p1NOTES

March-18-13
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Intersection
sNotesNEW

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see below

Intersection - Notes

Tentative TEST date Mon Apr. 8

Big idea/Learning Goals

In this unit, we will introduce perhaps the most important idea associated with vectors – the solution of systems of equations. In grade 10, the solution of systems of equations was introduced in situations dealing with two equations in two unknowns. Geometrically, the solution of two equations in two unknowns is the point of intersection between two lines on the *xy*-plane. In this unit, we are going to extend these ideas and consider systems of equations in R^3 and interpret their meaning. We will be working with systems of up to three equations in three unknowns, and we will demonstrate techniques for solving these systems. At university level you will learn how to use matrices to solve the many equations with less amount of writing than we will do in this unit.

Corrections for the textbook answers:
Review #9a non coplanar vectors, thus meet in a point $(\frac{42}{83}, \frac{-34}{83}, \frac{18}{83})$

Success Criteria

- ☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
Mar. 27	2-4	2 Lines & Line with a Plane 9.1 and 9.2	
	5-9	3 Planes & 2 Planes – 2 days 9.3 and 9.4	
	10-11	Distances 9.5 and 9.6	
		Review	

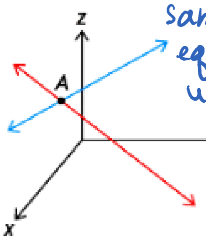
Reflect – previous TEST mark __________, Overall mark now__________.

2 Lines

Intersecting Lines

Case 1: Intersecting Lines at a Point

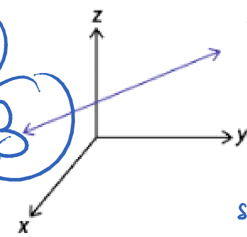
A



same # of equations as unknowns (or more eqtns than unknowns) + no contradiction

Case 2: Coincident Lines

B



less equations than unknowns + no contradictions

ex. $x-2y=3$

same ~~$2x-4y=6$~~

let $y=t$

$x=3+2t$

Non-intersecting Lines

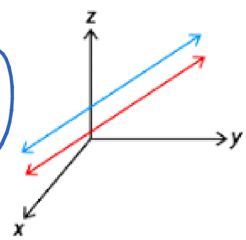
get a contradiction

ex. $3=0$

or $t=5$ and $t=1$

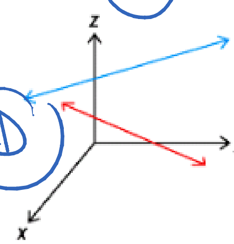
Case 3: Parallel Lines

C



Case 4: Skew Lines

D



1. Find the intersection of

make sure you use different letters for parameters of 2 different lines.

line 1 $\begin{cases} x = -1 + 3t \\ y = 1 + 4t \\ z = -2t \end{cases}$ and line 2 $\begin{cases} x = -1 + 2s \\ y = 3s \\ z = -7 + s \end{cases}$

