Exponential, Logarithmic &Trigonometric Derivatives

Tentative TEST date_______________

Big idea/Learning Goals

The world's population experiences exponential growth—the rate of growth becomes more rapid as the size of the population increases. Can this be explained in the language of calculus? Well, the rate of growth of the population is described by an exponential function, and the derivative of the population with respect to time is a constant multiple of the population at any time *t*. There are also many situations that can be modelled by trigonometric functions, whose derivative also provides a model for instantaneous rate of change at any time *t*. By combining the techniques in this unit with the derivative rules seen earlier, we can find the derivative of an exponential, logarithmic or trigonometric function that is combined with other functions.

<u>Corrections for the textbook answers:</u>

Success Criteria

- ☐ I <u>understand the new topics</u> for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	3-6	Natural Exponential and Logarithmic Derivatives 5.1 & Appendix of textbook **p 571-575**	
	7-9	Exponential and Logarithmic Derivatives of any Base 5.2 & 5.3 & Appendix of textbook **p 576-578**	
	10-12	Trigonometric Derivatives 5.4 & 5.5	
	13-15	Related Rates – 2 days Appendix of textbook **p 565-570**	
		Review of All Derivatives - Handouts online	

Reflect – previous TEST mark ___________, Overall mark now___________.

Derivative Formulas

In the following, *u* and *v* are functions of x, and *n*, *e*, *a*, and *k* are constants.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	The Definition of the Derivative
$\frac{d}{dx}(k) = 0$	The derivative of a constant is zero.
$\frac{d}{dx}\left(k\left(u(x)\right)\right) = k\frac{du}{dx}$	The derivative of a constant times a function.
$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$	The Power Rule (Variable raised to a constant).
$\frac{d}{dx}(uv) = uv' + vu'$	The Product Rule.
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$	The Quotient Rule.
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	The Chain Rule.
$\frac{d}{dx}\left(f(g(x)) = f'(g(x))g'(x)\right)$	Another Form of the Chain Rule.
$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$	The Derivative of the Sine.
$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$	The Derivative of the Cosine.
$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$	The Derivative of the Tangent.
$\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$	The Derivative of e raised to a variable
$\frac{d}{dx}\left(a^u\right) = a^u \ln a \frac{du}{dx}$	The Derivative of a constant raised to a variable.
$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$	The Derivative of the Natural Log.
$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}\frac{du}{dx}$	The Derivative of the log to base a

others:

$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx} \qquad \frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx} \qquad \frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\mathrm{Tan}^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\left(\mathrm{Cos}^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\left(\mathrm{Sin}^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\mathrm{Cot}^{-1} u\right) = \frac{-1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\left(\mathrm{Sec}^{-1} u\right) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}\left(\mathrm{Csc}^{-1} u\right) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

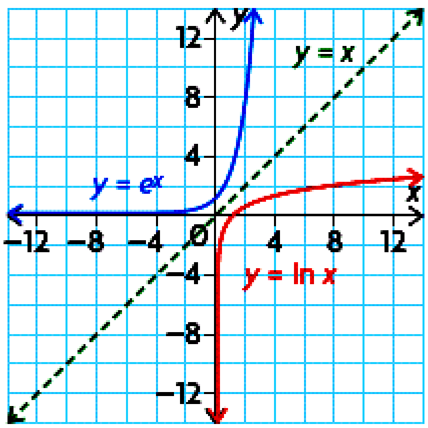
Natural Exponential and Logarithmic Derivatives

1. Show the applet of an exponential function and its derivative. For what value of the base does the derivative and the function become the same?

   2.71828 ...

2. This special value is given the name Euler's Constant, it is defined to be a number such that $\boxed{\frac{d}{dx}(e^x) = e^x}$.

3. Recall that exponentials and logarithms are inverses.
Here are some properties of the natural exponential and natural logarithm.



$y = e^x$	$y = \ln x$
• The domain is $\{x \in \mathbf{R}\}$.	• The domain is $\{x \in \mathbf{R} \mid x > 0\}$.
• The range is $\{y \in \mathbf{R} \mid y > 0\}$.	• The range is $\{y \in \mathbf{R}\}$.
• The function passes through (0, 1).	• The function passes through (1, 0).
• $e^{\ln x} = x, x > 0$.	• $\ln e^x = x, x \in \mathbf{R}$.
• The line $y = 0$ is the horizontal asymptote.	• The line $x = 0$ is the vertical asymptote.

