4}$

$\alpha = \tan^{-1}\left(\frac{-2}{4}\right) = -27°$

$\therefore \alpha = 153°$

ⓒ $Ax + By + C = 0$

$\sqrt{3}x + y + C = 0$

pt. $(6, -4)$

$\sqrt{3}(6) + -4 + C = 0$

$C = 4 - 6\sqrt{3}$

$\therefore \sqrt{3}x + y + 4 - 6\sqrt{3} = 0$

$\tan 120° = \frac{b}{a} = \frac{\sqrt{3}}{-1}$

$\vec{m} = (a, b)$

$\therefore \vec{m} = (-1, \sqrt{3})$

$\vec{n} = (\sqrt{3}, 1)$

8. Find the angle between the two lines

$\vec{r}_1 = (5, 2) + t(-3, 6)$

$\vec{r}_2 = (5, 2) + u(11, 2)$

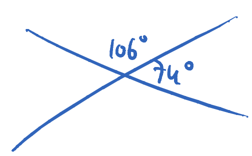
$\cos\theta = \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$

$\cos\theta = \frac{(-3, 6) \cdot (11, 2)}{\sqrt{3^2 + 6^2}\sqrt{11^2 + 2^2}}$

$\cos\theta = \frac{-33 + 12}{\sqrt{45}\sqrt{125}}$

$\cos\theta = \frac{-21}{75}$

$\theta \doteq 106°$ or $74°$

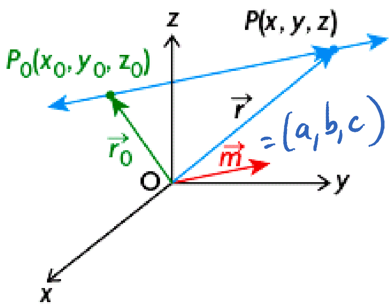


Angle between lines
$A_1x+B_1y+C_1=0$ and
$A_2x+B_2y+C_2=0$

$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}}{|\vec{n}_1||\vec{n}_2|}$

can also use direction vectors $\vec{m}_1$ and $\vec{m}_2$

Vector, Parametric and Symmetric Equations of a Line in R^3



Vector Equation of a Line $\vec{OP} = \vec{OP_0} + \vec{P_0P}$

$$(x,y,z) = (x_0,y_0,z_0) + t(a,b,c)$$

Parametric Equations of a Line

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

isolate for t to eliminate it.

Symmetric Equation of a Line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

- convient if don't want many variables

1. Write a vector equation for the line $\underset{-1}{\underline{-x}} = \underset{1}{\underline{y+2}} = \underset{1}{\underline{z}}$.

$(x,y,z) = (0,-2,0) + t(-1,1,1)$

2. Find vector, parametric, and symmetric equations of
   a. the line that passes through the points $P(6, -4, 1)$ and $Q(2, -8, -5)$
   b. a line perpendicular to the xz-plane

a) $\vec{m} = \vec{PQ} = (2-6, -8-(-4), -5-1) = (-4,-4,-6)$

$(x,y,z) = (6,-4,1) + k(-4,-4,-6)$ or $(2,2,3)$

$x = 6-4k$ $\quad z = 1-6k$

$y = -4-4k$

$$\frac{x-6}{-4} = \frac{y+4}{-4} = \frac{z-1}{-6}$$

(b) $(x,y,z) = (0,0,0) + t(0,1,0)$

$x=0$ $\quad y=t$ $\quad z=0$

$\therefore$ no symmetric equation possible. Since you cannot divide by zero.

3. Do the equations $\frac{x-5}{2} = \frac{y+4}{-5} = \frac{z+1}{3}$ and $\frac{x+1}{-4} = \frac{y-11}{10} = \frac{z+4}{-6}$ represent the same line?

$(x,y,z) = (5,-4,-1) + t(2,-5,3)$ $\quad \vec{m}_1$

$(x,y,z) = (-1,11,-4) + t(-4,10,-6)$ $\quad \vec{m}_2$

$-2\vec{m}_1 = \vec{m}_2$ parallel

& check the point.

$$\frac{-1-5}{2} = \frac{11+4}{-5} = \frac{-4+1}{3}$$

$-3 = -3 \neq -1$ contradiction

Not the same line.

8

4. Find the symmetric equations of the line that passes through the point $(-6, 4, 2)$ and is perpendicular to both of the lines $\frac{x}{-4} = \frac{y+10}{-6} = \frac{z+2}{3}$ and $\frac{x-5}{3} = \frac{y-5}{2} = \frac{z+5}{4}$.

