p1NOTES

April-04-13
7:54 PM

TO DO

☐ p.10 #3a, b fix questions!

☐ reprint pg 2,3,4 changed layout + added in proper way to record.

☐ p.9 ⓑ change to piecewise

☐ extra day on limits + add in graph from given conditions

☐ pg.7 #3 change quest. see below ↓

IntroCalculusNotesNEW

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Proper way to record limits:

① $\lim_{x\to3}\frac{x^2-9}{x-3}$

$=\lim_{x\to3}\frac{(x+3)(x-3)}{x-3}$

$=\lim_{x\to3} x+3$

$=3+3$

$=6$

- write equals in front of limit (not after)

meaningless to write $\boxed{\lim_{x\to3} = \frac{x^2-9}{x-3}}$ ✗

need a function

after equals is the output y limit's answer.

- drop limit upon substitution only.

② Don't write divide by zero

ex. $\lim_{x\to0}\frac{3}{x}=\cancel{\frac{3}{0}}$ write DNE instead

③ Don't write $\frac{0}{0}$ (can as rough work only)

since it's indeterminate form

must simplify to see what it is.

ex. $\lim_{x\to0}\frac{3x}{x}$

$=\lim_{x\to0} 3$

$=3$

looks like $\frac{0}{0}$ you may think

its $=1$

or $=0$

or $=$ undefined

NOPE for all!

Converting repeating decimals to fractions:

ex. $x=3.772772\ldots$

multiply by 1000 since 3 digits repeat

$1000x=3772.772772\ldots$

$x=\quad 3.772772\ldots$

subtract $999x=3769$

$x=\frac{3769}{999}$ is the fraction value of the repeating decimal

Introduction to Calculus Unit - Notes

Tentative TEST date ~~Thrs. Apr. 18~~ Fri. Apr. 19

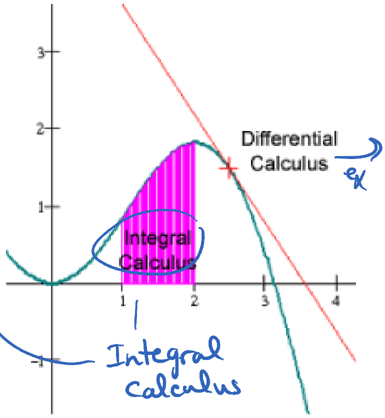
Big idea/Learning Goals

Calculus is an entire branch of mathematics. Calculus is built on two major complementary ideas.

At university

ex. $\int (3x^2 + 2)\,dx$

$= x^3 + 2x + C$



Integral calculus

High school

$y = x^3 + 2x$

ex $y' = \frac{dy}{dx} = 3x^2 + 2$

The first is differential calculus, which is concerned with the instantaneous rate of change. This can be illustrated by the slope of a tangent to a function's graph. The second is integral calculus, which studies the areas under a curve. These two processes act inversely to each other. The development of the mathematical methods of calculus has been credited to two great mathematicians; Sir Isaac Newton (1642 –1727) and Gottfried Wilhelm von Leibniz (1646 –1716). Although others as far back as 200B.C. had been working on solutions to these types of problems, Newton and Leibniz developed the process of differentiation and integration. Calculus allows you to find optimal solutions to mathematical expressions and is used in medicine, engineering, economics, computer science, business, physical sciences, statistics, and many more areas.

Corrections for the textbook answers:
1.2 #20 500

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	2-4	Limit of a Function 1.4	
	5-7	Properties of Limits 2days 1.5	
	8-10	Continuity 1.6	
	11-12	Slopes of Secants and Tangents 1.2	
	13-14	More Rates of Changes with Limits 1.3	
		Review	

Reflect – previous TEST mark ___________, Overall mark now___________.

Date:____________ Name: ______________________

Limit of a Function

1. Sometimes you can't work something out directly ... but you can see what it should be as you get closer and closer!

Why can't you find out the output of the function $f(x)=\frac{x^2-1}{x-1}$ at x=1?

$f(1)=\frac{1^2-1}{1-1}=\frac{0}{0}$ can't divide by zero! there is a hole or VA at $x=1$

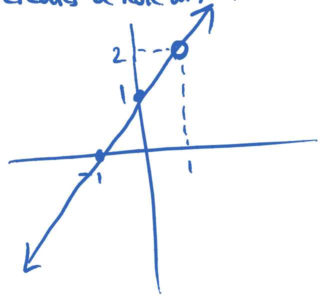
2. Sketch the function $f(x)=\frac{x^2-1}{x-1}=\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}=x+1$

line

factor that cancels creates a hole at $x=1$

x-int $x=-1$

y-int $y=1$



3. What is the output approaching as $x \to 1$?

approaching/tends

as $x\to 1$, $y \to 2$ like end behaviour notation

using limit notation:

$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$$

no arrows here set up as an equation.