Laws

4. Recall also the following exponent and log rules:

Exponent Rules

$a^n a^m = a^{n+m}$ $\qquad \frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$

$(a^n)^m = a^{nm}$ $\qquad a^0 = 1, \quad a \neq 0$

$(ab)^n = a^n b^n$ $\qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^{-n} = \frac{1}{a^n}$ $\qquad \frac{1}{a^{-n}} = a^n$

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ $\qquad a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}$ $\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^n} = a$, if n is odd

$\sqrt[n]{a^n} = |a|$, if n is even

If $a \neq 1, a^x = a^y \Leftrightarrow x = y$

If $x \neq 0, a^x = b^x \Leftrightarrow a = b$

Log Rules

The domain of $\log_b x$ is $x > 0$

$\log_b x = y \iff x = b^y$

$b^{\log_b x} = x$

$\log_b b^x = x$

$\log_b 1 = 0$

$\log_b b = 1$

$\log_b xy = \log_b x + \log_b y$

$\log_b \frac{x}{y} = \log_b x - \log_b y$

$\log_b x^y = y \log_b x$

$\log_b x = \frac{\log_a x}{\log_a b}$

Special Logarithms

$\ln x = \log_e x$ natural log

$\log x = \log_{10} x$ common log

where $e = 2.718281828\ldots$

If $b \neq 1, \log_b x = \log_b y \Leftrightarrow x = y$

If $x \neq 1, \log_a x = \log_b x \Leftrightarrow a = b$

Review how to work with the laws:

6. Solve for x, to three decimal places.

(a) $e^x = 16$ **(b)** $\ln (x-1)^2 = 4$

(c) $e^{2x} = 125$ **(d)** $\ln x = 2$

8. Express as a single logarithm.

(a) $2 \ln x + \ln 2x$ **(b)** $3 \ln 4x^2 - 2 \ln x$

(c) $2 \ln 3x + 3 \ln (2x-1)$ **(d)** $2 \ln x + 3 \ln y$

(e) $\left(\frac{1}{2}\right) \ln x - \left(\frac{1}{3}\right) \ln y$ **(f)** $-5 \ln 2x + 6 \ln x$

6ⓐ $e^x = 16$

$\Updownarrow$

$\log_e 16 = x$

$\ln 16 = x$

$2.773 \doteq x$

ⓑ $\ln(x-1)^2 = 4$

$\Updownarrow$

$\sqrt{e^4} = \sqrt{(x-1)^2}$

$\pm e^2 = x-1$

$1 \pm e^2 = x$

8.389 or $-6.389 \doteq x$

ⓒ $e^{2x} = 125$

$\Updownarrow$

$\ln 125 = 2x$

$\frac{\ln 125}{2} = x$

$2.414 \doteq x$

ⓓ $\ln x = 2$

$\Updownarrow$

$e^2 = x$

$7.389 \doteq x$

8ⓐ $\ln x^2 + \ln 2x$

$= \ln(2x^3)$

ⓑ $\ln(4x^2)^3 - \ln x^2$

$= \ln\left(\frac{64x^6}{x^2}\right)$

$= \ln 64x^4$

ⓒ $\ln(3x)^2 + \ln(2x-1)^3$

$= \ln[9x^2(2x-1)^3]$

ⓓ $\ln x^2 + \ln y^3$

$= \ln(x^2y^3)$

ⓔ $\ln x^{1/2} - \ln y^{1/3}$

$= \ln\left(\frac{\sqrt{x}}{\sqrt[3]{y}}\right)$

ⓕ $\ln(2x)^{-5} + \ln x^6$

$= \ln\left(\frac{x^6}{32x^5}\right)$

$= \ln\left(\frac{x}{32}\right)$

5. So derivative of $f(x) = e^x$ is the same as the function. (Remember the number e was chosen so that this occurs). Use this fact and the fact that logs are inverses of exponentials to figure out what the derivative of natural logarithm $f(x) = \ln x$ is. (The answer is on the formula page 2 but show a proof here)

$y = e^x$ inverse is $x = e^y \underset{\text{switch form}}{\Leftrightarrow} \log_e x = y$ or $\ln x = y$

Now take derivative implicity of $x = e^y$ which is same as $y = \ln x$

$1 = e^y \frac{dy}{dx}$

isolate y' $\frac{1}{e^y} = \frac{dy}{dx}$

$\frac{1}{x} = \frac{dy}{dx}$

$\therefore$ $\boxed{f(x) = \ln x \quad f'(x) = \frac{1}{x}}$

Now practice finding the derivative

7. Knowledge and Understanding: Find $\frac{dy}{dx}$ for each function.

(a) $y = \sqrt{\ln x}$ **(b)** $y = \frac{\ln x}{x^3}$

(c) $y = \ln e^{3x}$ **(d)** $y = \ln 6x + \ln 2x$

(e) $y = x^4 \ln x$ **(f)** $y = \frac{\ln 6x}{\ln 2x}$

(g) $y = \ln 10x^8$ **(h)** $y = \ln x + \ln x^2 + \ln x^3 + \ln x^4$

(i) $y = \ln (8x^2 + 2)^4$ **(j)** $y = \sqrt{e} \ln 3$

(k) $y = \ln 3x^7$ **(l)** $y = (e^{2x})(\ln x^3)$

13. Find $\frac{dy}{dx}$ for each function.

(a) $y = x^3 \ln 2x$ **(b)** $y = (\ln 6x)(\ln 2x)$

(c) $y = \frac{\ln x}{2x^3 - 4}$ **(d)** $y = \ln \left(\frac{2x^2 - 3}{2x^3}\right)$

