p1NOTES

May-03-13
9:13 AM

☐ p.12 add in MAX/MIN cusp/saddle ...

☐ reprint Blanks - unit 7 now

☐ add in textbook sections

☐ more space pg. 11

see below
↓

CurveSketch NotesNEW

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Curve Sketching Unit - Notes

Tentative TEST date Thrs. May 16

Big idea/Learning Goals

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at this point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century. In this unit, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

Corrections for the textbook answers:

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
May 6	2-3	Increasing and Decreasing intervals 4.1	
May 8	4-7	Critical Points & Local Extremes (first deriv test) 4.2	
9	8-9	Asymptotes 4.3	
10	10-11	Concavity and Points of Inflections (second deriv test) 4.4	
13	12-13	All together 4.5	
14		Review + journals DUE	

15 – start new unit

Reflect – previous TEST mark ___________, Overall mark now___________.

16 – Test

Increasing & Decreasing Intervals

Where Does the Function Increase & Decrease

$f(x)$ is increasing (or has positive slope) if $f'(x) > 0$

$f(x)$ is decreasing (" " neg. ") if $f'(x) < 0$

1.
   a. Find the intervals of increase and decrease for the function $f(x)=\overset{q}{2}x^3+3x^2-36x+\overset{p}{5}$
   b. Graph $f'(x)$ and explain how it also indicates intervals of increase and decrease

try $\pm\frac{p}{q}$

ⓐ $f'(x) = 6x^2+6x-36$

$= 6(x^2+x-6)$

$= 6(x+3)(x-2)$

need to solve $6x^2+6x-36 > 0$

and

$6x^2+6x-36 < 0$

* to solve non linear inequalities can't deal with factors separately!

$-\infty$ -3 2 ∞

	$x<-3$	$-3<x<2$	$2<x$
6	+	+	+
$(x+3)$	−	+	+
$(x-2)$	−	−	+
$f'(x)$	+	−	+
$f(x)$	↗ inc	↘ dec.	↗ inc

factors of function

* must be all mult/div.

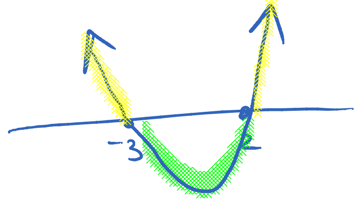
∴ intervals of increase are

$x \in (-\infty,-3)$ and $(2,\infty)$

$x<-3$ $x>2$

decrease interval is $x \in (-3,2)$

ⓑ $f'(x)$ graph



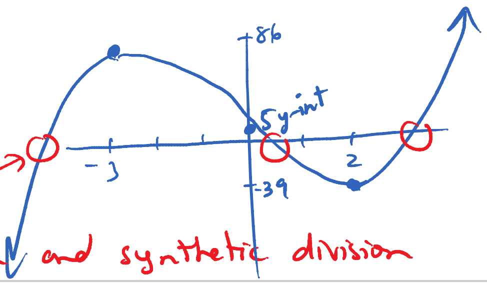
f' above x-axis means f is inc.

f' below x-axis means f is dec

$f(x)$ graph has t.p at

$x=-3$ and $x=2$

$y=86$ $y=-39$



zeros can be found by Factor Theorem and synthetic division

2. Find the intervals of increase and decrease for the function $f(x)=\frac{6-2x}{x^2-4}$

α in +/- chart place zeros and VA's in order

$$f'(x)=\frac{(x^2-4)(-2)-(6-2x)(2x)}{(x^2-4)^2}$$

$$=\frac{-2x^2+8-12x+4x^2}{(x^2-4)^2}$$

$$=\frac{2x^2-12x+8}{[(x+2)(x-2)]^2}$$

$$\boxed{=\frac{2(x-(3+\sqrt{5}))(x-(3-\sqrt{5}))}{(x+2)^2(x-2)^2}}$$ exact

$$\doteq\frac{2(x-5.2)(x-0.8)}{(x^2-4)^2}$$

$-\infty$ $\quad -2 \quad 0.8 \quad 2 \quad 5.2 \quad \infty$

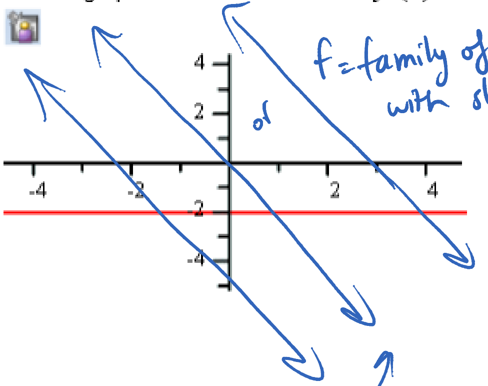
2	+	+	+	+	+
$x-5.2$	−	−	−	−	+
$x-0.8$	−	−	+	+	+
$(x+2)^2$	+	+	+	+	+
$(x-2)^2$	+	+	+	+	+
$f'(x)$	+	+	−	−	+
$f(x)$	↗	↗	↘	↘	↗

$\therefore$ f inc on $x\in(-\infty,-2),(-2,3-\sqrt{5}),(3+\sqrt{5},\infty)$

f dec on $x\in(3-\sqrt{5},2),(2,3+\sqrt{5})$

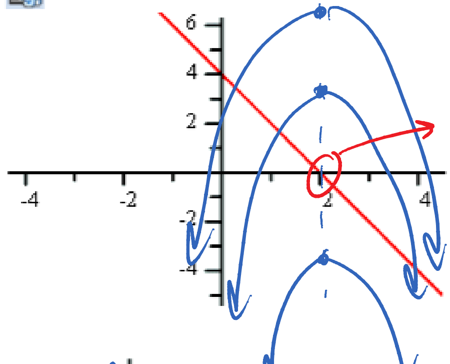
3. Use the graphs of the first derivative $f'(x)$ to sketch a possible functions $f(x)$

a.