4. The notation $\lim_{x\to a} f(x)=L$ is read as:

The limit of function f(x) as x tends to a is equal to L

5. $\lim_{x\to a} f(x)=L$ means that it EXISTS if:

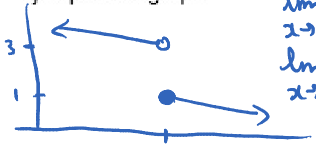
the following conditions are satisfied

① $\lim_{x\to a^+} f(x)$ exists ② $\lim_{x\to a^-} f(x)$ exists

③ $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)$

6. Limit will not exist if there is:

jump in the graph

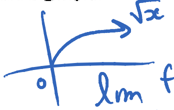


$\lim_{x\to 5^+} f(x)=1$

$\lim_{x\to 5^-} f(x)=3$

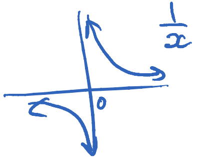
$\therefore \lim_{x\to 5} f(x)=$ D.N.E does not exist.

one sided graph



$\lim_{x\to 0^-} f(x)=$ D.N.E

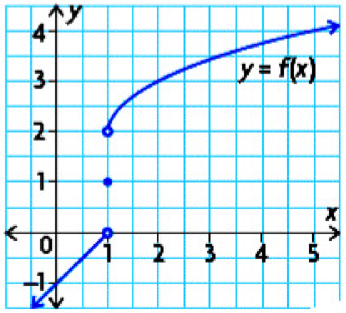
approaching ∞ because of a VA



$\lim_{x\to 0^+} f(x)=\infty$

not a definite # D.N.E.

7. Find the following limits.



explanation in words:

find $x=5$ then go where the graph is, follow the graph with your finger from left side, Ask yourself what output is approached as you do that.

a. $\lim_{x\to 5^-} f(x)=4$

b. $\lim_{x\to 5^+} f(x)=4$

c. $\lim_{x\to 1^-} f(x)=0$

d. $\lim_{x\to 1^+} f(x)=2$

e. $\lim_{x\to 5} f(x)=4$

f. $\lim_{x\to 1} f(x)=$ D.N.E

8. Determine if the limit exists both algebraically and graphically

Algebraic | **Graphical**

1. $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ -2 & \text{if } x=1 \\ -x+3 & \text{if } x>1 \end{cases}$ } left side of 1 limit at $x=1$

$\lim_{x\to 1} f(x) = ?$

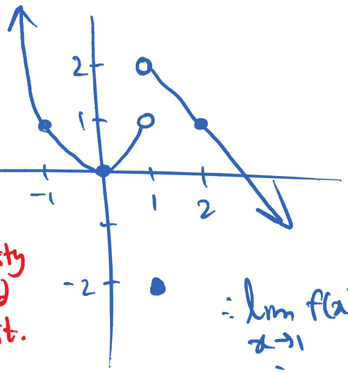
check all 3 conditions.

① $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x+3 = -1+3 = 2$

② $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$ ← not equal

③ since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ then $\lim_{x\to 1} f(x) = \text{D.N.E.}$

if you plug in the # and do not get error / infinity just do so and drop the limit.



$\therefore \lim_{x\to 1} f(x) = \text{D.NE}$ since approaches 2 different outputs.

Algebraic informal way | **Graphical**

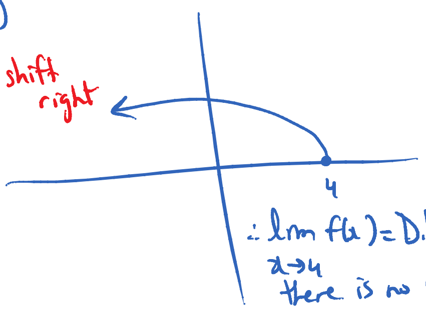
2. $f(x)=\sqrt{4-x}$ limit at $x=4$

$\lim_{x\to 4^-} \sqrt{4-x} \doteq \sqrt{4-3.999} = 0.001$

$\lim_{x\to 4^+} \sqrt{4-x} \doteq \sqrt{4-4.001} = \sqrt{\text{neg}} = \text{undefined}$

$\therefore \lim_{x\to 4} f(x) = \text{D.N.E.}$

$\sqrt{-(x-4)}$ flips in y-axis, shift right



$\therefore \lim_{x\to 4} f(x) = \text{D.NE}$ there is no right side to approach.

Algebraic | **Graphical**

3. $f(x)=\dfrac{1}{1-x}$ limit at $x=1$

$\lim_{x\to 1} \dfrac{1}{1-x} = \dfrac{1}{1-1} = \boxed{\dfrac{1}{0} = \infty}$

not proper way to record this.