$(x, y, z) = (-6, 4, 2) + k(-30, 25, 10)$

$\therefore \frac{x+6}{-30} = \frac{y-4}{25} = \frac{z-2}{10}$

$\vec{m}_1 = (-4, -6, 3)$

$\vec{m}_2 = (3, 2, 4)$

$-4 \quad -6 \quad 3 \quad -4 \quad -6 \quad 3$

$3 \quad 2 \quad 4 \quad 3 \quad 2 \quad 4$

$(-24-6,\ 9+16,\ -8+18)$

$(-30, 25, 10)$

could be

$\frac{x+6}{-6} = \frac{y-4}{5} = \frac{z-2}{2}$

5. Find the distance from the point $Q(1, -2, -3)$ to the line $\vec{r} = \underbrace{(3, 1, 0)}_{P} + t\underbrace{(1, 1, 2)}_{\vec{m}}$.

$\vec{PQ} = (1-3, -2-1, -3)$

$= (-2, -3, -3)$

$\frac{|\vec{m} \times \vec{PQ}|}{|\vec{m}|} = \text{distance}$

$\begin{matrix} 1 & 1 & 2 & 1 & 1 \\ -2 & -3 & -3 & -2 & -3 \end{matrix}$

$(-3+6,\ -4+3,\ -3+2)$

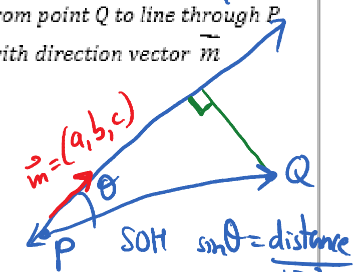
$= (3, -1, -1)$

$\frac{|(3, -1, -1)|}{|(1, 1, 2)|} = \text{distance}$

$\frac{\sqrt{3^2+1^2+1^2}}{\sqrt{1^2+1^2+2^2}} = \frac{\sqrt{11}}{\sqrt{6}}$

Shortest Distance $\mathbb{R}^3$

from point Q to line through P with direction vector $\vec{m}$



SOH $\sin\theta = \frac{\text{distance}}{|\vec{PQ}|}$

$\text{distance} = |\vec{PQ}|\sin\theta$

cross product formula

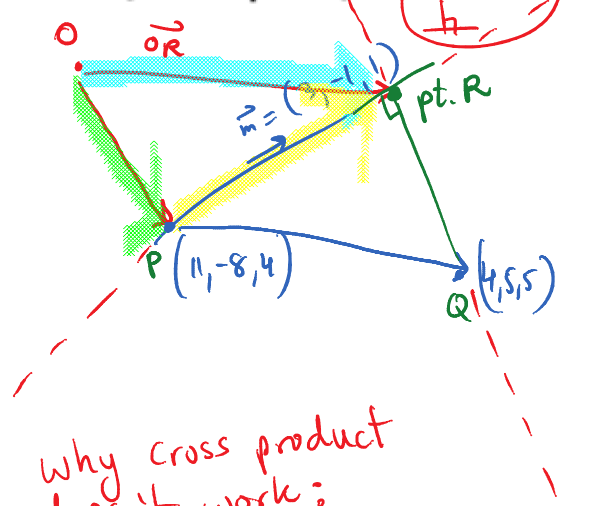
$|\vec{m} \times \vec{PQ}| = |\vec{m}||\vec{PQ}|\sin\theta$

$\therefore \frac{|\vec{m} \times \vec{PQ}|}{|\vec{m}|} = \text{distance}$

don't reduce here

9

6. Find an equation of the line through the point (4, 5, 5) that meets the line $\frac{x-11}{3} = \frac{y+8}{-1} = \frac{z-4}{1}$ at right angles.



"pt. R" = proj of $(\vec{PQ}$ on $\vec{m})$ + "pt. P"

position vector of a pt. R

$\vec{OR} = \vec{OP} + \text{proj}(\vec{PQ} \text{ on } \vec{m})$

once pt. R is known find $\vec{n} = \vec{RQ}$ and equation

$\vec{OR} = (11,-8,4) + \frac{\vec{PQ}\cdot\vec{m}}{|\vec{m}|}\hat{m}$

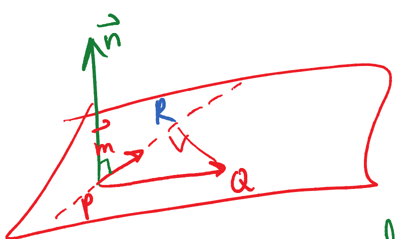
$+ \frac{\vec{PQ}\cdot\vec{m}}{|\vec{m}|}\frac{\vec{m}}{|\vec{m}|}$

$+ \frac{\vec{PQ}\cdot\vec{m}}{|\vec{m}|^2}\vec{m}$

$\vec{PQ} = (-7, 13, 1)$

$\vec{m} = (3,-1,1)$

why cross product doesn't work:



