p1NOTES

April-15-13
10:01 PM

DerivativesNotesNEW

Inserted from: <file://C:\Users\MrsK\Desktop\LacieOct9\2 Math\Math 12\MCB 4U Calc Vect\2013\6 Deriv\DerivativesNotesNEW.doc>

p.15 ☐ remove parts i for 2ⓐⓑ
remind students when deriv. = 0 p.7 bottom
☐ remove #2ⓒⓓ
☐ p.15 #3 more space
☐ p.16 #4 remove "both methods"
☑ p.17 already changed

↓ see below

Derivative Unit - Notes

Tentative TEST date Mon May 6

fog = f(g)

Big idea/Learning Goals

This unit is a continuation of the introduction to calculus. You will be introduced to rates of change as they apply to first monomial functions, then polynomials and simple rational functions, and finally compositions of functions. You will develop an understanding of the relationship between the numeric, graphical and algebraic representations of the derivative and the original function. This knowledge will later be used to sketch complex functions by identifying the turning points and points where the function changes concavity from Concave Up to Concave Down.

Corrections for the textbook answers:

Success Criteria

- ☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
Apr23	2-4	The Derivative Sketches - Handouts online	
	5-7	The Derivative Function for Monomials 2.1	
	8-9	Derivatives of Polynomials 2.2	
	10-11	Product Rule 2.3	
	12-13	Quotient Rule 2.4	
	14-16	Implicit Differentiation - Handout online - Appendix of textbook p 561-564	journal Unit #9 ques #3 do in this journal.
	17-19	Chain Rule for Composed Functions 2.5	
		Review	

Reflect – previous TEST mark __________, Overall mark now__________.

The Derivative Sketches

1. Differentiation was developed by Sir Isaac Newton and Gottfried Leibniz in 17^{th} century. The output of the differentiation operation is called the <u>derivative</u>. It can be used to calculate the <u>slope of the tangent</u> at <u>ANY</u> point. x

2. Prime notation

   read as: f prime at x $f'(x)$ or y'

3. Leibniz notation

   Read as "dee y by dee x" $\frac{dy}{dx}$

   comes from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

The First Principles Definition of the Derivative of f(x) is

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

4. Find the derivative using 1^{st} principles of $f(x) = x^2$
   a. Use the derivative to calculate quickly the slopes of tangents at $x = -3, 0, 1$
   b. Graph the function and the tangents found above to show graphical representations of the derivative

4. $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$f'(x) = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$

$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

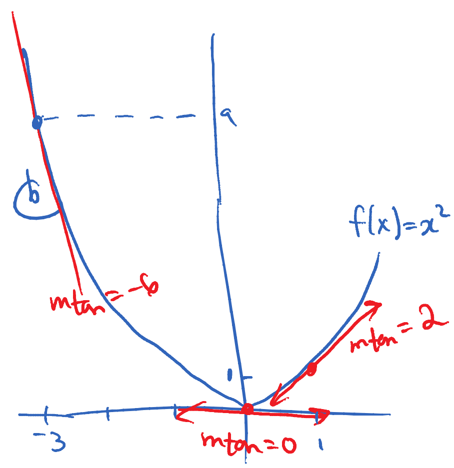
$\boxed{f'(x) = 2x}$ derivative = slope of x^2 at any pt. x

a) $f'(-3) = 2(-3) = -6 = m_{tan}$

$f'(0) = 2(0) = 0$

$f'(1) = 2(1) = 2$

b)



2. Sketch the derivative for each of the following functions (draw sketches on grids below).

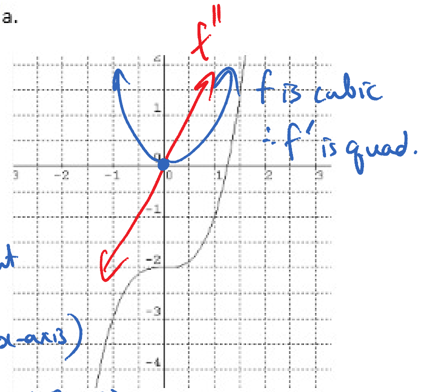
Key:

- at t.p/saddle pt. slope = 0
- pos slope means derivative output is positive (draw above x-axis)
- neg slope → below x-axis
- derivatives of polyn. always have 1 less degree

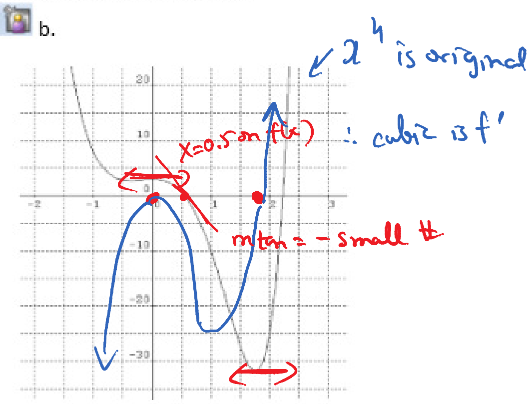
ex. $y = 6x^3$ cubic

$y' = 18x^2$ quad.

a.



b.



