 12CV Date:____________ Name: ________________________

Derivative Unit - Notes

Tentative TEST date_______________

Big idea/Learning Goals

This unit is a continuation of the introduction to calculus. You will be introduced to rates of change as they apply to first monomial functions, then polynomials and simple rational functions, and finally compositions of functions. You will develop an understanding of the relationship between the numeric, graphical and algebraic representations of the derivative and the original function. This knowledge will later be used to sketch complex functions by identifying the turning points and points where the function changes concavity from Concave Up to Concave Down.

Corrections for the textbook answers:

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	2-4	The Derivative Sketches - Handouts online	
	5-7	The Derivative Function for Monomials 2.1	
	8-9	Derivatives of Polynomials 2.2	
	10-11	Product Rule 2.3	
	12-13	Quotient Rule 2.4	
	14-16	Implicit Differentiation - Handout online - Appendix of textbook **p 561-564**	
	17-19	Chain Rule for Composed Functions 2.5	
		Review	

Reflect – previous TEST mark ___________, Overall mark now___________.

The Derivative Sketches

1. Differentiation was developed by Sir Isaac Newton and Gottfried Leibniz in 17th century. The output of the differentiation operation is called the ______________________. It can be used to calculate the _________________________________ at _________ point.

2. Prime notation

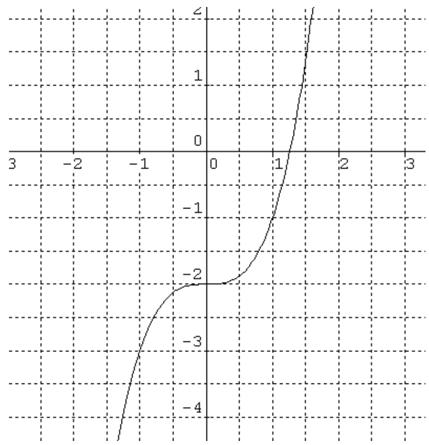
3. Leibniz notation

The First Principles Definition of the Derivative of f(x) is

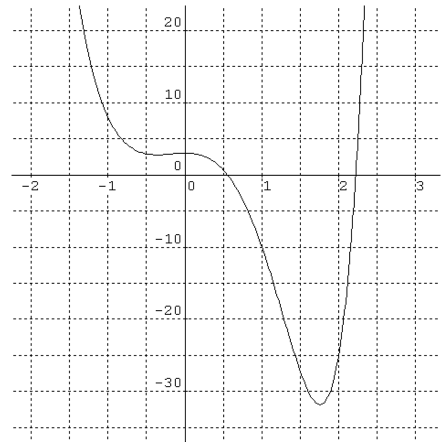
4. Find the derivative using 1st principles of $f(x) = x^2$
   a. Use the derivative to calculate quickly the slopes of tangents at $x = -3, 0, 1$
   b. Graph the function and the tangents found above to show graphical representations of the derivative

2. Sketch the derivative for each of the following functions (draw sketches on grids below).

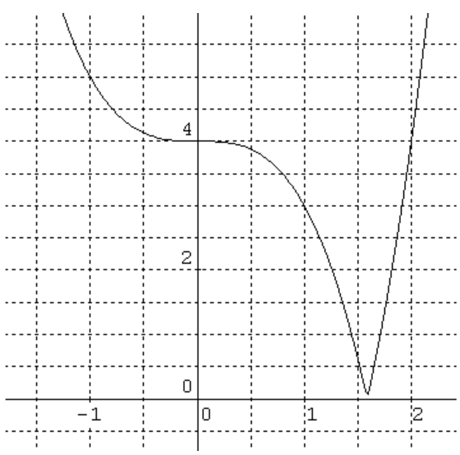
a.



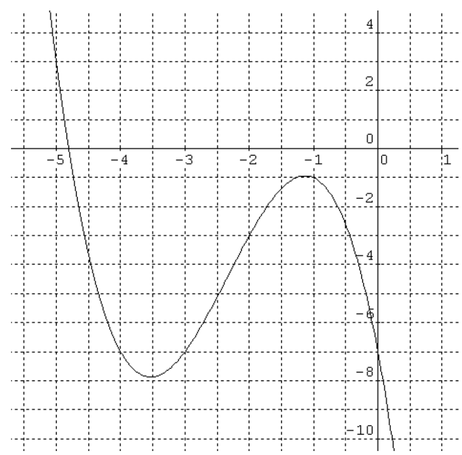
b.