$\vec{m} \times \vec{PQ}$ → get a normal sticking out of the plane at 90°

is ⊥ to $\vec{m}$ but wrong orientation

→ must go through Q AND meet line

$\vec{OR} = (11,-8,4) + \frac{-7(3)+13(-1)+1(1)}{\left(\sqrt{3^2+1^2+1^2}\right)^2}(3,-1,1)$

$= (11,-8,4) + \frac{-33}{11}(3,-1,1)$

$= (11,-8,4) + (-9,3,-3)$

$= (2,-5,1)$

Things to try for 90°
→ cross prod
→ projections
→ SOH CAH TOA
→ dot prod = 0

$\vec{m} = \vec{RQ} = (4-2, 5--5, 5-1)$
$= (2, 10, 4)$

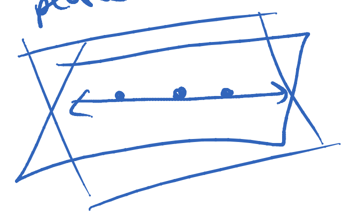
∴ eqn of line is
$(x,y,z) = (2,-5,1) + t(2,10,4)$
$(1, 5, 2)$

Vector and Parametric Equations of a Plane

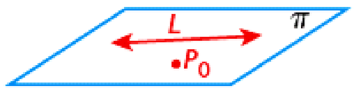
Plane – a flat surface that extends infinitely in all directions

Can define a unique plane from:

if collinear do not get a unique plane

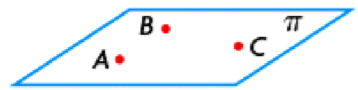


a.



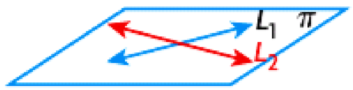
A line and a point not on the line

c.



Three noncollinear points

b.



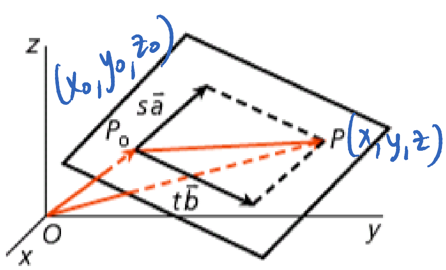
Two intersecting lines

d.

Two parallel and non-coincident lines

no plane if skew lines (not meet)

if 2 vectors always a plane since can move but lines can't be moved



Direction Vectors of a Plane

$\vec{m_1}$ $\vec{a} = (a_1, a_2, a_3)$ $\vec{m_2}$ $\vec{b} = (b_1, b_2, b_3)$

Vector Equation of a Plane

$\vec{OP} = \vec{OP_0} + \vec{P_0P} = \vec{OP_0} + s\vec{a} + t\vec{b}$

$(x, y, z) = \underbrace{(x_0, y_0, z_0)}_{\text{known "pt."}} + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$

must be non-zero and non collinear!!

Parametric Equations of a Plane

$$\begin{cases} x = x_0 + sa_1 + tb_1 \\ y = y_0 + sa_2 + tb_2 \\ z = z_0 + sa_3 + tb_3 \end{cases}$$

1. Find vector and parametric equations of the plane that contains the three points $A(1, 0, -3)$, $B(2, -3, 1)$, and $C(3, 5, -3)$.

$\vec{m_1} = \vec{AB} = (2-1, -3-0, 1-(-3)) = (1, -3, 4)$

$\vec{m_2} = \vec{BC} = (3-2, 5-(-3), -3-1) = (1, 8, -4)$

check if non collinear (if they are collinear it's a line not a plane!)

$\therefore (x, y, z) = (3, 5, -3) + t(1, -3, 4) + r(1, 8, -4)$

$$\begin{cases} x = 3 + t + r \\ y = 5 - 3t + 8r \\ z = -3 + 4t - 4r \end{cases}$$

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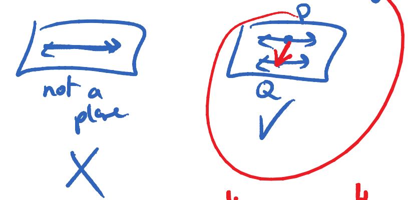
2. Find the vector equation of the plane that contains the two parallel lines

$$l_1: \vec{r} = (2, 4, 1) + t(3, -1, 1)$$
$$l_2: \vec{r} = (1, 4, 4) + t(-6, 2, -2)$$

$\vec{PQ} = (1-2, 4-4, 4-1)$

must check if same line since they are parallel



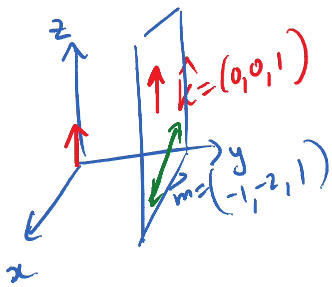
$\therefore (x, y, z) = (2, 4, 1) + t(3, -1, 1) + r(-1, 0, 3)$

$$\frac{x-2}{3} = \frac{y-4}{-1} = \frac{z-1}{1}$$

$-1/3 \neq 0 \neq 3$ contradiction

$\therefore$ parallel but not the same line

3. Find vector and parametric equations of the plane that contains the line $x = 7 - t$, $y = -2t$, $z = -7 + t$ and that does not intersect the z-axis.