equal

① $-1+3t=-1+2s$

② $1+4t=3s$

③ $-2t=-7+s$

③ ×3 $\quad 0-6t=-21+3s$

② $\quad 1+4t=0\ \ 3s$

subtract $-1\ \ -10t=-21\ \cancel{0s}$

$-10t=-20$

$t=2$ sub in ③

$-2(2)=-7+s$

$-4+7=s$

$3=s$

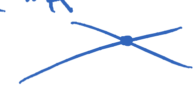
check in ①

$-1+3(2) \stackrel{?}{=} -1+2(3)$

✓

no contradictions

∴ situation #A



$\therefore POI = (-1+3t,\ 1+4t,\ -2t)$

$= (-1+3(2),\ 1+4(2),\ -2(2))$

$= (5,\ 9,\ -4)$

2

2. Find the intersection of

line 1 $\vec{r} = (2, 1, 0) + t(1, -1, 1)$ $\vec{m}_1$

line 2 $\vec{r} = (3, 0, -1) + s(2, 3, -1)$ $\vec{m}_2$

① $2+t = 3+2s$

② $1-t = 0+3s$

add $3 = 3+5s$

$0 = s$ sub in ① $2+t = 3+2(0)$ $t=1$

③ $0+t = -1-s$

check in ③

$1 = -1-0$

$1 \neq -1$

$\therefore$ contradiction

since $\vec{m}_1$ is not parallel to $\vec{m}_2$ the situation is skew lines

D

3. Find the intersection of each pair of lines. If they do not meet explain why.

a. $(x, y, z) = (1 + t, 2 + t, -t)$

$(x, y, z) = (3 - 2u, 4 - 2u, -1 + 2u)$

① $1+t = 3-2u$

② $2+t = 4-2u$

③ $-t = -1+2u$

add ②+③ $2+0 = 3+0$

$2 \neq 3$

contradiction

$\vec{m}_1 = (1,1,-1)$

$\vec{m}_2 = (-2,-2,2)$

these are multiples $\therefore$ parallel lines

situation C

b. $x = 1 + t, y = 1 + 2t, z = 1 - 3t$

$x = 3 - 2u, y = 5 - 4u, z = -5 + 6u$

① $1+t = 3-2u$

② $1+2t = 5-4u$

③ $1-3t = -5+6u$

2x① $2+2t = 6-4u$

② $1+2t = 5-4u$

subtract $1 = 1$

no contradiction

$\vec{m}_1 = (1,2,-3)$

$\vec{m}_2 = (-2,-4,6)$

multiples

$\therefore$ situation B

L.O.I

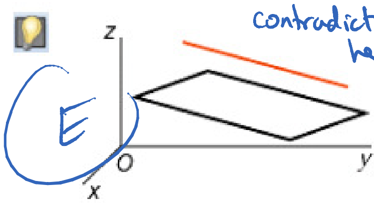
Line of intersection

$(x,y,z) = (1,1,1) + t(1,2,-3)$

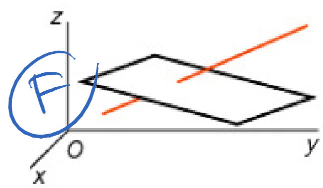
Continue same day next topic since same type of steps

Line with a Plane

contradiction here.

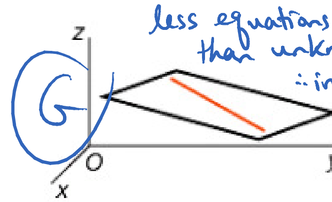


line is parallel to the plane



line intersects the plane

less equations than unknowns ∴ introduce one parameter



line lies in the plane

1. Find the intersection of the line with parametric equations $x = 1 + 2t$, $y = -6 + 3t$, $z = -5 + 2t$ and the plane whose scalar equation is $4x - 2y + z - 19 = 0$.

$4(1+2t) - 2(-6+3t) + (-5+2t) - 19 = 0$

$4 + 8t + 12 - 6t - 5 + 2t - 19 = 0$

$4t = 8$

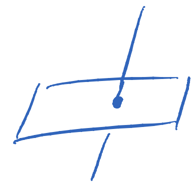
$t = 2$

sub into line to find pt.

$x = 1 + 2(2) = 5$

$y = -6 + 3(2) = 0$

$z = -5 + 2(2) = -1$



∴ POI is $(5, 0, -1)$

∴ situation F

2. Find the intersection of the line $x = 2t$, $y = 1 - t$, $z = -4 + t$ and the plane $x + 4y + 2z - 4 = 0$.

$2t + 4(1-t) + 2(-4+t) - 4 = 0$

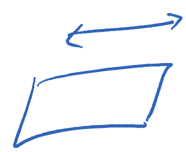
$2t + 4 - 4t - 8 + 2t - 4 = 0$

$0t = 8$

$0 \neq 8$

contradiction

∴ situation E



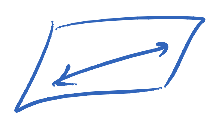
3. Find the intersection of the line $x = -4 + 3t$, $y = 0$, $z = t$ and the plane $x - 2y - 3z + 4 = 0$.