(e) $y = (x + \ln x)^2$ **(f)** $y = \frac{\ln x}{(x + 3)^3}$

(g) $y = e^{x \ln x}$ **(h)** $y = 3\left(\ln \sqrt{2x + 3}\right)^2$

7 (a) $y' = \frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right)$

$= \frac{1}{2x\sqrt{\ln(x)}}$

(b) $y' = \frac{x^3 \frac{1}{x} - 3x^2 \ln x}{x^6}$

$y' = \frac{x^2\left[1 - 3\ln x\right]}{x^6}$

$y' = \frac{1 - 3\ln x}{x^4}$

(c) $y = 3x$

$y' = 3$

(d) $y = \ln(12x^2)$

$y' = \frac{1}{12x^2}(24x)$

(e) $y' = x^4\frac{1}{x} + 4x^3 \ln x$ $= \frac{2}{x}$

$y' = x^3\left[1 + 4\ln x\right]$

(f) $y' = \frac{\ln(2x)\left(\frac{1}{\cancel{6}x}\right)(\cancel{6}) - (\ln 6x)\left(\frac{1}{\cancel{2}x}\right)(\cancel{2})}{[\ln(2x)]^2}$

$y' = \frac{1}{x}\left[\frac{\ln 2x - \ln 6x}{\ln^2(2x)}\right]$

$y' = \frac{\ln\left(\frac{1}{3}\right)}{x\ln^2(2x)}$

(g) $y' = \frac{1}{10x^8}(80x^7)$

$= \frac{8}{x}$

(h) $y = \ln(x^{10})$

$y' = \frac{1}{x^{10}}(10x^9)$

$y' = \frac{10}{x}$

(i) $y' = \frac{1}{(8x^2+2)^4}\left[4(8x^2+2)^3(16x)\right]$

$= \frac{64x}{8x^2+2}$

(j) $y' = 0$ since it's a constant!

(k) $y' = \frac{1}{3x^7}(21x^6)$

$= \frac{7}{x}$

(l) $y' = e^{2x}(2)\ln x^3 + e^{2x}\frac{1}{x^3}(3x^2)$

$= e^{2x}\left[2\ln x^3 + \frac{1}{x}\right]$

13 (a) $y' = 3x^2\ln 2x + x^3\frac{1}{2x}(2)$

$= x^2\left[3\ln 2x + 1\right]$

(b) $y' = \frac{1}{6x}(6)\ln 2x + \frac{1}{2x}(2)\ln 6x$

$= \frac{\ln 2x + \ln 6x}{x}$

$= \frac{\ln(12x^2)}{x}$

(c) $y' = \frac{(2x^3-4)\frac{1}{x} - (\ln x)6x^2}{(2x^3-4)^2}$

(d) $y' = \frac{2x^3}{2x^2-3}\left[\frac{2x^3(4x) - (2x^2-3)(6x^2)}{(2x^3)^2}\right]$

(e) $y' = 2(x + \ln x)\left[1 + \frac{1}{x}\right]$

(f) $y = \frac{(x+3)^3\left(\frac{1}{x}\right) - (\ln x)(3)(x+3)^2(1)}{(x+3)^6}$

(g) $y' = e^{x\ln x}\left[1\ln x + x\left(\frac{1}{x}\right)\right]$

$= e^{x\ln x}\left[\ln x + 1\right]$

(h) $y' = 6\left(\ln\sqrt{2x+3}\right)\frac{1}{\sqrt{2x+3}}\left(\frac{1}{\cancel{2}}\right)(2x+3)^{-1/2}(\cancel{2})$

$y' = \frac{6\ln\sqrt{2x+3}}{2x+3}$

14. Find the equation of the tangent line to the curve $y = \ln 2x$ at the point where $x = \frac{e}{2}$. Graph $y = \ln 2x$ and this tangent at that point.

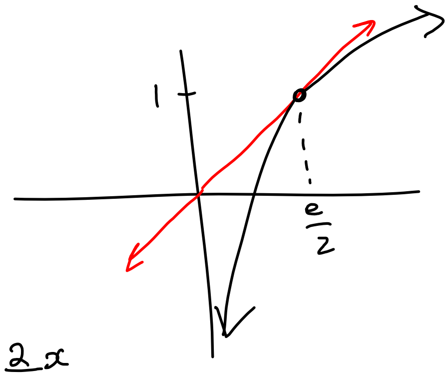
$y\left(\frac{e}{2}\right) = \ln\left(2\left(\frac{e}{2}\right)\right)$
$= \ln e$
$= 1$

pt. $\left(\frac{e}{2}, 1\right)$

$y' = \frac{1}{2x}(2) = \frac{1}{x}$

$y'\left(\frac{e}{2}\right) = \frac{1}{e/2} = \frac{2}{e} = m$

$\therefore y = mx + b$
$1 = \frac{2}{e}\left(\frac{e}{2}\right) + b$
$0 = b$
$\therefore$ tangent line $y = \frac{2}{e}x$



24. Let $f(x) = \ln(x^2e^x)$.

(a) Determine $f'(x)$ by first using the laws of logarithms to "expand" the expression.