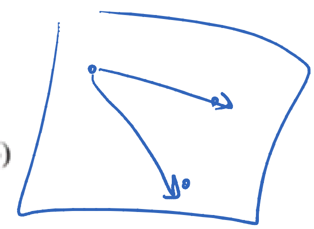


$\therefore (x, y, z) = (7, 0, -7) + t(0, 0, 1) + r(-1, -2, 1)$

$$\begin{cases} x = 7 - r \\ y = -2r \\ z = -7 + t + r \end{cases}$$

4. a. Explain why the three points (2, 3, −1), (8, 5, −5), and (−1, 2, 1) *do not* determine a plane.

b. Explain why the line $\vec{r} = (4, 9, -3) + t(1, -4, 2)$ and the point (8, −7, 5) *do not* determine a plane.

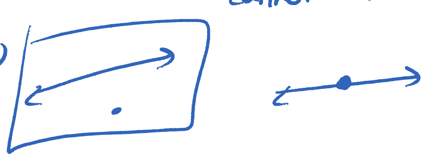


ⓐ $\vec{AB} = (8-2, 5-3, -5--1) = (6, 2, -4)$

$\vec{BC} = (-1-8, 2-5, 1--5) = (-9, -3, 6)$ $\times -1.5$

$\therefore$ Since $\vec{BC} = -1.5\vec{AB}$, $\vec{BC}$ and $\vec{AB}$ are collinear and cannot make a unique plane.

ⓑ



check if pt on the line

$$\frac{x-4}{1} = \frac{y-9}{-4} = \frac{z+3}{2}$$

$4 = 4 = 4$ ✓

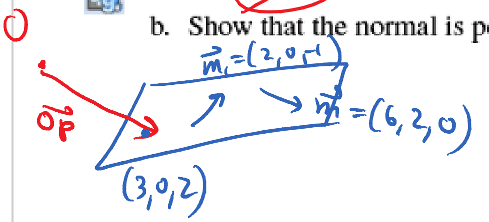
$\therefore$ pt is on the line $\therefore$ no plane possible

Cartesian Equation of a Plane

→ don't include this vector since $\vec{OP}$ is off the plane

1. a. Find a normal to the plane with vector equation
$\vec{r} = (3, 0, 2) + s(2, 0, -1) + t(6, 2, 0)$. $= (\cancel{3} + 2s + 6t,\ 2t,\ -s)$ = general plane vector $\vec{v}$

b. Show that the normal is perpendicular to every vector in the plane.



ⓐ $\vec{n} = \vec{m_1} \times \vec{m_2} = (2,-6,4) \sim (1,-3,2)$

$$\begin{matrix} 2 & 0 & -1 & 2 & 0 & -1 \\ 6 & 2 & 0 & 6 & 2 & 0 \end{matrix}$$

$(0+2, -6-0, 4-0)$

ⓑ $(2s+6t, 2t, -s) \cdot (1,-3,2) \stackrel{?}{=} 0$
$= 2s + 6t - 6t - 2s$
$= 0$

$\vec{v} \cdot \vec{n} = 0$ ∴ any vector on the plane is ⊥ to $\vec{n}$

2. a. Find the scalar equation of the plane with vector equation
$\vec{r} = (3, 0, 2) + p(\underbrace{2, 0, -1}_{\vec{m_1}}) + q(\underbrace{6, 2, 0}_{\vec{m_2}})$.

b. Show that $\vec{r} = (-1, -2, 1) + s(5, 3, 2) + t(2, 4, 5)$ is another vector equation of the same plane.

> ~~Cartesian~~ Scalar Equation of a Plane
>
> $Ax + By + Cz + D = 0$
>
> $\vec{n} = (A, B, C)$

intercept equation:

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1$$

x-int $(x_0, 0, 0)$
y-int $(0, y_0, 0)$
z-int $(0, 0, z_0)$

ⓐ $\vec{n} = \vec{m_1} \times \vec{m_2}$
as above
$\vec{n} = (1, -3, 2)$
A B C

$1x - 3y + 2z + D = 0$ sub pt $(3,0,2)$

$3 - 3(0) + 2(2) + D = 0$

$D = -7$

∴ scalar eqtn is
$x - 3y + 2z - 7 = 0$

★ To show coplanar
can ① $\vec{m_1} = t\vec{m_3} + r\vec{m_4}$ + check pt!
OR
② $\vec{m_1} \cdot (\vec{m_3} \times \vec{m_4}) = 0$