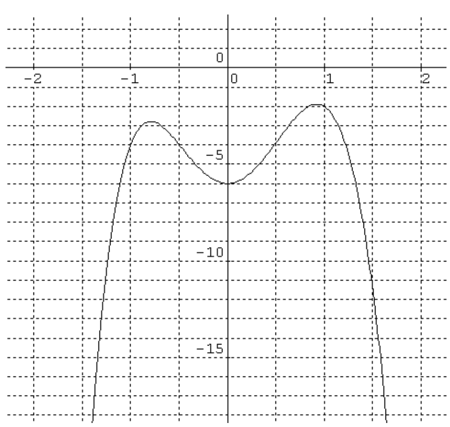
c.



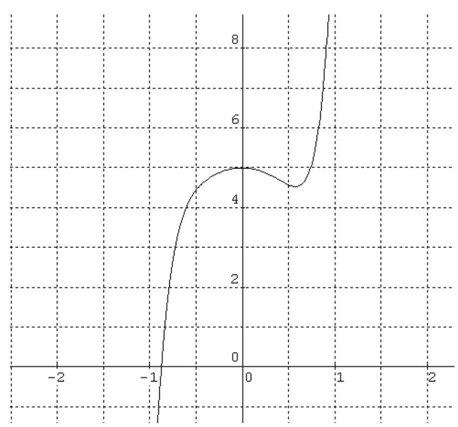
d.



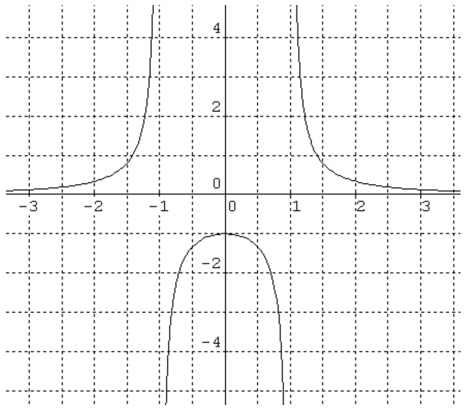
e.



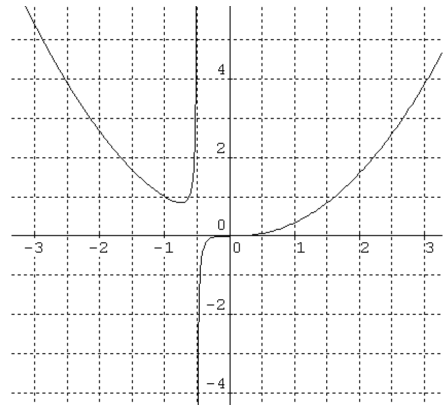
f.



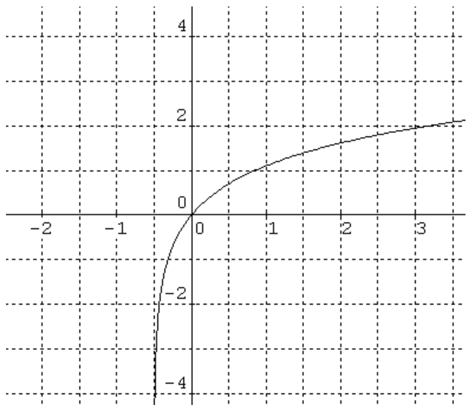
g.



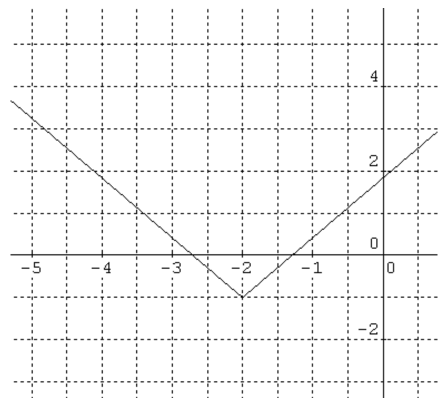
h.



i.