(b) Determine $f'(x)$ without first simplifying.

(c) Compare the results. Which method do you prefer? Why?

ⓐ $f = \ln x^2 + \ln e^x$
$f = 2\ln x + x$
$f' = \frac{2}{x} + 1$

ⓑ $f' = \frac{1}{x^2e^x}\left[2xe^x + x^2e^x\right]$
$= \frac{xe^x[2+x]}{x^2e^x}$
$= \frac{2+x}{x}$

ⓒ if you do LCD the answers are the same ⓐ is faster.

26. Use implicit differentiation to find $\frac{dy}{dx}$ for each function.

(a) $\ln(xy) = 2 - x - y$ **(b)** $\ln y + 2x = 1$
(c) $\ln(x + y) = 1$ **(d)** $\ln x + \ln y = x$

ⓐ $\frac{1}{xy}\left[1y + xy'\right] = 0 - 1 - y'$
$\frac{1}{x} + \frac{y'}{y} = -1 - y'$
$y'\left[\frac{1}{y} + 1\right] = -1 - \frac{1}{x}$
$y' = \frac{-1 - \frac{1}{x}}{\frac{1}{y} + 1}$

ⓑ $\frac{1}{y}y' + 2 = 0$
$y' = -2y$

ⓒ $\frac{1}{x+y}\left[1 + y'\right] = 0$
$\frac{1}{x+y} + \frac{y'}{x+y} = 0$
$y' = \left(-\frac{1}{x+y}\right)(x+y)$
$y' = -1$

ⓓ $\frac{1}{x} + \frac{1}{y}y' = 1$
$y' = \left[1 - \frac{1}{x}\right]y$

Exponential and Logarithmic Derivatives of any Base

1. Can we somehow use the chain rule in combination with our knowledge of how to differentiate $f(x)=e^x$ to help us to differentiate $f(x)=a^x$?

rewrite $f(x)=a^x$

$= e^{\ln(a^x)}$ since $e^{\ln \star} = \star$

$= e^{x\ln a}$

$= e^{x\ln a}$ chain rule

now $f'(x) = e^{x\ln a}(\ln a)$ } put back

$= e^{\ln a^x}(\ln a)$

$= a^x(\ln a)$

$\therefore \boxed{\begin{aligned} f(x) &= a^x \\ f'(x) &= a^x \ln a \end{aligned}}$

2. And the general derivative of any base of a logarithm proof:

let $y = \log_a x$

$\Updownarrow$

$a^y = x$

$\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$

$\ln a \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x\ln a}$ sub in

$\therefore \boxed{\begin{aligned} f(x) &= \log_a x \\ f'(x) &= \frac{1}{x\ln a} \end{aligned}}$

1. Differentiate.

(a) $y = 5^x$ **(b)** $y = (0.47)^x$

(d) $y = 5(2)^x$ **(e)** $y = 4(e)^x$

(a) $y' = 5^x \ln 5$ (b) $y' = 0.47^x \ln(0.47)$

(d) $y' = 5(2)^x \ln 2$ (e) $y' = 4e^x$

2. Find $\frac{dy}{dx}$ for each function.

(a) $y = \log_2 x$ **(b)** $y = \log_3 x$

(d) $y = -3\log_7 x$ **(e)** $y = -(\log x)$

(a) $y' = \frac{1}{x\ln 2}$ (b) $y' = \frac{1}{x\ln 3}$

(d) $y' = \frac{-3}{x\ln 7}$ (e) $y' = \frac{-1}{x\ln 10}$

5. Differentiate.

(a) $y = x^5 \times (5)^x$
(b) $y = \log_7 (x^2 + x + 1)$
(c) $y = 3^x \log_3 x$
(d) $y = \frac{2^{4x}}{x^3}$
(e) $y = 2x \log_4 x$
(f) $y = 3.2(10)^{0.2x}$
(g) $y = 2^x \ln x$
(h) $y = x(3x)^{x^2}$
(i) $y = 3^{\ln x}$
(j) $y = \frac{\log_5 3x^4}{5^{2x}}$

ⓒ $y' = 3^x \ln 3 \log_3 x + 3^x \frac{1}{x \ln 3}$

ⓓ $y' = \frac{x^3 2^{4x} \ln 2 (4) - 2^{4x}(3x^2)}{x^6}$

$y' = \frac{x^2 2^{4x} [4x \ln 2 - 3]}{x^6}$

$y' = \frac{2^{4x}[4x \ln 2 - 3]}{x^4}$

ⓐ $y' = 5x^4 5^x + x^5 5^x \ln 5$
$= 5^x x^4 [5 + x \ln 5]$

ⓑ $y' = \frac{1}{x^2+x+1}(2x+1)$

ⓔ $y' = 2\log_4 x + 2x \frac{1}{x \ln 4}$

$y' = 2\log_4 x + \frac{2}{\ln 4}$

ⓕ $y' = 3.2(10)^{0.2x} \ln 10 (0.2)$

$y' = 0.64(\ln 10) 10^{0.2x}$

ⓖ $y' = 2^x \ln 2 \ln x + 2^x \frac{1}{x}$

ⓗ don't know how to derive $(x's)^{x's}$!!