$(-4+3t) + 2(0) - 3(t) + 4 = 0$

$0t = 0$

no contradictions and no equation to solve than unknowns ie. less eqtn. than unknowns

∴ situation G



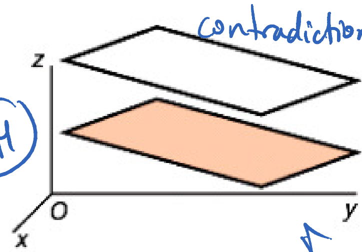
L.O.I is $(x, y, z) = (-4, 0, 0) + t(3, 0, 1)$

4

2 Planes

contradiction

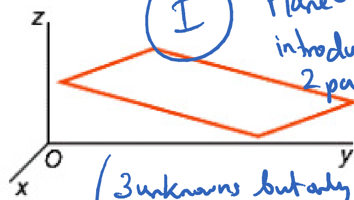
H



planes are parallel

I

Plane O.I.
introduce 2 parameters

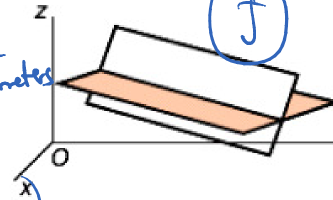


planes are coincident

3 unknowns but only 1 equation

J

L.O.I
introduce 1 parameter

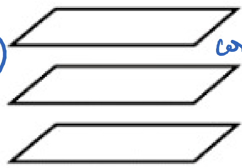


planes intersect in a line

3 Planes

When the normals of all three are parallel, the possibilities are

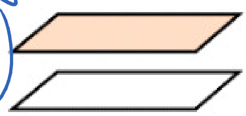
K



3 planes are
parallel and distinct;
no intersection

contradiction

H



2 planes are coincident,
the other parallel;
no intersection

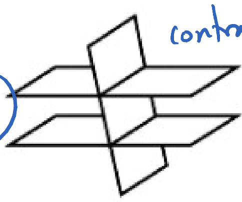
I

3 planes are coincident;.
intersection: a plane

When only two of the normals of the planes are parallel, the possibilities are

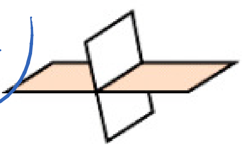
contradiction

L



two planes are
parallel and distinct,
the other crossing;
no common intersection

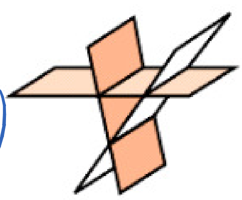
J



two planes are coincident;
the other crossing;
intersection: a line

When none of the normals are parallel, the possibilities are

M

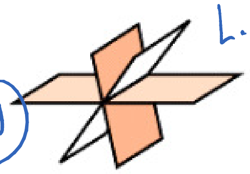


normals coplanar; $(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$
no intersection

contradiction

L.O.I

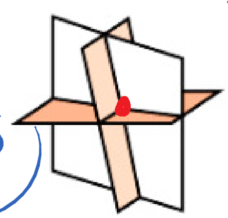
N



normals coplanar;
intersection: a line

P.O.I

P



normals are not parallel
and non-coplanar;
intersection: a point

$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 \neq 0$

System of Equations with 3 Planes

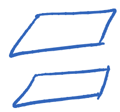
1. Solve each system and interpret what type of geometrical picture the system represents.

a.

Eg.

(1) $x + y + 2z = 2$
(2) $x - y - 2z = 5$
(3) $3x + 3y + 6z = 5$

2 planes are parallel but not the same



$\vec{n}_1 = (1, 1, 2)$ ×3 collinear
$\vec{n}_3 = (3, 3, 6)$

but $D_1 = 2$ is not the same multiple as $D_3 = 5$

(1)+(2) $2x = 7$

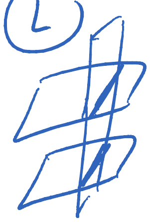
$x = 7/2$

contradiction

3×(2) $3x - 3y - 6z = 15$
(3) $3x + 3y + 6z = 5$
add $6x = 20$

$x = \frac{20}{6} = \frac{10}{3}$

∴ situation L



no solution

b.