ⓑ
$$\begin{matrix} 5 & 3 & 2 & 5 & 3 & 2 \\ 2 & 4 & 5 & 2 & 4 & 5 \end{matrix}$$

$(15-8, 4-25, 20-6)$
$(7, -21, 14) \sim (1, -3, 2)$

same normal
∴ planes are parallel
to see if they overlap sub in pt.

pt. $(-1, -2, 1)$

$-1 - 3(-2) + 2(1) - 7 \stackrel{?}{=} 0$
$0 = 0$ ✓

∴ planes are coincident

3. Find the distance from the point $Q(1, 3, -2)$ to the plane $4x - y - z + 6 = 0$.

Shortest Distance from point $Q(x_1,y_1)$ to line $Ax+By+C=0$

$P(x,y)$ is pt on line

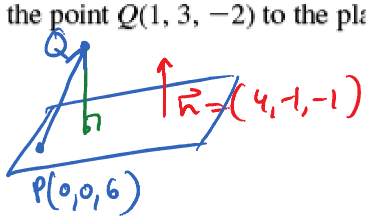
distance $= |\text{Proj}(\vec{PQ} \text{ onto } \vec{n})|$

$= \dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$

$= \dfrac{|(x_1-x, y_1-y)\cdot(A,B)|}{\sqrt{A^2+B^2}}$

$= \dfrac{|Ax_1+By_1-Ax-By|}{\sqrt{A^2+B^2}}$

$= \dfrac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$



to find pt. on plane choose $\underset{=0}{x}$ and $\underset{=0}{y}$ and solve for $z=6$

same formula: distance $= |\text{Proj}(\vec{PQ} \text{ on } \vec{n})|$

$\vec{PQ} = (1-0, 3-0, -2-6)$
$= (1,3,-8)$

$= \dfrac{|(1,3,-8)\cdot(4,-1,-1)|}{\sqrt{4^2+1^2+1^2}}$

$= \dfrac{|4-3+8|}{\sqrt{18}}$

$= \dfrac{9}{\sqrt{18}} = \dfrac{9}{3\sqrt{2}} = \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$

4. a. Find a vector equation for the plane with scalar equation $2x - y + 3z - 24 = 0$
b. Find an equation for the plane that passes through the origin and is perpendicular to plane in a.

(a) $2x - y + 3z - 24 = 0$

since there are 3 unknowns there will be 2 parameters, r, t

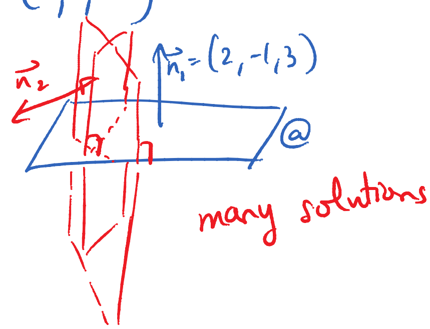
let $x = r$, $z = t$ } to keep it simple choose the ones with coefficients

$y = 2r + 3t - 24$

$\therefore$ vector eqtn.

$(x,y,z) = (0,-24,0) + r(1,2,0) + t(0,3,1)$

(b) pt. $(0,0,0)$



$\vec{n}_1 \cdot \vec{n}_2 = 0$ since $\vec{n}_1 \perp \vec{n}_2$

$(2,-1,3)\cdot(A,B,C) = 0$

let $A = 2$, $B = 4$ solve for C

$2(2) - 1(4) + 3C = 0$
$C = 0$

$\therefore \vec{n}_2 = (2,4,0)$

$2x + 4y + D = 0$ D=0 since thru origin

plane $\boxed{2x + 4y = 0}$

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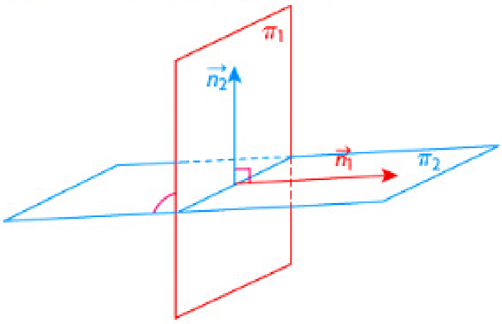
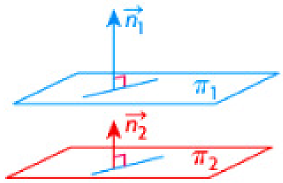
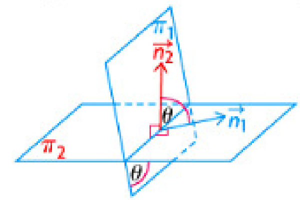
Perpendicular Planes



Parallel Planes



Angle between Intersecting Planes



5. Determine whether the following pairs of planes are coincident (same), parallel and distinct, or neither. If neither find the angle between the planes.