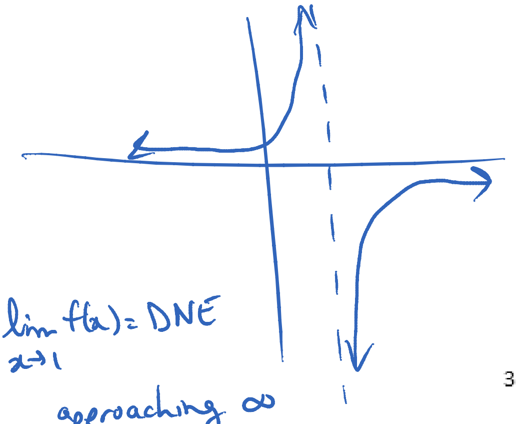
$\lim_{x\to 1^+} \dfrac{1}{1-x} \doteq \dfrac{1}{1-1.001} = -1000$

not the same

$\lim_{x\to 1^-} \dfrac{1}{1-x} \doteq \dfrac{1}{1-0.999} = 1000$

$\therefore \lim_{x\to 1} f(x) = \text{D.N.E.}$

$\dfrac{1}{-1(x-1)} = \dfrac{-1}{x-1}$ ← flip in x/y axis ← shift right.



$\lim_{x\to 1} f(x) = \text{DNE}$ approaching ∞

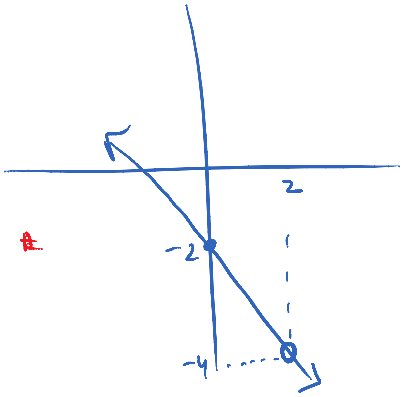
Determine if the limit exists both algebraically and graphically

Algebraic	**Graphical**
4. $f(x)=\dfrac{4-x^2}{x-2}$ limit at $x=2$	$f(x)=-(x+2)=-x-2$ hole at $x=2$

$$\lim_{x\to 2}\frac{4-x^2}{x-2}=\lim_{x\to 2}\frac{(2-x)(2+x)}{(x-2)}$$

$$=\lim_{x\to 2}\frac{-\cancel{(x-2)}(2+x)}{\cancel{(x-2)}}=\lim_{x\to 2}-(2+x)=-4$$

do $\boxed{\text{full}}$ limit (not one sided)
if not a piecewise function + can plug in the #
after simplifying.

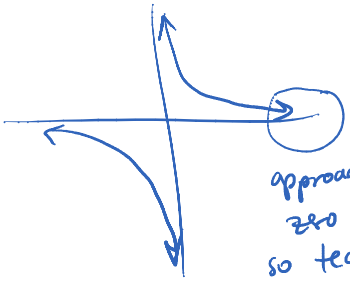


Algebraic *informal way*	**Graphical**
5. $f(x)=\dfrac{1}{x}$ limit at $x\to\infty$	

$$\lim_{x\to\infty}\frac{1}{x}\doteq\frac{1}{1000000}=0.000001$$

$$\frac{1}{1000\,0000000}=0.0000000001$$

$$\therefore \lim_{x\to\infty}\frac{1}{x}=0$$

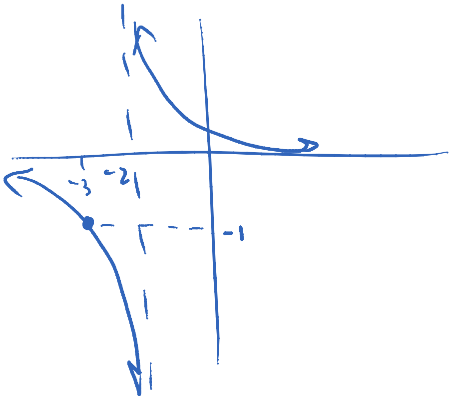


approaches zero from left.
so technically $\lim\limits_{x\to\infty^{-}} f(x)=0$
must write one sided limit.

Algebraic	**Graphical**
6. $f(x)=\dfrac{1}{x+2}$ limit at $x=-3$	

$$\lim_{x\to -3}\frac{1}{x+2}=\frac{1}{-3+2}=\frac{1}{-1}=-1$$

sub in right away
no errors.



Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$.

1. $\lim_{x\to a} k = k$, for any constant k
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$, for any constant c
5. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
6. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, for any rational number n

<u>THINGS to try</u>

- sub in what x approaches to right away, if defined, then that's the answer.
- If you have $\frac{0}{0}$ indeterminate form
  - * factor, expand, LCD
  - * rationalizing (numerator/denom)
  - * change of variable
- If you have $\frac{\#}{0}$ form or single square root (check if the pt. x approaches is where the graph starts) then try #'s close to the point on both sides to show the limit will not exist
- If you have piecewise or abs. val function - you must do both sides of the limit separately.
- If you have $\lim_{x\to\infty} f(x)$ ← rational x's in both numer/denom then divide by highest power.