$x + 2y + z = 12$ (1)
$2x - y + z = 5$ (2)
$3x + y - 2z = 1$ (3)

add (2)+(3) $5x - z = 6$ (4)

& eliminate y again!

2×(2) $4x - 2y + 2z = 10$
(1) $x + 2y + z = 12$
add $5x + 3z = 22$ (5)

(4)−(5) $-4z = -16$

$z = 4$ sub in (4)

$5x - 4 = 6$
$5x = 10$
$x = 2$ sub in (1)

$2 + 2y + 4 = 12$
$2y = 6$
$y = 3$

check in (3)
$3(2) + 3 - 2(4) \stackrel{?}{=} 1$ ✓
no contradiction

∴ solution is P.O.I $(2, 3, 4)$
situation P

c.

$\cancel{-2x + 4y + 6z = -2}$

$\cancel{4x - 8y - 12z = 4}$

$x - 2y - 3z = 1$

these are multiples of (3) (including "D") ∴ all 3 planes are identical.

Plane O.I. situation (I)

to find vector form:

introduce 2 parameters

let $y = t = 0 + 1t + 0r$

$z = r = 0 + 0t + 1r$

then $x = 1 + 2t + 3r$

∴ Plane OI: $(x,y,z) = (1,0,0) + t(2,1,0) + r(3,0,1)$

d.

$x + y - z = 5$

$\cancel{2x + 2y - 4z = 6}$ } multiple to (3)

$x + y - 2z = 3$

Plane(2) = Plane(3) 

(1) − (2) $\boxed{z = 2}$ sub in (1) and in (3)

$x + y - 2 = 5$ $\qquad$ $x + y - 2(2) = 3$

$\boxed{x + y = 7}$ ⟷ $\cancel{x + y = 7}$ identical.

have only 1 equation but 2 unknowns ∴ need 1 parameter

let $x = t = 0 + t$

$y = 7 - t = 7 - t$

$z = 2 = 2 + 0t$

∴ LOI $(x,y,z) = (0,7,2) + t(1,-1,0)$

situation (J)

e.

$x - 3y - 2z = 9$

$x + 11y + 5z = -5$ } none parallel

$2x + 8y + 3z = 4$

(1) − (2) $\quad -14y - 7z = 14$ (4)

same

2×(2) $\quad 2x + 22y + 10z = -10$

(3) $\quad 2x + 8y + 3z = 4$

subtract $\quad 14y + 7z = -14$ (5) ⟶ reduce $2y + z = -2$

∴ only one equation but 2 unknowns (need 1 parameter)

let $y = t$

$z = -2 - 2t$ } sub in (1)

$x = 3(t) + 2(-2-2t) + 9 = 5 - t$

∴ LOI is $(x,y,z) = (5,0,-2) + t(-1,1,-2)$

situation (N)

7

f.

$$x - y + z = 1$$
$$2x + y - z = 11$$
$$3x + y + 2z = 12$$

① + ② $3x = 12$

$x = 4$

Sub in ① or ② → $4 - y + z = 1$

sub in ③ → $12 + y + 2z = 12$

add $16 + 3z = 13$

$z = -1$

sub in ①

$4 - y + -1 = 1$

$y = 2$

∴ POI $(4, 2, -1)$

situation P

g.

$$x + y + 2z = 6$$
$$x - y - 4z = -2$$
$$3x + 5y + 12z = 27$$

$\vec{n}_1 = (1,1,2)$
$\vec{n}_2 = (1,-1,-4)$
$\vec{n}_3 = (3,5,12)$
} none are parallel

① + ② $2x - 2z = 4$ ④

5 × ② $5x - 5y - 20z = -10$

③ $3x + 5y + 12z = 27$

add $8x - 8z = 17$ ⑤

4 × ④ $8x - 8z = 16$

⑤ $8x - 8z = 17$

subtract $0 \neq -1$

contradiction

∴ situation M

System of Equations with 2 Planes

1. Solve each system and interpret what type of geometrical picture the system represents.

a.