not on test

$y = x(3x)^{x^2}$ } take ln of both sides

$\ln y = \ln(x(3x)^{x^2})$

$\ln y = \ln x + x^2 \ln(3x)$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\ln x + x^2 \ln(3x)]$

$\frac{1}{y} y' = \frac{1}{x} + 2x \ln(3x) + x^2 \frac{1}{3x}(3)$

$y' = \left[\frac{1}{x} + 2x\ln(3x) + x\right] y$

ⓘ $y' = 3^{\ln x}(\ln 3)\frac{1}{x}$

ⓙ $y = 5^{-2x} \log_5 3x^4$

$y' = 5^{-2x}(\ln 5)(-2)\log_5(3x^4) + 5^{-2x} \frac{1}{3x^4 \ln 5}(12x^3)$

10. For each function, find $f'(x)$. State the domains of $f(x)$ and $f'(x)$.

(a) $f(x) = \log(5 - 2x)$
(b) $f(x) = 50(1.02)^{4x}$
(c) $f(x) = \log_3 (x^2 - 4)$
(d) $f(x) = \ln(3^x)$

ⓐ $D_f: 5 - 2x > 0$
$5 > 2x$
$\frac{5}{2} > x$

$f' = \frac{1}{(5-2x)\ln 10}(-2)$ $D_{f'}: x \neq \frac{5}{2}$

actually can't be where f doesn't exist either

ⓑ $D_f: x \in \mathbb{R}$ $f' = 50(1.02)^{4x}(\ln 1.02)(4)$ $D_{f'}: x \in \mathbb{R}$

ⓒ $D_f: x^2 - 4 > 0$
$(x+2)(x-2) > 0$

$-2 \quad 2$
$+ \mid - \mid +$

$\therefore x < -2, x > 2$

$f' = \frac{1}{(x^2-4)\ln 3}(2x)$ $D_{f'}: x \neq \pm 2$

actually can't be where f doesn't exist either

ⓓ $D_f: x \in \mathbb{R}$ since 3^x always pos.

$f' = \frac{1}{3^x} 3^x \ln 3$
$= \ln 3$

$D_{f'}: x \in \mathbb{R}$
constant line

3. If $y = te^t - e^t - 2t^2$ represents the movement (distance to the origin) of a particle along a straight line.
   a. When does velocity equal to zero?
   b. Is there a maximum or minimum distance to the origin?
   c. Find the acceleration function.
   d. Is there a maximum or minimum velocity?

ⓐ velocity $= y' = 1e^t + te^t - e^t - 4t$

$y' = te^t - 4t$

$0 = t(e^t - 4)$

$t = 0$ or $e^t - 4 = 0$

$e^t = 4$

$\Updownarrow$

$\ln 4 = t$

$\therefore$ at time $\emptyset$ or 1.39 velocity is zero

ⓑ classify crit. pt.

	$-\infty$	0	$\ln 4$	∞
t	$-$	$+$	$+$	
$e^t - 4$	$-$	$-$	$+$	
vel $= y'$	$+$	$-$	$+$	
dist $= y$	↗	↘	↗	

$\therefore$ ~~Max~~, time starts at $x = 0$, $y = -1$

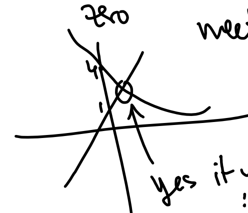
Min at $x \doteq \ln 4$, $y \doteq -2.3$

ⓒ accel $= y'' = 1e^t + te^t - 4$

ⓓ need $y'' = 0$ or $v' = 0$

try factoring $y'' = \underbrace{e^t}_{\text{never zero}}\underbrace{[1 + t - 4e^{-t}]}$

does line $1 + t = 4e^{-t}$ meet exponential?



yes it will $\therefore$ there is MAX/MIN velocity.

4. When a particular medication is swallowed by a patient, the concentration of the active ingredient, in parts per million, in the bloodstream is given by the equation $C(t) = 150t(0.5)^t$, after t hours.
   a. What is the highest concentration of the medicine?
   b. How fast is the concentration decreasing after 2 hours?