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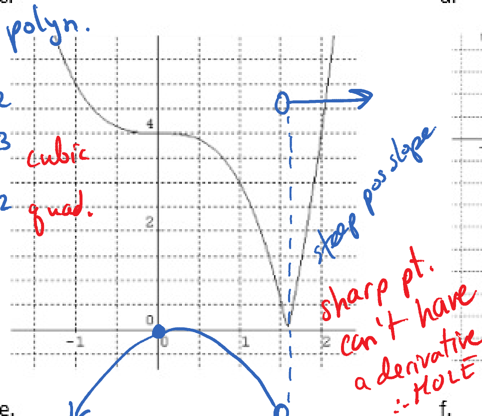

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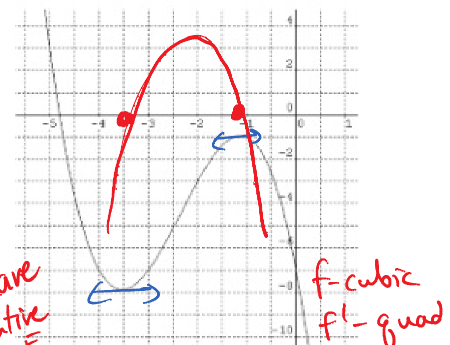
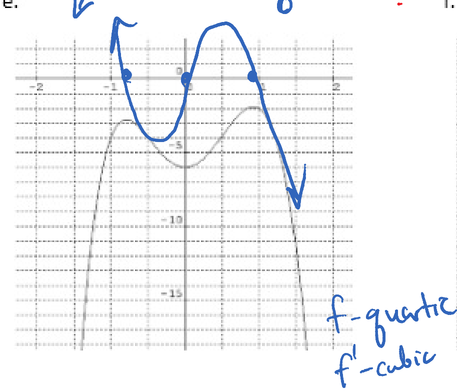
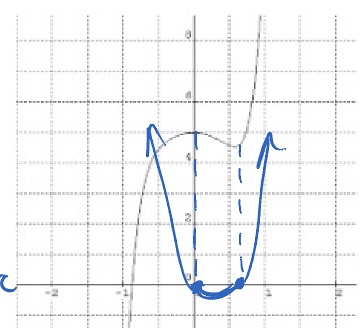


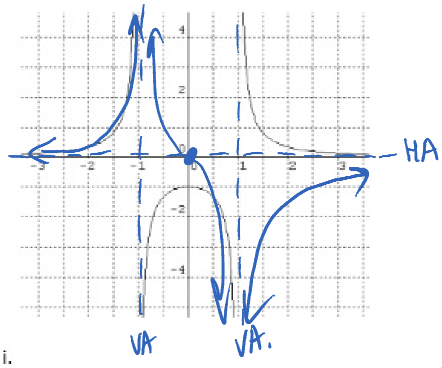
e.



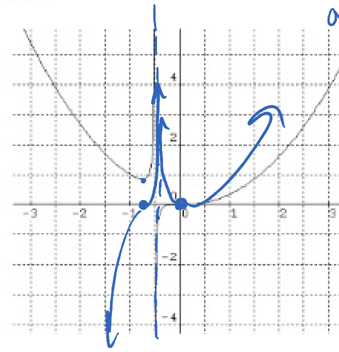
f.



g.



h.

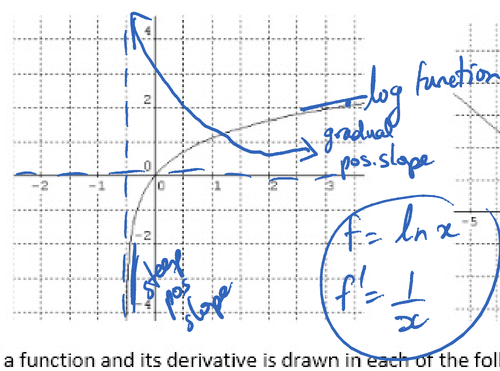


derivatives of undefined value are undefined.

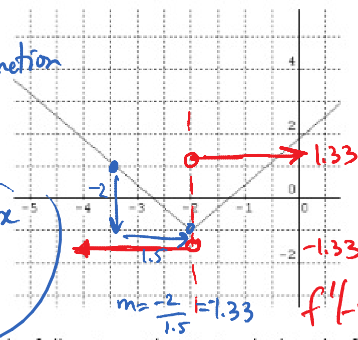
★ ∴ VA's stay where they are

★ sharp pts become holes

i.



j.

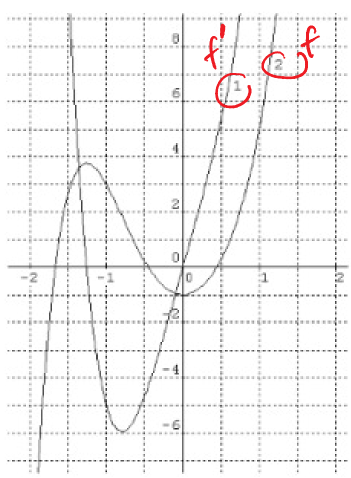


★ any HA. of the original function becomes HA y=0 of the derivative.

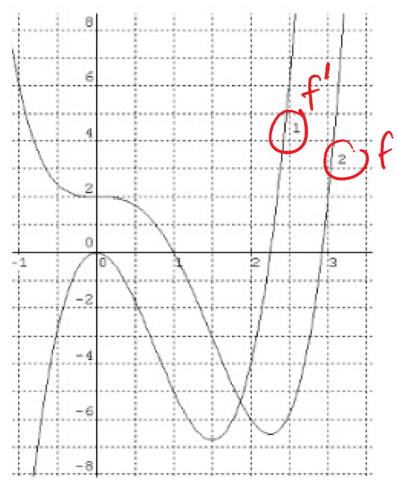
f'(-2) = undefined

3. The graph of a function and its derivative is drawn in each of the following grids. State which is the function and which is the derivative. Explain your reasoning for each.

a.



b.