Evaluate the limits

ex. $\lim_{x\to3}\sqrt{x-1}$ $=\sqrt{2}$

$\lim_{x\to1}\sqrt{x-1}$ $=$ D.N.E

1. $\lim_{x\to5}\sqrt{\frac{x^2}{x-1}}$

(use the properties above to explain why you can just substitute the value in if it is not indeterminate form.)

by 7. $= \sqrt{\lim_{x\to5}\left(\frac{x^2}{x-1}\right)}$

by 6 $= \sqrt{\frac{\lim_{x\to5}(x^2)}{\lim_{x\to5}(x-1)}}$

$= \sqrt{\frac{\left(\lim_{x\to5} x\right)^2}{\lim_{x\to5} x - \lim_{x\to5} 1}}$ by 7 by 3

by 2. $= \sqrt{\frac{5^2}{5-1}}$ by 1

$= \sqrt{\frac{25}{4}}$

$= \frac{5}{2}$

$$|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$$

2. $\lim_{x\to 2}\dfrac{\overset{f(x)}{|2x-4|}}{x-2}$ $\quad \frac{0}{0}$ form

$$|2(x-2)| = \begin{cases} 2(x-2) & \text{if } 2(x-2)>0 \quad \text{right of 2} \quad x>2^+ \\ -2(x-2) & \text{if } 2(x-2)<0 \quad \text{left side of 2} \quad x<2^- \end{cases}$$

$$= \lim_{x\to 2}\frac{|2(x-2)|}{(x-2)}$$

$$\lim_{x\to 2^+}\frac{2\cancel{(x-2)}}{\cancel{(x-2)}} \qquad \lim_{x\to 2^-}\frac{-2\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2^+} 2 \qquad = \lim_{x\to 2^-} -2$$

$$= 2 \qquad = -2$$

$\therefore \lim_{x\to 2} f(x) = \text{D.N.E.}$

3. $\lim_{x\to 3}\dfrac{\frac{1}{3}-\frac{1}{x}}{x-3}$ $\quad$ LCD $\quad \%$

$$= \lim_{x\to 3}\frac{\left(\frac{x-3}{3x}\right)}{\left(\frac{x-3}{1}\right)} \qquad \div\left(\frac{x-3}{1}\right) \quad \times\left(\frac{1}{x-3}\right)$$

$$= \lim_{x\to 3}\left(\frac{\cancel{x-3}}{3x}\right)\times\frac{1}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3}\frac{1}{3x}$$

$$= \frac{1}{9}$$

4. $\lim_{x\to 3}\sqrt{9-x^2}$

$$\lim_{x\to 3^+}\sqrt{9-x^2}$$

$$\doteq \sqrt{9-(3.001)^2}$$

$$= \sqrt{\text{neg}}$$

= D.N.E. can stop here without doing the left side.

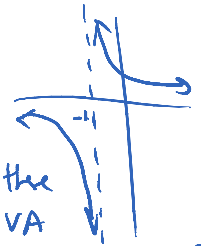
$\therefore \lim_{x\to 3}\sqrt{9-x^2} = \text{D.NE}$ since one sided limit didn't exist

5. $\lim_{x\to -1}\dfrac{x}{x^2+x}$ $\quad \frac{-1}{0}$

$$= \lim_{x\to -1}\frac{x}{x(x+1)}$$

$$= \lim_{x\to -1}\frac{1}{x+1}$$

= D.NE since there is a VA at $x=-1$ as seen in the graph.



6. $\lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$ try to use a substitution that will get rid of both roots

let $u^6 = 1+x \qquad u = \sqrt[6]{1+x}$

then $u^6 - 1 = x$

$$\lim_{u\to 1}\frac{\sqrt{1+u^6-1}\;-1}{\sqrt[3]{1+u^6-1}\;-1}$$

$$= \lim_{u\to 1}\frac{u^{6/2}-1}{u^{6/3}-1}$$

$$= \lim_{u\to 1}\frac{u^3-1}{u^2-1}$$

$$= \lim_{u\to 1}\frac{\cancel{(u-1)}(u^2+1u+1)}{(u+1)\cancel{(u-1)}}$$

$$= \lim_{u\to 1}\frac{u^2+u+1}{u+1}$$

$$= \frac{1^2+1+1}{1+1}$$

$$= \frac{3}{2}$$

7. $\lim_{x\to\infty}\dfrac{4-3x^3+2x}{8-x^2+7x^3}\;\dfrac{\div x^3}{\div x^3}$

$$= \lim_{x\to\infty}\frac{\frac{4}{x^3}-\frac{3x^3}{x^3}+\frac{2x}{x^3}}{\frac{8}{x^3}-\frac{x^2}{x^3}+\frac{7x^3}{x^3}}$$

$$= \lim_{x\to\infty}\frac{\overset{0}{\cancel{\frac{4}{x^3}}}-3-\overset{0}{\cancel{\frac{2}{x^2}}}}{\overset{0}{\cancel{\frac{8}{x^3}}}-\overset{0}{\cancel{\frac{1}{x}}}+7}$$

$$= -\frac{3}{7}$$

using the property of $\boxed{\lim_{x\to\infty}\frac{1}{x}=0}$

Note: this is HA of the rational function (end behaviour.)