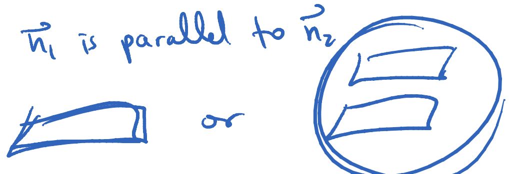
a. $x + 3y - z - 2 = 0$ and $2x + 6y - 2z - 8 = 0$

b. $2x + y + z - 3 = 0$ and $6x + 2y + 2z - 9 = 0$

c. $3x - 3y + z - 2 = 0$ and $6x - 6y + 2z - 4 = 0$

(a) $\vec{n_1} = (1, 3, -1)$ $\vec{n_2} = (2, 6, -2)$

$\vec{n_1}$ is parallel to $\vec{n_2}$

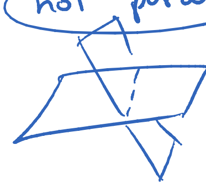


compare if D is the same multiple as $\vec{n_1}$ and $\vec{n_2}$

it's not $\therefore$ parallel + distinct

(b) $\vec{n_1} = (2, 1, 1)$ $\vec{n_2} = (6, 2, 2)$

not parallel



$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$

$(2,1,1) \cdot (6,2,2) = \sqrt{2^2+1^2+1^2}\sqrt{6^2+2^2+2^2} \cos\theta$

$12 + 2 + 2 = \sqrt{6}\sqrt{44} \cos\theta$

$\frac{16}{\sqrt{264}} = \cos\theta$

$10° \doteq \theta$

(c) $\vec{n_1} = (3, -3, 1)$ $\vec{n_2} = (6, -6, 2)$

$\vec{n_1}$ parallel to $\vec{n_2}$

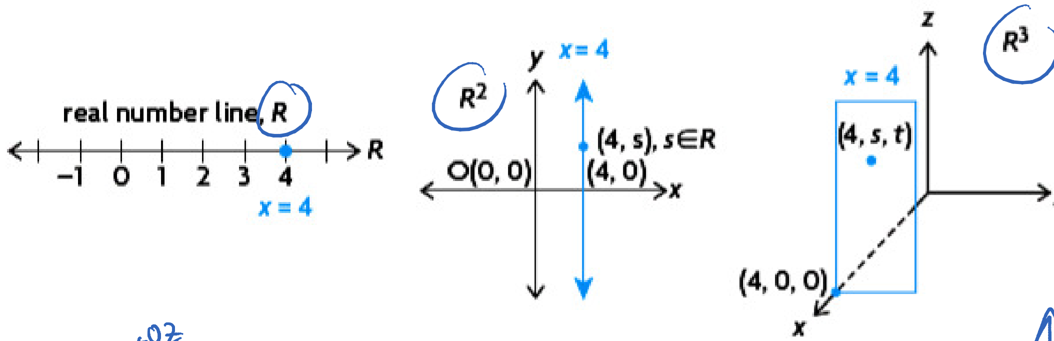
check D: the equation is a multiple

$\therefore$ same equation

$\therefore$ same planes.

Sketching & Review

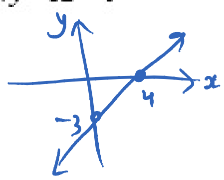
Different interpretations of the graph with equation $x = 4$



1. Sketch $3x-4y-12=0$ $+0z$

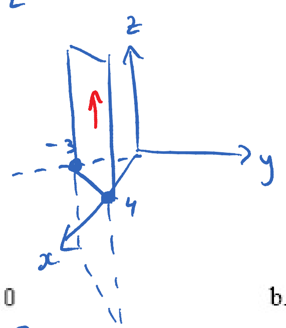
eg.

$\mathbb{R}^2$ – a line



x-int $=(4,0)$
y-int $=(0,-3)$

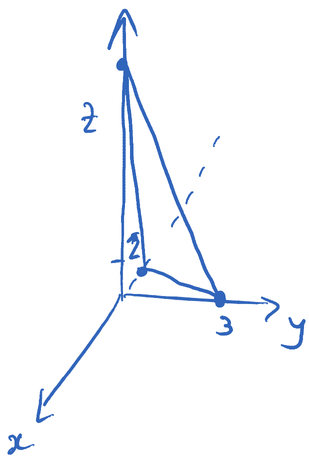
$\mathbb{R}^3$ – a plane
$\vec{n}=(3,-4,0)$



x-int $=(4,0,0)$
y-int $=(0,-3,0)$
z-int $=$ N/A
z is any #
$z=t$ $\quad t(0,0,1)$