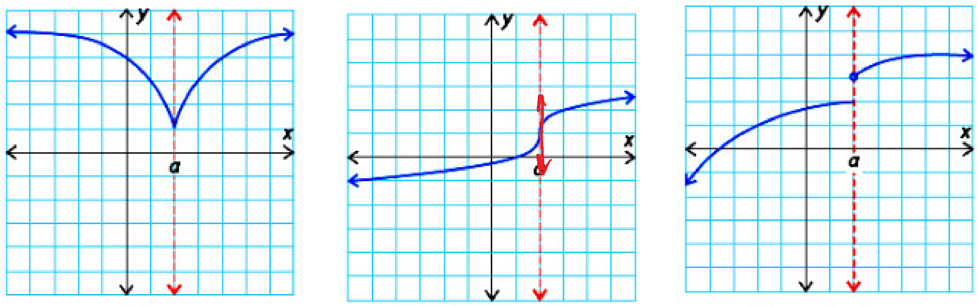


4

The Derivative Function for Monomials

Existence of Derivatives

A function *f* is said to be differentiable at *a* if *f'(x)* exists. At points where *f* is not differentiable, we say that the derivative does not exist. Three common ways for a derivative to fail to exist are shown.



Cusp | Vertical Tangent | Discontinuity

1. Prove that the function $f(x) = |x|$ is not differentiable at x=0, but differentiable at x=3.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h| - |0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{-h}{h} \qquad \lim_{h\to 0^+} \frac{h}{h}$$

$$= -1 \qquad = 1$$

$$\therefore f'(0) = \lim_{h\to 0} \frac{|h|}{h} = D.N.E$$

$\therefore$ f is not differentiable at $x=0$

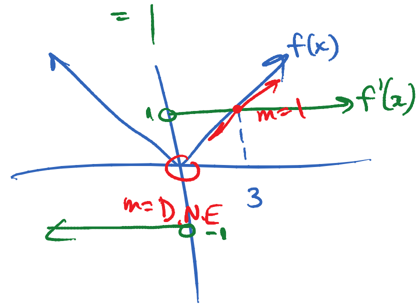
$$f'(3) = \lim_{h\to 0} \frac{|3+h| - |3|}{h}$$

Both sides of zero.

$$|h+3| = \begin{cases} h+3 & \text{if } h+3 \geq 0 \quad h \geq -3 \\ -(h+3) & \text{if } h+3 < 0 \quad h < -3 \end{cases}$$

$$= \lim_{h\to 0} \frac{h+\cancel{3} - \cancel{3}}{h}$$

$$= 1$$



LOOKING FOR PATTERNS:

2. Determine the derivatives of the following, from first principles.

1. $f(x)=x^4$ $\quad f'(x)=4x^3$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$$
$$=\lim_{h\to 0}\frac{\cancel{x^4}+4x^3h+6x^2h^2+4xh^3+h^4-\cancel{x^4}}{h}$$
$$=\lim_{h\to 0}\frac{\cancel{h}(4x^3+6x^2h+4xh^2+h^3)}{\cancel{h}}$$
$$=4x^3$$

2. $f(x)=\frac{1}{x^2}=x^{-2}$ $\quad f'(x)=\frac{-2}{x^3}$

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{(x+h)^2}-\frac{1}{x^2}\right]\frac{1}{h}$$
$$=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2h}$$
$$=\lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{x^2(x+h)^2h}$$
$$=\lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{x^2(x+h)^2\cancel{h}}=\frac{-2x}{x^2x^2}$$
$$=\frac{-2}{x^3}$$

3. $f(x)=\sqrt[3]{x}$ $\quad x^{1/3}$ $\quad f'(x)=\frac{1}{3x^{2/3}}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$$

let $u=\sqrt[3]{x+h}$

then $u^3=x+h$ $\therefore h=u^3-x$

as $h\to 0$

$u\to\sqrt[3]{x}$

$$f'(x)=\lim_{u\to\sqrt[3]{x}}\frac{u-\sqrt[3]{x}}{(u^3-x)}$$
$$=\lim_{u\to\sqrt[3]{x}}\frac{\cancel{(u-\sqrt[3]{x})}}{\cancel{(u-\sqrt[3]{x})}(u^2+u\sqrt[3]{x}+\sqrt[3]{x^2})}$$
$$=\lim_{u\to\sqrt[3]{x}}\frac{1}{u^2+u\sqrt[3]{x}+\sqrt[3]{x^2}}$$
$$=\frac{1}{(\sqrt[3]{x})^2+(\sqrt[3]{x})(\sqrt[3]{x})+\sqrt[3]{x^2}}$$
$$=\frac{1}{3\sqrt[3]{x^2}}=\frac{1}{3x^{2/3}}$$

4. $f(x)=\frac{4}{x}+5x^2$ $\quad 4x^{-1}+5x^2$ $\quad f'(x)=\frac{-4}{x^2}+10x$

$$f'(x)=\lim_{h\to 0}\frac{\left[\frac{4}{x+h}+5(x+h)^2\right]-\left[\frac{4}{x}+5x^2\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\left[\frac{4}{(x+h)}+\cancel{5x^2}+10xh+5h^2-\frac{4}{x}-\cancel{5x^2}\right]\frac{1}{h}$$
$$=\lim_{h\to 0}\left[\frac{4x-4(x+h)}{x(x+h)}+(10xh+5h^2)\right]\frac{1}{h}$$
$$=\lim_{h\to 0}\left[\frac{\cancel{4x}-\cancel{4x}-4\cancel{h}}{x(x+h)}\right]\frac{1}{\cancel{h}}+\left[\cancel{h}(10x+5h)\right]\frac{1}{\cancel{h}}$$
$$=\lim_{h\to 0}\frac{-4}{x(x+h)}+10x+5h$$
$$f'(x)=\frac{-4}{x^2}+10x$$

3. What is the pattern that you notice, describe it in words.

The coefficient is multiplied by exponent and subtract one from the exponent.