Day 2

Rationalize.

8. $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\times\frac{(\sqrt{4+x}+2)}{(\sqrt{4+x}+2)}$

$=\lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)}$

$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)}$

$=\frac{1}{\sqrt{4+0}+2}=\frac{1}{2+2}=\frac{1}{4}$

Foil top

$(\sqrt{4+x})^2+2\sqrt{4+x}-2\sqrt{4+x}-4$

9. $\lim_{x\to -3}\frac{(1+x)^2-4}{x+3}$

$=\lim_{x\to -3}\frac{x^2+2x+1-4}{x+3}$

$=\lim_{x\to -3}\frac{x^2+2x-3}{x+3}$

$=\lim_{x\to -3}\frac{\cancel{(x+3)}(x-1)}{\cancel{(x+3)}}=-3-1$

$=-4$

10. $\lim_{x\to -\infty}\left(\frac{(1-x)\div x^2}{(5+x^2)\div x^2}-2\right)$

divide by highest power x^2

$=\lim_{x\to -\infty}\frac{\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{1}{x}}^{\to 0}}{\cancel{\frac{5}{x^2}}^{\to 0}+1}-2$

$=\frac{0}{1}-2$

$=-2$

11. $\lim_{x\to 9}\frac{2x}{x^2-9}$

$=\frac{2(9)}{9^2-9}$

$=\frac{18}{81-9}=\frac{18}{72}=\frac{1}{4}$

12. $\lim_{x\to 8}\frac{2-\sqrt[3]{x}}{8-x}$ change of variable

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

let $u=\sqrt[3]{x}$

then $u^3=x$

as $x\to 8$

$u\to 2$

rewritten all with u's

$=\lim_{u\to 2}\frac{2-u}{8-u^3}$

$=\lim_{u\to 2}\frac{\cancel{(2-u)}}{\cancel{(2-u)}(4+2u+u^2)}$

$=\lim_{u\to 2}\frac{1}{4+2u+u^2}$

$=\frac{1}{4+4+4}=\frac{1}{12}$

13. $\lim_{x\to -2}\frac{2+x}{16-x^4}$ change question! so can practice factoring

$=\lim_{x\to -2}\frac{(2+x)}{(4+x^2)(4-x^2)}$

$=\lim_{x\to -2}\frac{\cancel{(2+x)}}{(4+x^2)(2-x)\cancel{(2+x)}}$

$=\frac{1}{(4+4)(2--2)}=\frac{1}{8(4)}=\frac{1}{32}$

Try 1.5 # 10ⓒ

8ⓓ

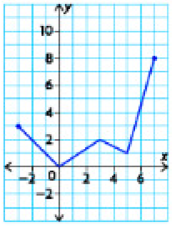
Discuss → $\lim_{x\to 3}|x-3|$ vs. $\lim_{x\to 3}|3-x|$

→ $f(x)=\begin{cases}x^2+1 & \text{if } x\neq 2\\ 1 & \text{if } x=2\end{cases}$ vs $f(x)=\begin{cases}x^2+1 & \text{if } x>2\\ 1 & \text{if } x\leq 2\end{cases}$

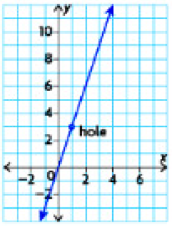
Continuity

quote only domain (x's)

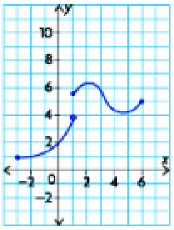
A. Continuous for all values of the domain



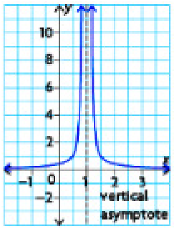
B. Discontinuous at $x = 1$ (point discontinuity)



C. Discontinuous at $x = 1$ (jump discontinuity)



D. Discontinuous at $x = 1$ (infinite discontinuity)



1. State the 3 conditions you must meet for the function to be continuous.

Check the equation for any undefined values: $\sqrt{neg}$ or $\frac{\#}{0}$

For piecewise check the pts $\boxed{a}$ where pieces may not connect

① $f(a)$ is defined

② $\lim_{x \to a} f(x)$ must exist

③ $\lim_{x \to a} f(x) = f(a)$

2. Determine if the function is continuous, if not, identify where the discontinuity is. Also show a graphical representation.

a. $f(x) = x^3 - x$

this is continuous everywhere on $x \in \mathbb{R}$ since there are no sq. roots or denominators to make it undefined.