ⓐ $C'(t) = 150(0.5)^t + 150t(0.5)^t \ln(0.5)$

$= \underbrace{150(0.5)^t}_{\text{never zero}}\underbrace{[1 + t\ln 0.5]}$

$-1 = t\ln 0.5$

$\frac{-1}{\ln 0.5} = t$

$1.44 \doteq t$

$\therefore$ at $t = 1.44$ there is a crit. pt.
Can be MAX/MIN/saddle

→ show it's MAX → 3 ways to do this!
- using f
- using f'
- using f''

	$-\infty$	1.44	∞
150	$+$	$+$	
0.5^t	$+$	$+$	
$1 + t\ln 0.5$	$+$	$-$	
f'	$+$	$-$	
f	↗	↘	

$\therefore$ MAX at $t \doteq 1.44$
and concentration is $C(1.44) \doteq 79.6$ $\frac{\text{parts}}{\text{million}}$

ⓑ $C'(2) \doteq -14.5$ 'parts per million' per hour

Trigonometric Derivatives

Here are some formulas you should have seen before:

Cosine Laws

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Pythagorean

$$a^2 + b^2 = c^2$$

Circles

Arc length: $s = r\theta$

Area: $A = \frac{1}{2}r^2\theta$

$$x^2 + y^2 = r^2$$

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Double Angle

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Sum/Difference

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Proving the derivatives for sine and cosine involves the following limit properties. (Check out derivatives unit in AP course online if you're interested)

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0}$$

These are not necessary to know for this course, so I'll just show you the visual proof using technology.

Now using sine and cosine derivatives prove the derivative of tangent shown on the formula page.

$y = \tan x = \frac{\sin x}{\cos x}$

$y' = \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$

$y' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$y' = \frac{1}{\cos^2 x} = \sec^2 x$

from formula pag

$\frac{d}{dx}(\sin x) = \cos x$

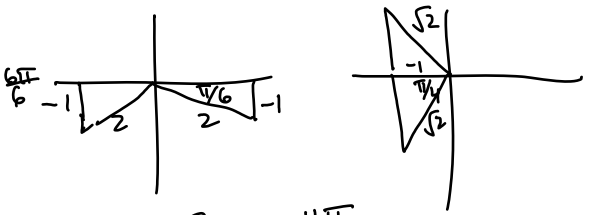
$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

Review how to solve trig equations

a. $(2\sin\theta+1)(2\cos\theta+\sqrt{2})=0$ b. $(\sqrt{3}\tan\theta-1)(\sec\theta+1)=0$ c. $3\sin x = 2\cos^2 x$

(a) $\sin\theta = -\frac{1}{2}$ $\cos\theta = -\frac{\sqrt{2}}{2} = \frac{-1}{\sqrt{2}}$

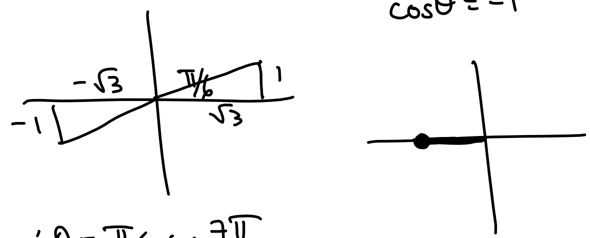


$\therefore \theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ $\therefore \theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$

add/subt period 2π to get more solutions

(b) $\tan\theta = \frac{1}{\sqrt{3}} = \frac{-1}{-\sqrt{3}}$ $\sec\theta = -1$

or $\cos\theta = -1$



$\therefore \theta = \pi/6$ or $\frac{7\pi}{6}$ $\theta = \pi$

+ more periodic solutions

(c)

$0 = 2\cos^2 x - 3\sin x$

$0 = 2(1-\sin^2 x) - 3\sin x$

$0 = 2 - 2\sin^2 x - 3\sin x$

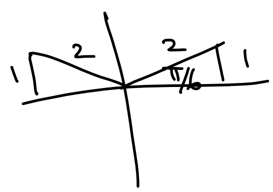
$0 = -(2\sin^2 x + 3\sin x - 2)$

$\binom{2}{1}$ $\binom{-1}{2}$

$0 = -(2\sin x - 1)(\sin x + 2)$

$\sin x = \frac{1}{2}$ or $\sin x = -2$

Never



$\therefore x = \pi/6,\ 5\pi/6$

Differentiate

1. $y=\cos 3x$
2. $y = 2\sin \pi x$
3. $y=\cos^3(x^2+\pi x)$
4. $y = 2\sin^3 x - 4\cos^2 x$

(1.) $y' = -\sin(3x)(3)$

$= -3\sin(3x)$

(2.) $y' = 2\cos(\pi x)\pi$

$y = 2\pi\cos(\pi x)$

(3.) $y' = 3\cos^2(x^2+\pi x)[-\sin(x^2+\pi x)][2x+\pi]$

(4.) $y' = 6\sin^2 x(\cos x) - 8\cos x(-\sin x)$

$y' = \cos x \sin x\,[6\sin x + 8]$

Differentiate

5. $y = x^3 \cos x$
6. $y = x^{-1} \tan(\pi - x)$
7. $y = 0.5 \tan 2\pi x$
8. $y = \tan x \sin 2x$
9. $y = \left(\sin\left(5x + e^x\right)\right)^4$
10. $y = \sin\left(5x^3 - 7.2x^2 + 3.8\right)$
11. $y = e^{\sin\left(x^2\right)}$
12. $y = \ln\left(2^x + \sin\left(5x\right)\right)$
13. $y = e^{x^2} \cos\left(8x - 4.2\right)$
14. $y = \sin\left(\sqrt{x}\right)$
15. $y = \cos\left(\ln\left(5x + 2\right)\right)$
16. $y = \sin\left(x^2\right)\cos\left(5^x\right)$