$\frac{2}{5} - 1 = \frac{2}{5} - \frac{5}{5} = \frac{-3}{5}$

4. Use the pattern to predict the derivatives of

$-2x^{-3}$ $\quad x^{2/5}$

$f(x) = x^{57} - 6x^{14} - \frac{2}{x^3} + \sqrt[5]{x^2}$

$f'(x) = 57x^{56} - 84x^{13} + \frac{6}{x^4} + \frac{2}{5x^{3/5}}$

Power Rule

$f(x) = x^n$

$f'(x) = \frac{df}{dx} = nx^{n-1}$

Try the following examples to see the importance of knowing what variable you're taking the derivative 'with respect to' and the need for Leibniz notation.

w.r.t

5. $f = \frac{a^3x^2}{z}$

$f = a^3x^2z^{-1}$

$f' = -1a^3x^2z^{-2}$

a) $\frac{df}{dx} = \frac{2a^3x}{z}$ b) $\frac{df}{dy} = 0$ c) $\frac{df}{da} = \frac{3a^2x^2}{z}$ d) $\frac{df}{dz} = -\frac{a^3x^2}{z^2}$

find deriv. of f w.r.t x

6. $f = x^2$ $\quad p = t^3 + 2t$

a) $\frac{df}{dx} = 2x$ b) $\frac{df}{dt} = 0$ c) $\frac{dp}{dx} = 0$ d) $\frac{dp}{dt} = 3t^2 + 2$

Derivatives of Polynomials

Develop/Prove some short cut RULES instead of doing 1st principles all the time.

Constant Rule

If $f(x) = cx^0$

then $f'(x) = \frac{df}{dx} = 0$

$$f'(x) = \lim_{h\to 0} \frac{c - c}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= 0 \left(\lim_{h\to 0} \frac{1}{h}\right)$$

$$= 0$$

informally

Draw $f(x) = c$

slope $= 0$

Power Rule

If $f(x) = x^n$

Then $f'(x) = \frac{df}{dx} = nx^{n-1}$

1
1 1
1 2 1
1 3 3 1
$1x^4 \; 4x^3h \; 6x^2h^2 \; 4xh^3 \; 1h^4 \leftarrow (x+h)^4$
1 n # . . . .

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \#x^{n-2}h^2 + \dots - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + \#x^{n-2}h + \dots h^2 \; h^3 \dots\right.}{\cancel{h}}$$

$$= nx^{n-1} + 0 + 0 \dots$$

Constant Multiple Rule

If $f(x) = Kg(x)$

Then $f'(x) = Kg'(x)$

$$f'(x) = \lim_{h\to 0} \frac{kg(x+h) - kg(x)}{h}$$

$$= k \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}$$

$$= k \quad g'(x)$$

Sum&Difference Rule

If $f(x) = g(x) \pm H(x)$

Then $f'(x) = g'(x) \pm H'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[g(x+h) + H(x+h)] - [g(x) + H(x)]}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \frac{H(x+h) - H(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{H(x+h) - H(x)}{h}$$

$$= g'(x) + H'(x)$$

1. Differentiate each function. Simplify to create an expanded polynomial first.

a. $y = 5x^6 - 4x^3 + 6$

$y' = \frac{dy}{dx} = 30x^5 - 12x^2$

b. $f(x) = -3x^5 + 8\sqrt{x} - \frac{9}{x}$

$f(x) = -3x^5 + 8x^{1/2} - 9x^{-1}$ $\qquad \frac{1}{2} - \frac{2}{2} = \frac{-1}{2}$

$f'(x) = \frac{df}{dx} = -15x^4 + 4x^{-1/2} + 9x^{-2}$

$= -15x^4 + \frac{4}{\sqrt{x}} + \frac{9}{x^2}$

c. $g(x) = (2x-3)(x+1)$

$g(x) = 2x^2 + 2x - 3x - 3$

$= 2x^2 - x - 3$

$g'(x) = 4x - 1$

d. $h(x) = \frac{-8x^6 + 8x^2}{4x^5} = (-8x^6 + 8x^2)(\frac{1}{4}x^{-5})$

$= -2x + 2x^{-3}$

$h'(x) = -\frac{6}{x^4} - 2$

2. Determine the equation of the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$

$f'(x) = 12x^2 + 6x$

$f'(-1) = 12(-1)^2 + 6(-1)$

$= 6$

$m_{tan} = 6$

$f(-1) = 4(-1)^3 + 3(-1)^2 - 5$

$= -4 + 3 - 5$

$y = -6$

$y = mx + b$

$-6 = 6(-1) + b$

$0 = b$

$\therefore y = 6x$

3. Determine the point(s) on the graph of $y = x^2(x+3)$ where the slope of the tangent is 24.

$y = x^3 + 3x^2$

$y' = 3x^2 + 6x$

$24 = 3x^2 + 6x$

$0 = 3x^2 + 6x - 24$

$0 = 3(x^2 + 2x - 8)$

$0 = 3(x+4)(x-2)$

$x = -4, 2$

$y = (-4)^2(-4+3)$

$= -16$

$(-4, -16)$

$y = (2)^2(2+3)$

$= 20$

$(2, 20)$

$\therefore (-4, -16)$ & $(2, 20)$

Product Rule

1. Show that the product rule is not just the derivatives of the two factors multiplied.

$h(x) = \underbrace{(3x+2)}_{f(x)}\underbrace{(x-4)}_{g(x)} = 3x^2 - 10x - 8$

$f' = 3$

$g' = 1$

not the same $h' = 6x - 10$ to do derivatives separately!