$x + 3y - z - 4 = 0$

~~$2x + 6y - 2z - 8 = 0$~~

let $x = -3t + r + 4$
$y = t$
$z = r$

situation I

Plane OI $(x, y, z) = (4, 0, 0) + t(-3, 1, 0) + r(1, 0, 1)$

b.

(1) $5x - 2y + 2z + 1 = 0$

(2) $5x - 2y + 2z - 3 = 0$

$\vec{n}_1 = (5, -2, 2) = \vec{n}_2$

but equations not multiples

(1) - (2) $4 \neq 0$ contradiction

situation H

c.

(1) $x + y - 3z = 4$

(2) $x + 2y - z = 1$

$\vec{n}_1 = (1, 1, -3)$

$\vec{n}_2 = (1, 2, -1)$

not collinear

$\therefore$ meet in a line

eliminate one variable 1st.

(1) - (2) $-y - 2z = 3$

introduce a parameter

let $z = t$

$y = -3 - 2t$ } sub in (1)

$x = 4 + 3(t) - (-3 - 2t)$

$x = 5t + 7$

$\therefore$ LOI $(x, y, z) = (7, -3, 0) + t(5, -2, 1)$

situation J

d.

$\vec{r} = (1, 1, 1) + p\underset{\vec{m}_1}{(0, 0, 1)} + q\underset{\vec{m}_2}{(0, 1, 0)}$

and $\vec{r} = (0, 0, 0) + s\underset{\vec{m}_3}{(0, 0, 1)} + t\underset{\vec{m}_4}{(1, 0, 0)}$

plane (1) $x = 1$
$y = 1 + q$
$z = 1 + p$

plane (2) $x = t$
$y = 0$
$z = s$

$\vec{n}_1 = \vec{m}_1 \times \vec{m}_2$
$= (-1, 0, 0)$

$0-1, 0-0, 0-0$

$\vec{n}_2 = \vec{m}_3 \times \vec{m}_4$
$= (0, 1, 0)$

$0-0, 1-0, 0-0$

normals not collinear

$\therefore$ plane will meet in a line

$x = 1$
$y = 0$
$z = t$ need one parameter

$\therefore$ LOI $(x, y, z) = (1, 0, \#) + t(0, 0, 1)$

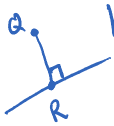
or $(1, 0, 5) + t(0, 0, 3)$

situation J

Distances

Shortest Distance
from point $Q(x_1,y_1)$ to line in R^2
$Ax+By+C=0$

$dist = \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|} = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$



- convert line into parametric form
- general form of pt. R using parameter
- $\vec{m}\cdot\vec{QR} = 0$ solve for parameter
- $dist = |\vec{QR}|$

Shortest Distance
from point Q to line in R^3
through P with direction
vector $\vec{m}$

$dist = \frac{|\vec{PQ}\times\vec{m}|}{|\vec{m}|}$

this can work in R3

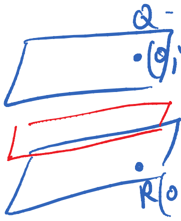
Shortest Distance
from point $Q(x_1,y_1,z_1)$
to plane in R^3 $Ax+By+Cz+D=0$

$dist = \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|} = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$

- convert plane into parametric form
- general form of pt. R
- $\vec{m_1}\cdot\vec{QR} = 0$, $\vec{m_2}\cdot\vec{QR} = 0$ } solve this system for 2 parameters
- $dist = |\vec{QR}|$

1. a. Determine the distance between the two planes π_1: $2x - y + 2z + 4 = 0$ and π_2: $2x - y + 2z + 16 = 0$.
   b. Determine the equation of the plane that is equidistant from π_1 and π_2.