5. $y' = 3x^2 \cos x + x^3(-\sin x)$
$y' = x^2\left[3\cos x - x\sin x\right]$

6. $y' = -x^{-2}\tan(\pi - x) + x^{-1}\sec^2(\pi - x)(-1)$
$y' = -x^{-2}\left[\tan(\pi - x) + x\sec^2(\pi - x)\right]$

7. $y' = 0.5\sec^2(2\pi x)(2\pi)$
$y' = \pi\sec^2(2\pi x)$

8. $y' = \sec^2 x \sin 2x + \tan x \cos 2x\,(2)$

9. $y' = 4\left[\sin(5x + e^x)\right]^3 \cos(5x + e^x)\left[5 + e^x\right]$

10. $y' = \cos(5x^3 - 7.2x^2 + 3.8)\left[15x^2 - 14.4x\right]$

11. $y' = e^{\sin(x^2)}\left[\cos(x^2)\right](2x)$

12. $y' = \dfrac{1}{(2^x + \sin 5x)}\left[2^x \ln 2 + \cos(5x)(5)\right]$

13. $y' = e^{x^2}(2x)\cos(8x - 4.2) + e^{x^2}(-\sin(8x - 4.2))(8)$

14. $y' = \cos\sqrt{x}\left(\frac{1}{2}\right)x^{-1/2}$

15. $y' = -\sin\left(\ln(5x + 2)\right)\left(\dfrac{1}{5x + 2}\right)(5)$

16. $y' = \cos(x^2)(2x)\cos(5^x) + \sin(x^2)(-\sin(5^x))\,5^x \ln 5$

Related Rates

1. Review the importance of Leibniz notation and find the following derivatives for $a^2+3a=b^3-ab^2$

a. $\frac{d}{da}\left[a^2+3a=b^3-ab^2\right]$ $\rightsquigarrow$ $2a+3=0-b^2$

b. $\frac{d}{db}\left[a^2+3a=b^3-ab^2\right]$ $\rightsquigarrow$ $0+0=3b^2-2ab$

c. $\frac{d}{dx}\left[a^2+3a=b^3-ab^2\right]$ if $a(x)$ and $b(x)$ $\rightsquigarrow$ $2aa'+3a'=3b^2b'-\underbrace{a'b^2-a(2bb')}_{\text{product}}$

or $2a\frac{da}{dx}+3\frac{da}{dx}=3b^2\frac{db}{dx}-b^2\frac{da}{dx}-2ab\frac{db}{dx}$

2. Air leaks out of a balloon at a rate of ***3 cubic feet per minute***. How fast is the surface area shrinking when the radius is 10 feet? (Note: ***SA = 4πr²*** & ***V = 4/3 πr³***)

$\frac{dV}{dt}=-3\frac{ft^3}{min}$

$\frac{dS}{dt}=?$ at $r=10$

as air leaks from balloon V, r, S all change with time
∴ they all have time inside them
$V(t), r(t), S(t)$

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=\frac{4}{3}\pi(3)r^2\frac{dr}{dt}$

$-3=4\pi(10)^2\frac{dr}{dt}$

$-\frac{3}{400\pi}=\frac{dr}{dt}$

now use $S=4\pi r^2$

$\frac{dS}{dt}=8\pi r\frac{dr}{dt}$

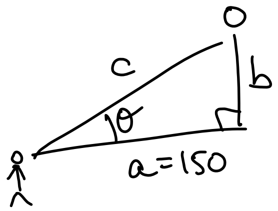
$\frac{dS}{dt}=8\pi(10)\left(\frac{-3}{400\pi}\right)$

$=-\frac{3}{5}\ ft^2/min$

surface area is shrinking at the rate $3/5\ ft^2/min$

Steps
(1.) identify what's given and what you need to find (pay attention to units "per" is a rate of change)
(2.) Find an equation to differentiate
(3.) ask yourself what variables change with time $\frac{d?}{dt}$ and which are actually constant! $\rightarrow \phi$
(4.) sub in all givens AFTER you do derivative
(5.) use another equation if still too many unknowns

3. A small balloon is released at a point 150 feet away from an observer, who is on level ground. If the balloon goes straight up at a rate of ***8 feet per second***,
   a. How fast is the distance from the observer to the balloon increasing when the balloon is 50 feet high?
   b. How fast is the angle of elevation increasing?