h' using product rule

$h' = 3(x-4) + 1(3x+2)$

$= 6x - 10$ ✓

2. Develop/Prove the product rule.

let $H(x) = f(x)g(x)$

$$H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\left[\frac{f(x+h) - f(x)}{h}\right] + \lim_{h\to 0} f(x)\left[\frac{g(x+h) - g(x)}{h}\right]$$

$$= \underbrace{\lim_{h\to 0} g(x+h)}_{g(x)} \underbrace{\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]}_{f'(x)} + \underbrace{\lim_{h\to 0} f(x)}_{f(x)} \underbrace{\lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right]}_{g'(x)}$$

$$= g(x) \quad f'(x) \quad + \quad f(x) \quad g'(x)$$

Product Rule

$$(fg)' = f'g + g'f$$

$$\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right)g + \left(\frac{d}{dx}g\right)f$$

3. Differentiate

a. $f = (x^2+1)(1-x)$

$$f' = (x^2+1)\frac{d}{dx}(1-x) + (1-x)\frac{d}{dx}(x^2+1)$$

$$f' = (x^2+1)(-1) + (1-x)(2x)$$

b. $f = (x^3+4x^2-6x+1)(-3x^2+9x-2)$

$$f' = (x^3+4x^2-6x+1)\frac{d}{dx}(-3x^2+9x-2) + (-3x^2+9x-2)\frac{d}{dx}(x^3+4x^2-6x+1)$$

$$f' = (x^3+4x^2-6x+1)(-6x+9) + (-3x^2+9x-2)(3x^2+8x-6)$$

4. Develop the power of a function rule by looking at the pattern from these two questions

a. $f(x) = g^2(x) = g \cdot g$

$f' = gg' + gg'$

$= 2gg'$

b. $f(x) = g^3(x)$

$= g \cdot g \cdot g$

$= g(g^2)$

$f' = g(2gg') + g^2(g')$

(leave 1st alone)(deriv of 2nd part) + (leave 2nd alone)(deriv of 1st)

$= 2g^2g' + g^2g'$

$= 3g^2g'$

Power of a Function Rule

Special case of Chain Rule

$f(x) = g^n(x)$, $f'(x) = ng^{n-1}(x)\,g'(x)$

$$\frac{d}{dx} g^n = n g^{n-1} \frac{dg}{dx}$$

$f = g^{10}(x)$

$f' = 10g^9 g'$

5. Differentiate

a. $f = (4x^3 - 2x^2 + x - 10)^5$

$f' = 5(4x^3 - 2x^2 + x - 10)^4(12x^2 - 4x + 1)$

(exponent comes down minus one from exponent but keep function the same!)(deriv. of inside)

b. $f = \sqrt{6 - 5x^3} = (6 - 5x^3)^{1/2}$

$f' = \frac{df}{dx} = \frac{1}{2}(6 - 5x^3)^{-1/2}(-15x^2)$

$= \dfrac{-15x^2}{2\sqrt{6 - 5x^3}}$

6. Differentiate

c. $f = (x^3 + 5x)(x^2 - 3)^4$

$f' = (x^3 + 5x)[4(x^2 - 3)^3(2x)] + (x^2 - 3)^4(3x^2 + 5)$

(leave 1st alone)(deriv of 2nd) + (leave 2nd)(deriv of 1st)

d. $f = \dfrac{5x - 3}{x^2 + 4x} = (5x - 3)(x^2 + 4x)^{-1}$

$f' =$

Quotient Rule

1. Develop/Prove the quotient rule by using the power rule.

$h(x) = \frac{f(x)}{g(x)} = fg^{-1}$ $\quad \frac{g^{-1}}{g^{-2}} = g$

$h' = f[-1g^{-2}g'] + g^{-1}f'$

$h' = g^{-2}[-fg' + gf']$ common factor g^{-2} out

$$\boxed{h' = \frac{gf' - fg'}{g^2}}$$

"leave bottom alone times deriv. of top — leave top alone times deriv. of bottom divided by bottom squared"

*as you see it separately

Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

Power Rule: Bring the power down subtract one in exponent (keep inside the same) times deriv. of inside.

2. Differentiate

a. $q(x) = \frac{6x-5}{x^3+4}$

$$q' = \frac{(x^3+4)(6) - (6x-5)(3x^2)}{(x^3+4)^2}$$

$$q' = \frac{6x^3 + 24 - 18x^3 + 15x^2}{(x^3+4)^2}$$

$$q' = \frac{-12x^3 + 15x^2 + 24}{(x^3+4)^2}$$

b. $p(x) = \frac{x+3}{\sqrt{x^2-1}} = \frac{x+3}{(x^2-1)^{1/2}}$

$$p'(x) = \frac{dp}{dx} = \frac{(x^2-1)^{1/2}(1) - (x+3)\left(\frac{1}{\cancel{2}}(x^2-1)^{-1/2}(\cancel{2}x)\right)}{(x^2-1)^1}$$

to simplify: ① common factor the lowest power out.

$$p' = \frac{(x^2-1)^{-1/2}\left[(x^2-1) - x(x+3)\right]}{(x^2-1)}$$

$\rightarrow (x^2-1)^{1/2}$

② expand to collect like terms only the top.

$$p' = \frac{x^2 - 1 - x^2 - 3x}{(x^2-1)^{3/2}}$$

$$p' = \frac{-3x-1}{(x^2-1)^{3/2}}$$

3. Determine the equation of the tangent to the curve $y=\frac{x^2-3}{5-x}$ at $x=2$

$$y'=\frac{(5-x)(2x)-(x^2-3)(-1)}{(5-x)^2}$$

$$y'=\frac{-2x^2+10x+x^2-3}{(5-x)^2}$$

$$y'(x)=y'=\frac{-x^2+10x-3}{(5-x)^2} = \text{slope at any pt. } x$$

need slope at $x=2$

$$\therefore y'(2)=\frac{-2^2+10(2)-3}{(5-2)^2}=\frac{13}{9}=m$$

$y=mx+b$

need y-value of pt.

$$y(2)=\frac{2^2-3}{5-2}=\frac{1}{3}$$

pt. $(2,\frac{1}{3})$

$$\frac{1}{3}=\frac{13}{9}(2)+b$$

$$\frac{-23}{9}=b$$

$\therefore y=\frac{13}{9}x-\frac{23}{9}$ is the tangent line.