Practice sketching:

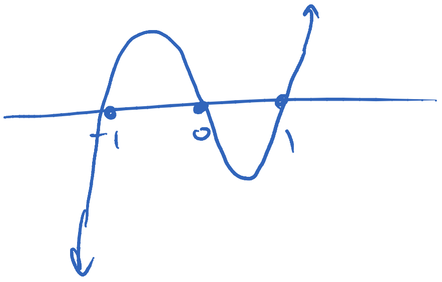
$f(x) = x(x^2-1)$

$= (x)(x+1)(x-1)$

zeros at $x = 0, -1, 1$

all factors of power one

∴ graph will look linear near them (cut)



b. $g(x) = \begin{cases} x & \text{if } x \le -4 \\ 5 & \text{if } -4 < x \le 3 \\ x^2 - 4 & \text{if } x > 3 \end{cases}$

checked that all pieces are defined everywhere (no sq. root, no denom.)

check at $x = -4$

① $f(-4) = -4$

② $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} x = -4$

$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} 5 = 5$

$\therefore \lim_{x \to -4} f(x) = \text{D.N.E}$

∴ $f(x)$ not continuous at $x = -4$

at $x = 3$

① $f(3) = 5$

② $\lim_{x \to 3^-} 5 = 5$

$\lim_{x \to 3^+} x^2 - 4 = 5$ } exists ☺

③ $\lim_{x \to 3} f(x) = 5 = f(3)$

∴ $f(x)$ is continuous at $x = 3$

Determine if the function is continuous, if not, identify where the discontinuity is. Also show a graphical representation.

c. $h(x)=\frac{2x+2}{3x-12}=\frac{2(x+1)}{3(x-4)}$

discontinuous at $x=4$ (VA)

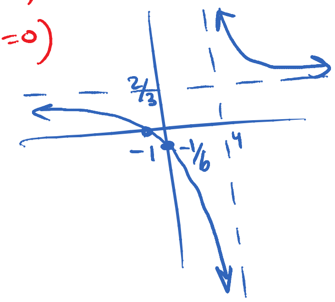
x-int $=-1$ (sub $y=0$)

y-int $=-\frac{1}{6}$ (sub $x=0$)

VA $=\boxed{4=x}$

HA/OA $=\boxed{\frac{2}{3}=y}$

look at leading coefficient if degrees are the same.



d. $i(x)=\begin{cases}\frac{x^2-9}{x+3} & \text{if } x<0\\ -3 & \text{if } x\geq 0\end{cases}$

1st piece is undefined at $x=-3$ (hole) and this is part of the domain of that piece

at $x=0$

① $f(0)=-3$

② $\lim_{x\to 0^-}\frac{x^2-9}{x+3}=-3$
$\lim_{x\to 0^+}-3=-3$ } exists

③ $\lim_{x\to 0}f(x)=-3=f(0)$

$\therefore$ continuous at $x=0$

only discont. at $x=-3$

e. $f(x)=\frac{1+0x}{x+2}$

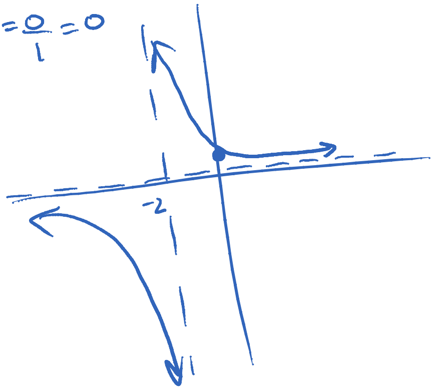
discont. at $x=-2$ (VA)

x-int = N/A

y-int $=\frac{1}{2}$

VA $x=-2$

HA $y=\frac{0}{1}=0$



f. $f(x)=\frac{1+0x^2}{x^2+4}$

TEAMS.

no discontinuity

since x^2+4 never zero

can try $x^2+4=0$

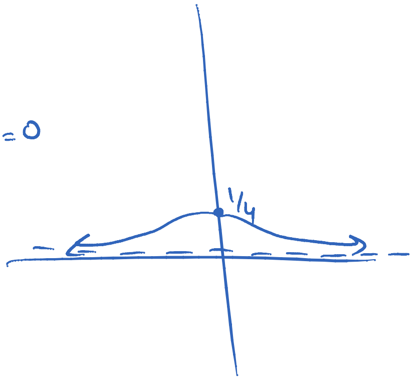
$x=\sqrt{-4}$ can't!

x-int = N/A

y-int $=\frac{1}{4}$

VA = N/A

HA $y=\frac{0}{1}=0$



3. Find the values of each constant that would make this function continuous.

a.