2. Sketch $-3x+2y+z-6=0$

plane in $\mathbb{R}^3$
x-int $=(-2,0,0)$
y-int $=(0,3,0)$
z-int $=(0,0,6)$

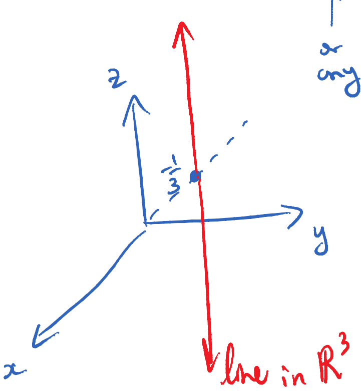


3. Sketch

in $\mathbb{R}^3$

a. $3xy+y=0$

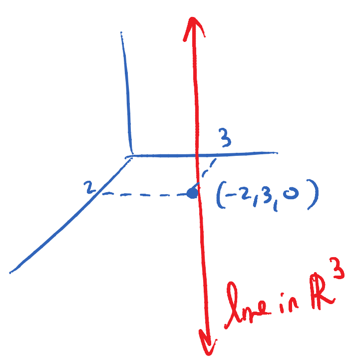
$y(3x+1)=0$
$y=0 \quad x=-\frac{1}{3}$
z can be anything
$z=t$
$(x,y,z)=(-\frac{1}{3},0,0)+t(0,0,1)$
or any #



line in $\mathbb{R}^3$

b. $xy+2y-3x-6=0$

$y(x+2)-3(x+2)=0$
$(x+2)(y-3)=0$
$x=-2 \quad y=3$
$z=t$
$(x,y,z)=(-2,3,0)+t(0,0,1)$



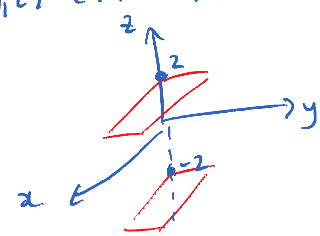
line in $\mathbb{R}^3$

c. $z^2-4=0$

$(z+2)(z-2)=0$
$z=-2$ or $z=2$
x and y can be anything
2 planes in $\mathbb{R}^3$

$z=-2$, $y=t$, $x=r$ or $z=2$, $y=p$, $x=q$

$(x,y,z)=(0,0,-2)+t(0,1,0)+r(1,0,0)$
$(x,y,z)=(0,0,+2)+p(0,1,0)+q(1,0,0)$



4. Identify what each model represents, then give another version for the model.

a. 1-x=z, y=5

$\frac{x-1}{-1} = \frac{y-5}{0} = \frac{z-0}{1}$ line in $\mathbb{R}^3$

$(x,y,z)=(1,5,0)+t(-1,0,1)$

b. (x, y, z)=(2, 3, 5) + t(7, 8, 9) + r(-3, 2, 1) not collinear

plane in $\mathbb{R}^3$

$x=2+7t-3r$
$y=3+8t+2r$
$z=5+9t+r$

c. x=2-t, y=7+2t

line in $\mathbb{R}^2$

$(x,y)=(2,7)+t(-1,2)$

d. 2x+10y-z+7=0

plane $x=t$
$y=r$
$z=2t+10r+7$

e. (x, y, z)=(3, 6, 5) + t(1, 2, 3) + r($\cancel{2, 4, 6}$) collinear ignore

$\therefore$ line in $\mathbb{R}^3$

$\frac{x-3}{1} = \frac{y-6}{2} = \frac{z-5}{3}$

f. x=2, y=7+t, z=-4+5r

plane

$(x,y,z)=(2,7,-4)+t(0,1,0)+r(0,0,5)$

g. (x, y, z)=(1, 1, 1) +t(0, 0, 1)

line $x=1$
$y=1$
$z=1+t$

h. x=7-y=z+1 symmetric

$\frac{x-0}{1} = \frac{y-7}{-1} = \frac{z--1}{1}$ line in $\mathbb{R}^3$

$\therefore (x,y,z)=(0,7,-1)+t(1,-1,1)$

i. (x, y)=(2, 0) +t(7, 5)

$x=2+7t$
$y=5t$ line in $\mathbb{R}^2$

j. 2x-4y-10=0

reduce $x-2y-5=0$

line in $\mathbb{R}^2$ $y=t$, $x=2t+5$ OR plane in $\mathbb{R}^3$ $z=r$, $y=t$, $x=2t+5$

5. A line goes through the points (9, 2) and (3, 4). Determine

a. its vector equation
b. its parametric equations
c. its symmetric equation
d. its scalar equation

(A = (9, 2), B = (3, 4))

ⓐ $\vec{AB}=(3-9,4-2)=(-6,2)$

$\therefore \vec{m}=(-3,1)$ can reduce dir. vectors.