4. Suppose the function $V(t)=\frac{50000+6t}{1+0.4t}$ represents the value, in dollars, of a new car t years after it is purchased.

   a. What is the rate of change of the value of the car at 2years? 5years? 7years?
   b. What is the initial value of the car?
   c. Explain how the values in a. can be used to support an argument in favour of purchasing a used car, rather than a new one.

ⓐ
$$V'(t)=\frac{(1+0.4t)(6)-(50\,000+6t)(0.4)}{(1+0.4t)^2}$$

$$=\frac{6+2.4t-20\,000-2.4t}{(1+0.4t)^2}$$

$$=\frac{-19994}{(1+0.4t)^2}$$

$V'(2)=-6170.99$ \$/yr

$V'(5)=-2221.56$ \$/yr

$V'(7)=-1384.63$ \$/yr

ⓑ initially $t=0$

asking for $V(0)$

not $V'(0)$ (value of car not slope/rate of change)

$V(0)=\$50\,000$ is the initial value of the car

ⓒ New cars fall in value a lot at the beginning and less later.

y is not isolated

Implicit Differentiation

y is isolated

Up to now, we have worked explicitly, solving an equation for one variable in terms of another. For example, if you were asked to find $\frac{dy}{dx}$ for $2x^2 + y^2 = 4$, you would solve for y and get $y = \pm\sqrt{4 - 2x^2}$ and then take the derivative.

Sometimes it is inconvenient or difficult to solve for y. In this case, we use implicit differentiation. You assume y could be solved in terms of x and treat it as a function in terms of x. Thus, you must apply the chain rule because you are assuming y is defined in terms of x.

Recall differentiating with respect to x:

$\frac{d}{dx}[x^3] = 3x^2$

variables agree

$\frac{d}{dx}[y^3] = \frac{d}{dx}\left[(y(x))^3\right]$

$= 3y^2 y'$ or $3y^2\frac{dy}{dx}$

variables disagree

ie. $\frac{d}{dx}\left(x^2+1\right)^{3/2}$

$\frac{3}{2}\left(x^2+1\right)^{1/2}(2x)$

deriv. of inside must be multiplied

1. Find derivative both ways for $2x^2 + y^2 = 4$

Explicit

solve for y, then take derivative.

$y^2 = 4 - 2x^2$

$y = \pm\sqrt{4-2x^2} = \pm\left(4-2x^2\right)^{1/2}$

$y' = \frac{dy}{dx} = \pm\frac{1}{\cancel{2}}\left(4-2x^2\right)^{-1/2}\left(-\cancel{4}^{\,2}x\right)$

$y' = -2x\left(4-2x^2\right)^{-1/2}$

$= \frac{-2x}{\sqrt{4-2x^2}}$

or

$y' = +2x\left(4-2x^2\right)^{-1/2}$

$= \frac{2x}{\sqrt{4-2x^2}}$

Implicit

$2x^2 + y^2 = 4$

exponent comes down, leave inside the same, subtract one in exponent times deriv. of inside

$4x + 2y\,y' = 0$

now just solve for y'

$2yy' = -4x$

$y' = \frac{-4x}{2y}$

$y' = \frac{-2x}{y}$

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b is inside b here ∴ product rule. product rule!!

2. Find the indicated derivative of the following expressions, these questions are unusual with different variables. The reason I'm asking you to try them is so that you get the hang of how Leibniz notation works with implicit differentiation.

can skip

a. $-4a^3+6b+a^2b=7$ where $a(b)$

i. Take the derivative $\frac{d}{da}$ consider b is #

$-12a^2+0+2ab=0$

ii. Take the derivative $\frac{d}{db}$ consider a has b inside of it.

$-12a^2a'+6+2aa'b+a^2(1)=0$ if you do deriv. of a add a'

b. $4t^5+y-2y^{-3}-2ty-t^4y^2-9=0$ where $y(t)$

can skip

i. Take the derivative $\frac{d}{dy}$ conside t is a #

$0+1+6y^{-4}-2t+2t^4y+0=0$

ii. Take the derivative $\frac{d}{dt}$ everytime you do deriv of y write y'

$20t^4+y'+6y^{-4}y'-2ty'-2y+t^4 2yy'+y^2 4t^3+0=0$

c. $y^5+3x^2y^2+5x^4=12$ where $y(x)$ Take the derivative with respect to x.

$5y^4y'+3x^2 2yy'+y^2 6x+20x^3=0$

$5y^4\frac{dy}{dx}+3x^2 2y\frac{dy}{dx}+y^2 6x+20x^3=0$

d. $x^4y^2-x^3y+2xy^3=0$ where $x(y)$ Take the derivative with respect to y.