$$f(x)=\begin{cases}-2x+a & \text{if } x\le -1\\ x^2+b & \text{if } -1<x\le 2\\ \frac{1}{x}+2a & \text{if } 2<x\end{cases}$$

$\frac{1}{x}$ → undefined at $x=0$ but it's not in domain $2<x$ (not a problem)

focus on condition ②

at $x=-1$

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^+} f(x)$$

$$\lim_{x\to -1^-} -2x+a = \lim_{x\to -1^+} x^2+b$$

$$2+a=1+b$$

$$1+a=b$$

at $x=2$

$$\lim_{x\to 2^-} x^2+b = \lim_{x\to 2^+} \frac{1}{x}+2a$$

$$4+b=\frac{1}{2}+2a$$

$$4+(1+a)=\frac{1}{2}+2a$$

$$4+1-1/2=2a-a$$

$$\boxed{4.5=a}$$

$$\therefore \boxed{b=5.5}$$

these values will make f(x) continuous.

b.

$$j(x)=\begin{cases}2x+a & \text{if } x\le -1\\ 22 & \text{if } -1<x\le 3\\ -bx^2+31 & \text{if } 3<x\le 5\\ \sqrt{cx}+b & \text{if } 5<x\end{cases}$$

check each piece!

4th piece maybe undefined if c is neg. Solve 1st to see

at $x=-1$

$$\lim_{x\to -1^-} 2x+a=\lim_{x\to -1^+} 22$$

$$-2+a=22$$

$$\boxed{a=24}$$

at $x=3$

$$\lim_{x\to 3^-} 22=\lim_{x\to 3^+} -bx^2+31$$

$$22=-9b+31$$

$$-9=-9b$$

$$\boxed{1=b}$$

at $x=5$

$$\lim_{x\to 5^-} -bx^2+31=\lim_{x\to 5^+} \sqrt{cx}+b$$

$$-1(25)+31=\sqrt{5c}+1$$

$$5=\sqrt{5c}$$

$$25=5c$$

$$\boxed{5=c}$$

10

c is not negative so j(x) is cont.

Slope of Secants and Tangents

1. What is the slope equation for straight lines?

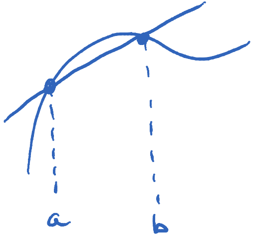
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

2. How is rate of change of a straight line different from rate of change of a curve?

straight lines have constant slopes

curves have changing slopes.

3. Recall from advanced functions,
   a. the picture and formula of the **average** rate of change of a curve.



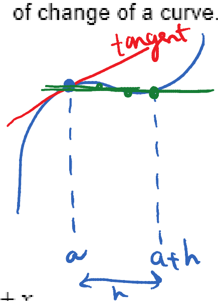
$$m_{sec} = a.r.o.c = \frac{f(b) - f(a)}{b - a}$$

on

$x \in [a, b]$

$a \le x \le b$

   b. the picture and formula of the **instantaneous** rate of change of a curve.



$$m_{tan} = iroc = \lim_{h \to 0} \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}}$$

at $x = a$

4. Determine the slope of the tangent for rational function $f(x) = \dfrac{5+x}{x^2}$ at $x = 5$

$$m_{tan} = iroc = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \left[\frac{10+h}{(5+h)^2} - \frac{10}{25}\right]\frac{1}{h}$$

$$= \lim_{h \to 0} \left[\frac{10+h}{25+10h+h^2} - \frac{2}{5}\right]\frac{1}{h}$$

LCD

$$= \lim_{h \to 0} \left[\frac{5(10+h) - 2(25+10h+h^2)}{5(25+10h+h^2)}\right]\frac{1}{h}$$

expand top only.

$$= \lim_{h \to 0} \frac{\cancel{50} + 5h - \cancel{50} - 20h - 2h^2}{5(25+10h+h^2)h}$$

* all terms left should have h !!
→ common factor it

$$= \lim_{h \to 0} \frac{\cancel{h}(5 - 20 - 2h)}{5(25+10h+h^2)\cancel{h}}$$

$$= \frac{-15 - 2(0)}{5(25+10(0)+0^2)}$$

$$= \frac{-15}{125} = \frac{-3}{25}$$

To find eqtn of tangent line:

$y = mx + b$

① sub x into original to y

② sub x, y, m to find b

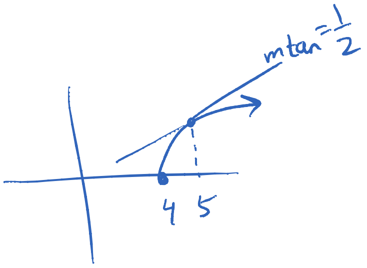
5. Determine the slope of the tangent for square root function $g(x)=\sqrt{x-4}$ at $x=5$

$$\text{iroc} = \lim_{h\to 0} \frac{\sqrt{5+h-4} - \sqrt{5-4}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$ is the slope of the tangent.