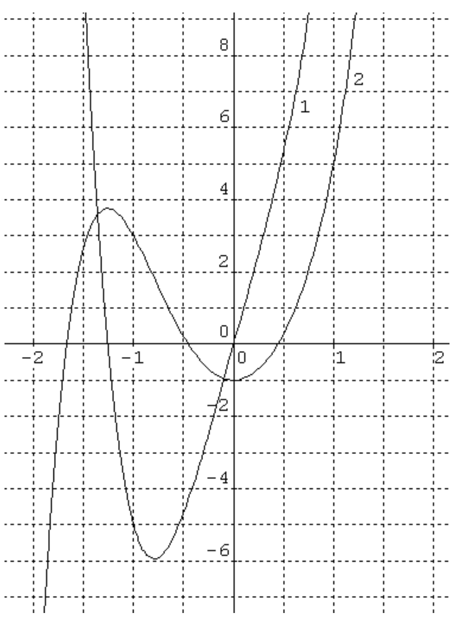


j.

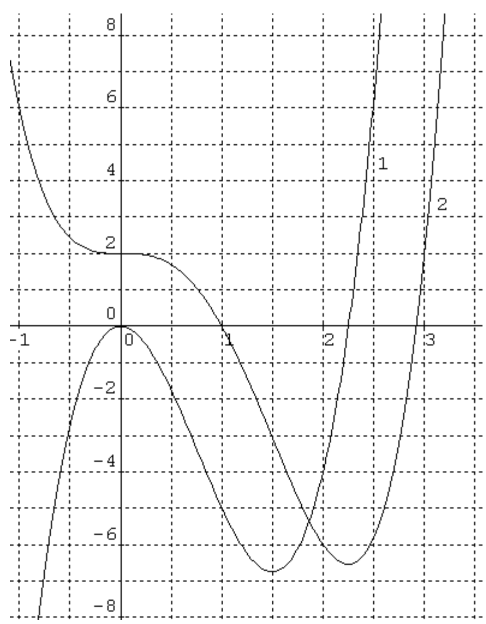


3. The graph of a function and its derivative is drawn in each of the following grids. State which is the function and which is the derivative. Explain your reasoning for each.

a.



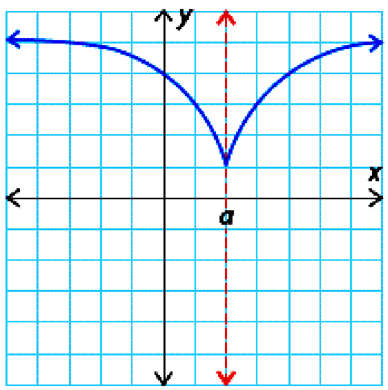
b.



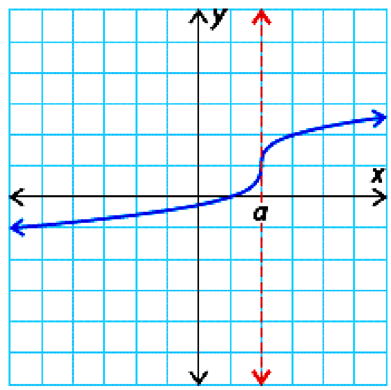
The Derivative Function for Monomials

Existence of Derivatives

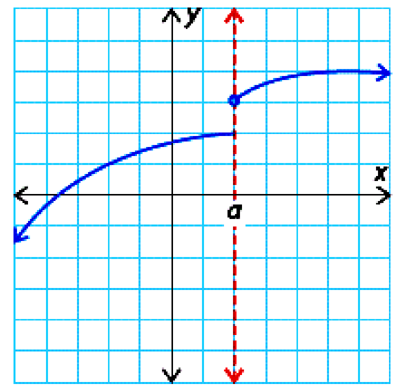
A function *f* is said to be differentiable at *a* if *f'(x)* exists. At points where *f* is not differentiable, we say that the derivative does not exist. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