$4x^3x'y^2+x^4 2y-3x^2x'y-x^3(1)+2x'y^3+2x3y^2=0$

can skip

3. Now try these questions which are more like what you'd see on a test or at university.

a. Find $\frac{dy}{dx}$ from $x^3+y^3=6xy$ at the point (3, 3)

$3x^2+3y^2\frac{dy}{dx}=6x\frac{dy}{dx}+y6$

$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2$

$\frac{dy}{dx}(3y^2-6x)=6y-3x^2$

$y'=\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$ at pt(3,3) $\frac{dy}{dx}=\frac{-9}{9}=-1$

b. Find $\frac{dy}{dx}$ from $\sqrt{x+y}=1+x^2y^2$ $(x+y)^{1/2}$

$\frac{1}{2}(x+y)^{-1/2}[1+y']=0+x^2 2yy'+y^2 2x$

$\frac{1}{2}(x+y)^{-1/2}+\frac{1}{2}(x+y)^{-1/2}y'=2x^2yy'+2xy^2$

$\frac{1}{2}(x+y)^{-1/2}y'-2x^2yy'=2xy^2-\frac{1}{2}(x+y)^{-1/2}$

$y'=\frac{2xy^2-\frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2}-2x^2y}$

c. Find $\frac{dy}{dx}$ from $x^{2/3}+y^{2/3}=4$ at the point $(-3\sqrt{3},1)$

$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$

$y'=\frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$

$y'=-\frac{y^{1/3}}{x^{1/3}}$ or $-\sqrt[3]{\frac{y}{x}}$

d. Find $\frac{dy}{dx}$ from $x^4(x+y)=y^2(3x-y)$

$x^5+x^4y=3xy^2-y^3$

$5x^4+4x^3y+x^4y'=3y^2+3x2yy'-3y^2y'$

$y'(x^4+3y^2-6xy)=3y^2-5x^4-4x^3y$

$y'=\frac{3y^2-5x^4-4x^3y}{x^4+3y^2-6xy}$

new way of specifying "evaluated at"

y' at $(-3\sqrt{3},1)$ is $-\sqrt[3]{\frac{1}{-3\sqrt{3}}}$

$y'\Big|_{x=-3\sqrt{3},\ y=1}=\frac{1}{\sqrt[3]{3\sqrt{3}}}$

4. Suppose that the equation $\frac{2}{x}+\frac{3}{y}=x$ defines a function f with $y=\cancel{f(x)}$ $y(x)$. Find $\frac{dy}{dx}$ and the slope of the tangent line at the point (2, 3). (~~show both methods~~)

$$\frac{d}{dx}\left[2x^{-1}+3y^{-1}=x\right]$$

$$-2x^{-2}-3y^{-2}\frac{dy}{dx}=1$$

$$-2x^{-2}-1=3y^{-2}\frac{dy}{dx}$$

$$\frac{\frac{-2}{x^2}-1}{\frac{3}{y^2}}=\frac{dy}{dx}=y'$$

$$y'(2)=\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=3}}=\frac{\frac{-2}{2^2}-1}{\frac{3}{3^2}}=-4.5$$

$\therefore$ slope of tangent line at (2,3) is $m=-4.5$

5. At a certain factory, approximately $q(t)=t^3-\frac{2}{\sqrt{t}}$ ($2t^{-1/2}$) units are manufactured during the first t hours of a production run, and it is estimated that the total cost of producing q units is $C(q)=300q+0.2\sqrt{q}+\frac{20}{q}$ dollars. Find the rate at wich the cost is changing with respect to time 4 hours after production commences.

since q has t inside
if you find deriv. of q
record q' or $\frac{dq}{dt}$

$$\frac{dq}{dt}=3t^2-2\left(\frac{-1}{2}\right)t^{-3/2}$$
$$=3t^2+t^{-3/2}$$

$$\frac{dC}{dt}=300\frac{dq}{dt}+0.1q^{-1/2}\frac{dq}{dt}-20q^{-2}\frac{dq}{dt}$$

$$\therefore \left.\frac{dq}{dt}\right|_{t=4}=3(4)^2+\frac{1}{(\sqrt{4})^3}=\frac{385}{8}$$

$$\left.\frac{dC}{dt}\right|_{\substack{t=4\\q=63}}=300\left(\frac{385}{8}\right)+\frac{0.1}{\sqrt{63}}\left(\frac{385}{8}\right)-\frac{20}{63^2}\left(\frac{385}{8}\right)$$

$$\doteq 14437.86$$

$\therefore$ Cost is changing at \$14437.86/hr (increasing)

Chain Rule for Composed Functions

This whole page is just to understand patterns.

1. Prime notation

$f'(x)$ or $y'(x)$

2. Leibniz notation

$\frac{df}{dx}$ or $\frac{dy}{dx}$

3. Composition of functions notation

$f \circ g(x) = f(g(x))$

4. For $u(x) = 4x^2 - 3x$ and $p(x) = \sqrt{x}$ (Again these examples use different variables to get the concept across the next page is what your test will be like.)

a. Find $p(u)$

$p(u) = \sqrt{u}$

b. Find $p(u(x)) = \sqrt{4x^2 - 3x}$

c. Find $\frac{d}{du}p = p'(u)$

$= \frac{1}{2}u^{-1/2}$

$= \frac{1}{2\sqrt{u}}$

d. Find $\frac{d}{dx}u = u'(x)$

$= 8x - 3$

e. Find $\frac{d}{dx}p$ using answer in b.

$p'(x) = \underbrace{\frac{1}{2}(4x^2-3x)^{-1/2}}_{\frac{dp}{du}}\underbrace{(8x-3)}_{\frac{du}{dx}}$

f. What do you notice?

The derivative of the outside function is done (kept the inside same) then times deriv. of inside function

Chain Rule

$[f(g(x))]' = \underbrace{f'(g(x))}_{\text{deriv. of outside}}\underbrace{g'(x)}_{\text{deriv of inside}}$

(keep inside same)

$\frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$

5. For $p(x) = x^3 + 5x$ and $b(x) = \frac{1}{x}$

a. Find $b(p) = \frac{1}{p}$

b. Find $b(p(x)) = \frac{1}{x^3+5x}$

c. Find $\frac{d}{dp}b = -p^{-2} = \frac{-1}{p^2}$

d. Find $\frac{d}{dx}p = 3x^2 + 5$

e. Find $\frac{d}{dx}b$ using answer in b.