a remains constant at 150

b changes
c changes
θ changes
} all have time inside the function

$$\frac{db}{dt} = 8\ \text{ft/sec}$$

ⓐ $\frac{dc}{dt} = ?$ at $b=50$ use $\boxed{a^2+b^2=c^2}$

$c=\sqrt{50^2+150^2}$

$c=\sqrt{25000}$

$$0 + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$\cancel{2}(50)(8) = \cancel{2}\sqrt{25000}\,\frac{dc}{dt}$$

$$\frac{400}{\sqrt{25000}} = \frac{dc}{dt}$$

∴ distance increases at ~ 2.53 ft/sec

ⓑ $\frac{d\theta}{dt} = ?$ at $b=50$ use $\boxed{\tan\theta = \frac{b}{150}}$

$\theta = \tan^{-1}\left(\frac{50}{150}\right)$

$\theta \doteq 18^\circ$

or

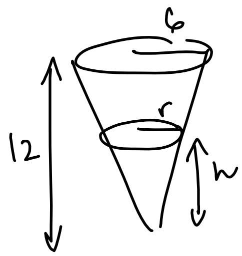
$\theta = 0.3217$ radians

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{150}\frac{db}{dt}$$

$$\sec^2 18^\circ\,\frac{d\theta}{dt} = \left(\frac{1}{150}\right)(8)$$

$$\frac{d\theta}{dt} = \frac{8}{150}\cos^2 18^\circ \doteq 0.048\ \text{degrees/sec}$$

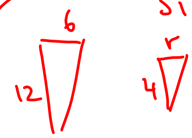
4. Water is pouring into a conical cistern at the rate of ***8 cubic feet per minute***. If the height of the cistern is 12 feet and the radius of its circular opening is 6 feet, how fast is the water level rising when the water is 4 feet deep?



$\frac{dV}{dt} = 8 \frac{ft^3}{min}$

$\frac{dh}{dt} = ?$ at $h = 4$, $r = 2$

Similar Δ's



$\frac{r}{6} = \frac{4}{12}$

$r = 6\left(\frac{1}{3}\right)$

$r = 2$

as water is rising

h, r, V all change with time

use $\boxed{V = \frac{1}{3}\pi r^2 h}$

have to do product rule

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}\right)h + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$8 = \frac{1}{3}\pi(2)(2)\frac{dr}{dt}(4) + \frac{1}{3}\pi(2)^2\frac{dh}{dt}$$

still too many unknowns

use $\boxed{\frac{r}{6} = \frac{h}{12}}$

$$\frac{1}{6}\frac{dr}{dt} = \frac{1}{12}\frac{dh}{dt}$$

$$\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$8 = \frac{16}{3}\pi\left(\frac{1}{2}\frac{dh}{dt}\right) + \frac{4}{3}\pi\frac{dh}{dt}$$

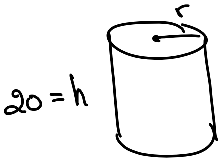
$$8 = \left(\frac{8}{3}\pi + \frac{4}{3}\pi\right)\frac{dh}{dt}$$

$$\frac{8}{4\pi} = \frac{dh}{dt}$$

$$\frac{2}{\pi} = \frac{dh}{dt}$$

∴ height is increasing at 0.64 ft/min

5. The radius of a cylinder is decreasing at a rate of 1cm/min. The height remains the same at 20cm. How fast is the volume changing when the radius is 12cm?



$\frac{dr}{dt} = -1\text{cm/min}$

$\frac{dV}{dt} = ?$ at $r = 12$

only r(t) and V(t)
h is constant

use $\boxed{V = \pi r^2 h}$

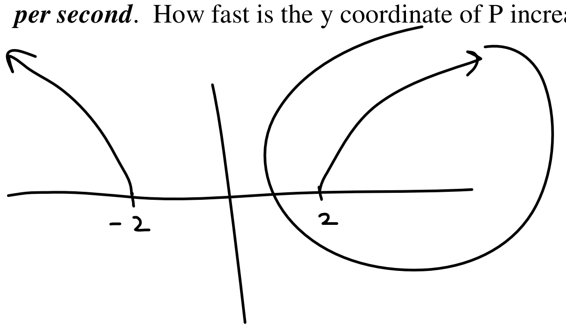
$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h$ (no product rule here)

$\frac{dV}{dt} = 2\pi(12)(-1)(20)$

$= -480\pi$

∴ Volume is decreasing at 1507 cm³/min

6. A particle P is moving along the graph of $y = \sqrt{x^2 - 4}$, $x \geq 2$, so that the x coordinate of P is increasing at the rate of ***5 units per second***. How fast is the y coordinate of P increasing when $\boldsymbol{x = 3}$?



$\frac{dx}{dt} = 5 \text{ units/sec}$

$\frac{dy}{dt} = ?$ at $x = 3$

$y = \sqrt{3^2 - 4}$

$y = \sqrt{5}$

use $y = \sqrt{x^2 - 4}$

$\frac{dy}{dt} = \frac{1}{2}(x^2 - 4)^{-1/2}\left(2x \frac{dx}{dt}\right)$

easier to use: $y^2 = x^2 - 4$

$2y \frac{dy}{dt} = 2x \frac{dx}{dt}$

$2(\sqrt{5})\frac{dy}{dt} = 2(3)(5)$

$\frac{dy}{dt} = \frac{15}{\sqrt{5}}$ or $3\sqrt{5}$ ~ 6.71 units/sec