Discontinuity

1. Prove that the function $f(x)=|x|$ is not differentiable at x=0, but differentiable at x=3.

LOOKING FOR PATTERNS:

2. Determine the derivatives of the following, from first principles.

1. $f(x) = x^4$

2. $f(x) = \dfrac{1}{x^2}$

3. $f(x) = \sqrt[3]{x}$

4. $f(x) = \dfrac{4}{x} + 5x^2$

3. What is the pattern that you notice, describe it in words.

4. Use the pattern to predict the derivatives of

$$f(x) = x^{57} - 6x^{14} - \frac{2}{x^3} + \sqrt[5]{x^2}$$

Power Rule

Try the following examples to see the importance of knowing what variable you're taking the derivative 'with respect to' and the need for Leibniz notation.

5. $f = \dfrac{a^3x^2}{z}$

a) $\dfrac{df}{dx}$ b) $\dfrac{df}{dy}$ c) $\dfrac{df}{da}$ d) $\dfrac{df}{dz}$

6. $f = x^2$ $p = t^3 + 2t$

a) $\dfrac{df}{dx}$ b) $\dfrac{df}{dt}$ c) $\dfrac{dp}{dx}$ d) $\dfrac{dp}{dt}$

Derivatives of Polynomials

Develop/Prove some short cut RULES instead of doing 1st principles all the time.

Constant Rule

Power Rule

Constant Multiple Rule

Sum&Difference Rule

1. Differentiate each function. Simplify to create an expanded polynomial first.

   a. $y = 5x^6 - 4x^3 + 6$

   b. $f(x) = -3x^5 + 8\sqrt{x} - \dfrac{9}{x}$

   c. $g(x) = (2x-3)(x+1)$

   d. $h(x) = \dfrac{-8x^6 + 8x^2}{4x^5}$

2. Determine the equation of the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$

3. Determine the point(s) on the graph of $y = x^2(x+3)$ where the slope of the tangent is 24.

Product Rule

1. Show that the product rule is not just the derivatives of the two factors multiplied.

2. Develop/Prove the product rule.

Product Rule

3. Differentiate

   a. $(x^2+1)(1-x)$

   b. $(x^3+4x^2-6x+1)(-3x^2+9x-2)$

4. Develop the power of a function rule by looking at the pattern from these two questions

a. $f(x) = g^2(x)$

b. $f(x) = g^3(x)$

Power of a Function Rule

Special case of Chain Rule

5. Differentiate

a. $(4x^3 - 2x^2 + x - 10)^5$

b. $\sqrt{6 - 5x^3}$

6. Differentiate

c. $(x^3 + 5x)(x^2 - 3)^4$

d. $\dfrac{5x - 3}{x^2 + 4x}$

Quotient Rule

1. Develop/Prove the quotient rule by using the power rule.

Quotient Rule

2. Differentiate

   a. $q(x)=\dfrac{6x-5}{x^3+4}$

   b. $p(x)=\dfrac{x+3}{\sqrt{x^2-1}}$

3. Determine the equation of the tangent to the curve $y = \dfrac{x^2 - 3}{5 - x}$ at $x = 2$

4. Suppose the function $V(t) = \dfrac{50000 + 6t}{1 + 0.4t}$ represents the value, in dollars, of a new car *t* years after it is purchased.
   a. What is the rate of change of the value of the car at 2years? 5years? 7years?
   b. What is the initial value of the car?
   c. Explain how the values in a. can be used to support an argument in favour of purchasing a used car, rather than a new one.

Implicit Differentiation

Up to now, we have worked explicitly, solving an equation for one variable in terms of another. For example, if you were asked to find $\frac{dy}{dx}$ for $\boldsymbol{2x^2 + y^2 = 4}$, you would solve for y and get ______________________ and then take the derivative.

Sometimes it is inconvenient or difficult to solve for y. In this case, we use <u>implicit differentiation</u>. You assume y could be solved in terms of x and treat it as a function in terms of x. Thus, you must apply the ________________ rule because you are assuming y is defined in terms of x.