$= -(x^3+5x)^{-2}(3x^2+5)$

f. Find $\frac{d}{dx}b$ using the Chain Rule from answer in a.

$\frac{db}{dx} = \frac{db}{dp}\cdot\frac{dp}{dx}$

$= \frac{-1}{p^2}\cdot(3x^2+5)$

usually here you'd sub in $p = \#$, $x = \#$

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The following questions are more like test.

6. If $y=-\sqrt{u}$ and $u=4x^3-3x^2+1$ determine $\frac{dy}{dx}$ at $x=0$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$
$$=-\frac{1}{2}u^{-1/2}\cdot\left(12x^2-6x\right)$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\u=1}}=-\frac{1}{2}(1)^{-1/2}\left(12(0)^2-6(0)\right)$$
$$=0$$

7. If $a=b(2-b^2)$ and $b=\frac{1}{c}$ determine $\frac{da}{dc}$ at $c=2$

$2b-b^3$

$$\frac{da}{dc}=\frac{da}{db}\cdot\frac{db}{dc}$$
$$=\left(2-3b^2\right)\left(-c^{-2}\right)$$

$$\left.\frac{da}{dc}\right|_{\substack{c=2\\b=\frac{1}{2}}}=\left[2-3\left(\tfrac{1}{2}\right)^2\right]\left[\frac{-1}{2^2}\right]$$
$$=\left(2-\frac{3}{4}\right)\left(-\frac{1}{4}\right)$$
$$=-\frac{1}{2}\frac{\times 8}{\times 8}+\frac{3}{16}$$
$$=-5/16$$

8. If $y=\sqrt[4]{u}$ and $u=1+2x+x^3$ determine $\frac{dy}{dx}$ at $x=1$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$
$$=\frac{1}{4}u^{-3/4}\cdot\left(2+3x^2\right)$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\u=4}}=\frac{1}{4(4)^{3/4}}\left(2+3(1)^2\right)$$
$$=\frac{5}{4(4)^{3/4}}=\frac{5}{4^{7/4}}$$

9. If $y=u\sqrt{1-u}$ and $u=3x-x^2$ determine $\frac{dy}{dx}$ at $x=-1$ $\quad -3-1$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$
$$=\left[u\left(\tfrac{1}{2}(1-u)^{-1/2}(-1)\right)+1(1-u)^{1/2}\right]\cdot(3-2x)$$
$$=(1-u)^{-1/2}\left[-\frac{1}{2}u+(1-u)\right]\cdot(3-2x)$$
$$=\frac{\left(1-\frac{3}{2}u\right)(3-2x)}{\sqrt{1-u}}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\u=-4}}=\frac{(1+6)(3+2)}{\sqrt{5}}$$
$$=\frac{35}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{35\sqrt{5}}{5}=7\sqrt{5}$$

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10. $f'(x) = -5x^4 + 15$ $\therefore f'(3) = -65$

$f(x) = -x^5 + 15x$, Find $h'(-2)$ if $h(x) = f(g(x))$, $g(-2) = 3$, and $g'(-2) = -6$

$\left.\frac{dh}{dx}\right|_{x=-2} = ?$

$$\boxed{\begin{aligned}\frac{dh}{dx} &= \frac{dh}{dg}\cdot\frac{dg}{dx} \\ h'(x) &= f'(g(x))g'(x)\end{aligned}}$$

$$h'(-2) = f'(g(-2))g'(-2)$$
$$= f'(3)(-6)$$
$$= -65(-6)$$
$$= 390$$

11. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(x) = f'(g(x))g'(x)$$
$$F'(5) = f'(g(5))g'(5)$$
$$= f'(-2)(6)$$
$$= 4(6)$$
$$= 24$$

Differentiate these complex functions using several rules, Then practice simplifying by pulling out a factor with the lowest exponent:

12. $y = \left(\frac{1+x^3}{2x-x^2}\right)^7$

$$y' = 7\left(\frac{1+x^3}{2x-x^2}\right)^6\left(\frac{(2x-x^2)(3x^2)-(1+x^3)(2-2x)}{(2x-x^2)^2}\right)$$
$$= \frac{7(1+x^3)^6}{x^6(2-x)^6}\left[\frac{6x^3 - 3x^4 - (2-2x+2x^3-2x^4)}{x^2(2-x)^2}\right]$$
$$= \frac{7(1+x^3)^6}{x^6(2-x)^6}\left[\frac{-x^4+4x^3+2x-2}{x^2(2-x)^2}\right]$$
$$= \frac{7(1+x^3)^6(-x^4+4x^3+2x-2)}{x^8(2-x)^8}$$

to sketch will need to factor using Factor Thrm and synthetic divion (OR use tech)

13. $(x^2-3x)^5(3-2x)^4 = y$

$$y' = 5(x^2-3x)^4(2x-3)(3-2x)^4 + (x^2-3x)^5\,4(3-2x)^3(-2)$$
$$= (x^2-3x)^4(3-2x)^3\left[5(2x-3)(3-2x) - 8(x^2-3x)\right]$$
$$= x^4(x-3)^4(3-2x)^3\left[5(6x-4x^2-9+6x) - 8x^2+24x\right]$$
$$= x^4(x-3)^4(3-2x)^3\left[-20x^2+60x-45-8x^2+24x\right]$$
$$= x^4(x-3)^4(3-2x)^3\left[-28x^2+84x-45\right]$$

14. Why it's called the chain rule example:

$$\frac{d}{dx}\left[\left(\left[3x^2+x\right]^{-1}-5\right)^3+10\right]^6$$

$$= 6\left[\left((3x^2+x)^{-1}-5\right)^3+10\right]^5(3)\left[\left((3x^2+x)^{-1}-5\right)^3+10\right]^2(-1)\left[3x^2+x\right]^{-2}(6x+1)$$