6. Determine the equation of the line that is perpendicular to the tangent to $y=x^5$ at $x=-2$, and which passes through the tangent point.

$$m_{tan} = \text{iroc} = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h}$$

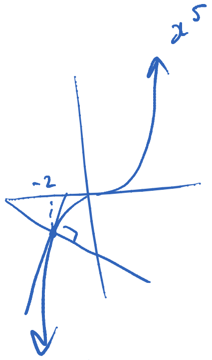
$$= \lim_{h\to 0} \frac{(\cancel{x^5}+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5)-\cancel{x^5}}{h}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$= \lim_{h\to 0} \frac{\cancel{h}(5x^4+10x^3h+10x^2h^2+5xh^3+h^4)}{\cancel{h}}$$

$$m_{tan}(x) = 5x^4 + 10x^3(0) + 0 + 0$$

$$m_{tan}(-2) = 5(-2)^4$$
$$= 80$$



∴ perpendicular slope is $\frac{-1}{80}$

$y = mx+b$

$-32 = \frac{-1}{80}(-2) + b$

$-32 = \frac{1}{40} + b$

$-\frac{1279}{40} = b$

at $x=-2$
$y=-32$

∴ perpendicular line through pt. $(-2,-32)$ is $y = \frac{-1}{80}x - \frac{1279}{40}$

More Rates of Change with Limits

Average Rate of Change

$$m_{sec} = aroc = \frac{f(b)-f(a)}{b-a}$$

average rate of change on interval $a \le x \le b$

or $x \in [a,b]$

Instantaneous Rate of Change

$$m_{tan} = iroc = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

at $x=a$

OR

change of variable : let $x = a+h$

$$iroc = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

1. The surface area of a snowball, in square centimeters, is modelled by the equation $S = 4\pi r^2$ where r is the radius, in cm. The volume, in cubic centimeters, is modelled by $V = \frac{4}{3}\pi r^3$.

    a. Determine the average rate of change of the surface area as the radius changes from 20cm to 25cm.
    b. Determine the instantaneous rate of change of volume when the radius is 10cm.

$$aroc = \frac{S(25)-S(20)}{25-20}$$

$$= \frac{4\pi(25)^2 - 4\pi(20)^2}{5}$$

$$= 4\pi\left[\frac{625-400}{5}\right]$$

$$= 180\pi$$

$$\doteq 565.2 \ \frac{cm^2 \text{ of area}}{cm \text{ of radius}} \quad \frac{\Delta S}{\Delta r}$$

$$iroc = \lim_{r \to 10} \frac{V(r)-V(10)}{r-10}$$

$$= \lim_{r \to 10} \frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi(10)^3}{r-10}$$

$$= \lim_{r \to 10} \frac{4}{3}\pi\left[\frac{r^3-10^3}{r-10}\right]$$

$$= \lim_{r \to 10} \frac{4}{3}\pi\left[\frac{\cancel{(r-10)}(r^2+10r+100)}{\cancel{(r-10)}}\right]$$

$$= \frac{4}{3}\pi\left[10^2+10(10)+100\right]$$

$$= \frac{4}{3}\pi(300)$$

$$= 400\pi$$

$$\sim 1256 \ \frac{cm^3 \text{ of volume}}{cm \text{ of radius}} \quad \frac{\Delta V}{\Delta r}$$

2. An outdoor hot tub holds 2700L of water. When a valve at the bottom of the tub is opened, it takes 3h for the water to completely drain. The volume of water in the tub is modelled by the function $V(t)=\frac{1}{12}(180-t)^2$, where V is the volume of water in the hot tub, in litres, and t is the time, in minutes, that the valve is open. Determine the instantaneous rate of change of the volume of water at 60min. (~~Use the second version of the definition for rate here)~~

$$\text{iroc} = \lim_{h\to 0} \frac{V(60+h)-V(60)}{h}$$

$$= \lim_{h\to 0} \left[\frac{1}{12}(180-60-h)^2 - \frac{1}{12}(180-60)^2\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{1}{12h}\left[(120-h)^2 - 120^2\right]$$

$$= \lim_{h\to 0} \frac{1}{12h}\left[\cancel{120^2} - 240h + h^2 - \cancel{120^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{12\cancel{h}}\left[\frac{\cancel{h}(-240+h)}{1}\right]$$

$$= \frac{1}{12}(-240+0)$$

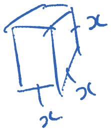
$$= -20 \ L/\text{min}$$

$$\frac{\Delta V}{\Delta t}$$

3. Show that the rate of change in the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

Volume of cube: $V = x^3$

$V = lwh$



$$\text{iroc of } V = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= 3x^2 + 3x(0) + 0$$

$$\boxed{\therefore \text{rate of change of Volume} = 3x^2}$$

Surface Area : $S = 6x^2$

half of it :)