ⓐ



Q - find a random pt. Q on π_1

•(0,4,0) let $x=0$
$z=0$
then $y=4$

R(0,16,0)

$dist. = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$ ← from the different plane than where Q is

$= \frac{|2(0)+-1(4)+2(0)+16|}{\sqrt{2^2+1^2+2^2}}$

$= 4$

ⓑ find midvalue of the D's but the normals must be identical (not multiples)

$\frac{4+16}{2} = \frac{20}{2} = 10$

∴ equation of middle plane.

$2x - y + 2z + 10 = 0$

OR normals same

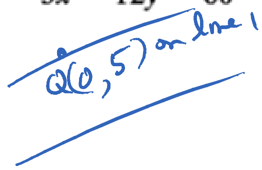
$2x - y + 2z + D = 0$

midpt. of QR $= \left(\frac{0+0}{2}, \frac{4+16}{2}, \frac{0+0}{2}\right)$

$= (0, 10, 0)$
sub in

10

2. Calculate the distance between the two parallel lines $5x - 12y + 60 = 0$ and $5x - 12y - 60 = 0$.



$$\text{dist} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$$ ← use C from line where pt. Q is NOT on.

$$= \frac{|5(0) + -12(5) + -60|}{\sqrt{5^2+12^2}} = \frac{120}{13}$$

3. Determine the distance between point $(-2, 1, 0)$ and line $\vec{r} = (0,1,0) + t(1,1,2), t \in R$

P Q

$$\text{dist} = \frac{|\vec{PQ} \times \vec{m}|}{|\vec{m}|}$$

$\vec{PQ} = (0+2, 1-1, 0-0)$
$= (2,0,0)$

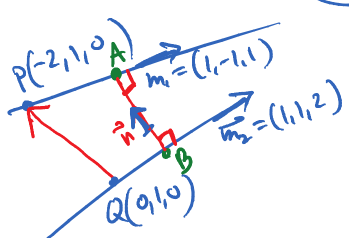
$\vec{PQ}$: 2 0 0 2 0 0
$\vec{m}$: 1 1 2 1 1 2

$(0,-4,2)$

$$= \frac{\sqrt{0^2+4^2+2^2}}{\sqrt{1^2+1^2+2^2}} = \frac{\sqrt{20}}{\sqrt{6}} = \frac{\sqrt{10}}{\sqrt{3}} = \frac{\sqrt{30}}{3}$$

4. Determine the distance between L_1: $\vec{r} = (-2, 1, 0) + s(1, -1, 1), s \in \mathbf{R}$, and L_2: $\vec{r} = (0, 1, 0) + t(1, 1, 2), t \in \mathbf{R}$.

not parallel.
if they meet dist = 0
if they don't meet dist ≠ 0 skew lines.



another way (longer)
- general pt. $A(-2+s, 1-s, s)$
- general pt. $B(t, 1+t, 2t)$
- find $\vec{AB}$
- do $\vec{AB} \cdot \vec{m_1} = 0$, $\vec{AB} \cdot \vec{m_2} = 0$ } solve for t and s

then find $|\vec{AB}|$

$\text{dist} = |\text{proj } \vec{PQ} \text{ on } \vec{n}|$ where $\vec{n} = \vec{m_1} \times \vec{m_2}$

$$= \frac{|(2,0,0) \cdot (-3,-1,2)|}{|(-3,-1,2)|}$$

$$= \frac{|-6 + 0 + 0|}{\sqrt{3^2+1^2+2^2}}$$

$$= \frac{6}{\sqrt{14}} = \frac{3\sqrt{14}}{7}$$

1 −1 1 1 −1 1
1 1 2 1 1 2

$(-2-1, 1-2, 1+1)$

$\vec{n} = (-3,-1,2)$

$\vec{PQ} = (0+2, 1-1, 0-0) = (2,0,0)$