$\therefore (x,y)=(9,2)+t(-3,1)\quad t\in\mathbb{R}$

ⓑ $\frac{x-9}{-3} = \frac{y-2}{1}$

ⓒ $x=9-3t$
$y=2+t$

ⓓ $\vec{n}=(1,3)$ AB

$Ax+By+C=0$ pt. $(9,2)$

$1(9)+3(2)+C=0$

$C=-15$

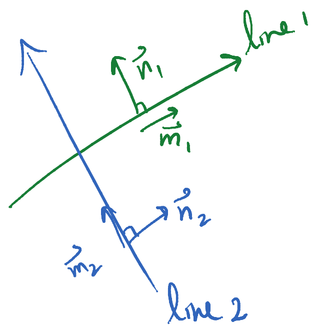
$\therefore x+3y-15=0$

6. Find the scalar equation of the line which is perpendicular to the line $2x - 3y + 18 = 0$ and has the same y-intercept as the line $(x, y) = (0, 1) + t(-3, 4)$.

y-int

$\vec{n}_1 = (2,-3)$

$\therefore \vec{n}_2 = (3,2)$



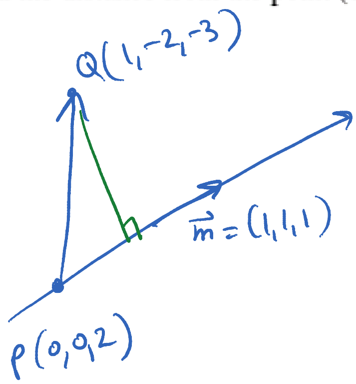
$\therefore Ax + By + C = 0$ sub pt. $(0,1)$

$3(0) + 2(1) + C = 0$

$C = -2$

$\boxed{\therefore 3x + 2y - 2 = 0}$

7. Find the distance from the point $(1, -2, -3)$ to the line $x = y = z - 2$.



$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-2}{1}$

$(x,y,z) = (0,0,2) + t(1,1,1)$

$\text{distance} = \frac{|\vec{PQ} \times \vec{m}|}{|\vec{m}|}$

can't use projection on $\vec{n}$ since in $\mathbb{R}^3$ don't have a unique $\vec{n}$

$\vec{PQ} = (1-0, -2-0, -3-2)$
$= (1,-2,-5)$

$$\begin{array}{cccccc} 1 & -2 & -5 & 1 & -2 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

$(-2+5, -5-1, 1+2)$

$(3,-6,3)$ don't reduce for distances

$= \frac{|(3,-6,3)|}{|(1,1,1)|}$

$= \frac{\sqrt{3^2+6^2+3^2}}{\sqrt{1^2+1^2+1^2}}$

$= \sqrt{\frac{54}{3}} = \sqrt{18} = 3\sqrt{2}$

8. A line through the origin has direction angles $\beta = 120°$ and $\gamma = 45°$. Find a vector equation for the line.

let $\hat{m}$ unit vector, $\hat{m} = (\cos\alpha, \cos\beta, \cos\gamma)$

$\hat{m} = (\cos\alpha, \cos 120°, \cos 45°)$

$|\vec{m}| = 1$

$\left(\sqrt{\cos^2\alpha + \cos^2 120 + \cos^2 45}\right)^2 = (1)^2$

$\cos^2\alpha + \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$

$\cos^2\alpha = \frac{1}{4}$

$\alpha = 60°$

$\therefore \hat{m} = (\cos 60°, \cos 120°, \cos 45°)$

$= \left(\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

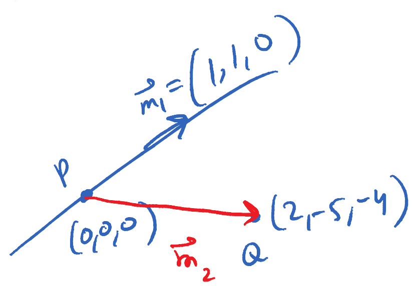
or

$(1, -1, \sqrt{2})$

$\therefore (x, y, z) = (0,0,0) + t(1, -1, \sqrt{2})$

$t \in \mathbb{R}$

9. Find the scalar equation of the plane containing the line $x = y$, $z = 0$ and the point $(2, -5, -4)$.



symmetric part

$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{0}$

line: $(x,y,z) = (0,0,0) + t(1,1,0)$

$\vec{PQ} = (2-1, -5-1, -4-0)$

$= (1, -6, -4)$

$\therefore$ plane $(x,y,z) = (0,0,0) + t(1,1,0) + r(1,-6,-4)$

convert to scalar

$\begin{matrix} 1 & 0 & 1 & 1 & 0 \\ -6 & -4 & 1 & -6 & -4 \end{matrix}$

$(-4-0, 0+4, -6-1)$

$\vec{n} = (-4, 4, -7)$

A B C

$Ax + By + Cz + D = 0$ pt. $(0,0,0)$

$-4(0) + 4(0) - 7(0) + D = 0$

$D = 0$

$\boxed{\therefore -4x + 4y - 7z = 0}$