Recall differentiating with respect to x:

$\frac{d}{dx}\left[x^3\right] =$

variables agree

$\frac{d}{dx}\left[y^3\right] =$

variables disagree

1. Find derivative both ways for ***2x² + y² = 4***

2.
Find the indicated derivative of the following expressions, these questions are unusual with different variables. The reason I'm asking you to try them is so that you get the hang of how Leibniz notation works with implicit differentiation.

a. $-4a^3 + 6b + a^2b = 7$ where $a(b)$

i. Take the derivative $\frac{d}{da}$

ii. Take the derivative $\frac{d}{db}$

b. $4t^5 + y - 2y^{-3} - 2ty + t^4y^2 - 9 = 0$ where $y(t)$

i. Take the derivative $\frac{d}{dy}$

ii. Take the derivative $\frac{d}{dt}$

c. $y^5 + 3x^2y^2 + 5x^4 = 12$ where $y(x)$ Take the derivative with respect to x.

d. $x^4y^2 - x^3y + 2xy^3 = 0$ where $x(y)$ Take the derivative with respect to y.

3. Now try these questions which are more like what you'd see on a test or at university.

a. Find $\frac{dy}{dx}$ from $x^3 + y^3 = 6xy$ at the point (3, 3)

b. Find $\frac{dy}{dx}$ from $\sqrt{x+y} = 1 + x^2y^2$

c. Find $\frac{dy}{dx}$ from $x^{2/3} + y^{2/3} = 4$ at the point $(-3\sqrt{3}, 1)$

d. Find $\frac{dy}{dx}$ from $x^4(x+y) = y^2(3x-y)$

4. Suppose that the equation $\frac{2}{x}+\frac{3}{y}=x$ defines a function *f* with $y=f(x)$. Find $\frac{dy}{dx}$ and the slope of the tangent line at the point (2, 3). (show both methods)

5. At a certain factory, approximately $q(t)=t^3-\frac{2}{\sqrt{t}}$ units are manufactured during the first *t* hours of a production run, and it is estimated that the total cost of producing *q* units is $C(q)=300q+0.2\sqrt{q}+\frac{20}{q}$ dollars. Find the rate at wich the cost is changing with respect to time 4 hours after production commences.

Chain Rule for Composed Functions

1. Prime notation
2. Leibniz notation
3. Composition of functions notation

4. For $u(x)=4x^2-3x$ and $p(x)=\sqrt{x}$ (Again these examples use different variables to get the concept across the next page is what your test will be like.)

a. Find $p(u)$

b. Find $p(u(x))$

c. Find $\frac{d}{du}p$

d. Find $\frac{d}{dx}u$

e. Find $\frac{d}{dx}p$ using answer in a.

f. What do you notice?

Chain Rule

5. For $p(x)=x^3+5x$ and $b(x)=\frac{1}{x}$

a. Find $b(p)$

b. Find $b(p(x))$

c. Find $\frac{d}{dp}b$

d. Find $\frac{d}{dx}p$

e. Find $\frac{d}{dx}b$ using answer in b.

f. Find $\frac{d}{dx}b$ using the Chain Rule from answer in a.

6. If $y = -\sqrt{u}$ and $u = 4x^3 - 3x^2 + 1$ determine $\frac{dy}{dx}$ at $x = 0$

7. If $a = b(2 - b^2)$ and $b = \frac{1}{c}$ determine $\frac{da}{dc}$ at $c = 2$

8. If $y = \sqrt[4]{u}$ and $u = 1 + 2x + x^3$ determine $\frac{dy}{dx}$ at $x = 1$

9. If $y = u\sqrt{1-u}$ and $u = 3x - x^2$ determine $\frac{dy}{dx}$ at $x = -1$

10.
$f(x) = -x^5 + 15x$, Find $h'(-2)$ if $h(x) = f(g(x))$, $g(-2) = 3$, and $g'(-2) = -6$

11.
If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

Differentiate these complex functions using several rules, Then practice simplifying by pulling out a factor with the lowest exponent:

12. $\left(\dfrac{1+x^3}{2x-x^2}\right)^7$

13. $(x^2 - 3x)^5 (3 - 2x)^4$

14. Why it's called the chain rule example:

$$\left[\left(\left[3x^2 + x\right]^{-1} - 5\right)^3 + 10\right]^6$$