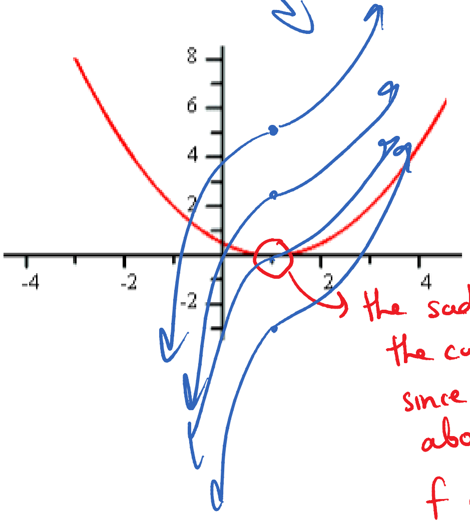
f = family of lines with slope = -2

b.



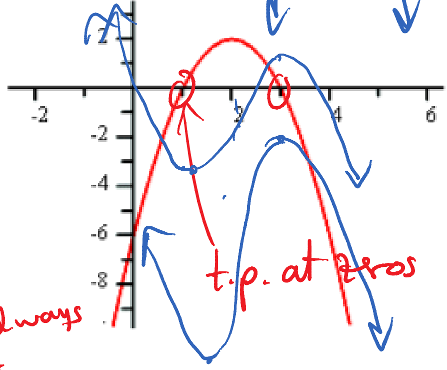
at x = 2 there's a turning pt. since f' linear then f will be quadratic

c.



the saddle pt of the cubic since f' is always above x-axis f will alway increase

d.

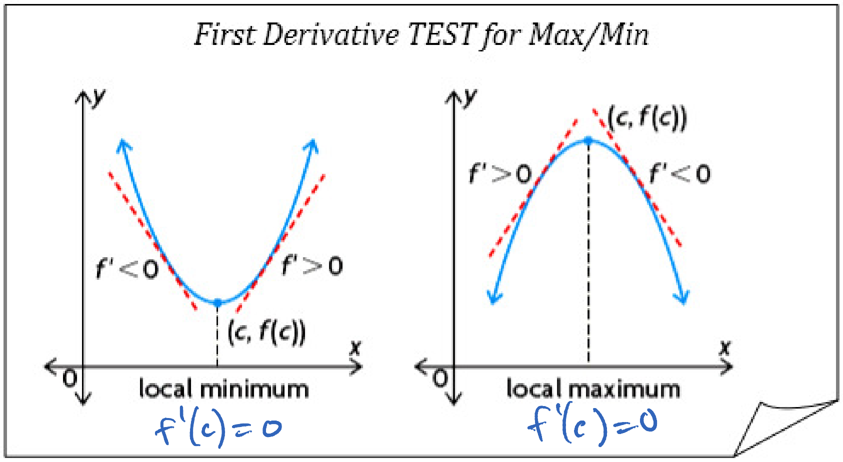


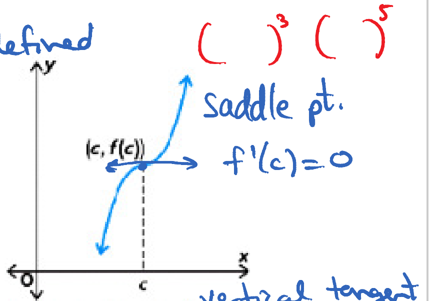
t.p. at zeros

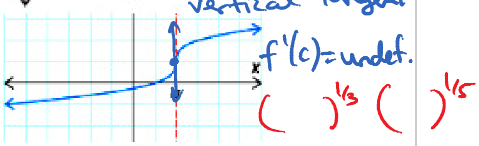
12CV Date:____________ Name: ________________________

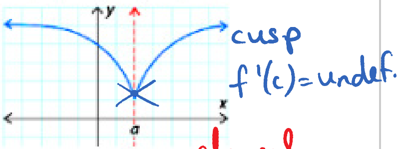
Critical Points & Local Extremes

Critical points are found from $f'(x)=0$ OR $f'(x)=\text{undefined}$









abs. val

$|(\ )^{1/3}|$

1. Find and classify all the critical points for

a. $y=2x^3-3x^2-12x+5$

$y'=6x^2-6x-12$

$0=6x^2-6x-12$

$0=6(x^2-x-2)$

$0=6(x-2)(x+1)$

deriv. never undefined (no $\sqrt{\ }$ no denom)

* must be all mult/div. for +/- chart

$\therefore$ crit. numbers are $x=2$ and $x=-1$

	$-\infty$ … -1 $x<-1$	-1 … 2 $-1<x<2$	2 … ∞ $x>2$
6	+	+	+
$(x-2)$	−	−	+
$(x+1)$	−	+	+
f'	+	−	+
f	↗	↘	↗

$\therefore$ at $x=-1$ local MAX since $f'(-1)=0$ and f' was pos then neg.

at $x=2$ local MIN

b. $y=x-\sqrt{x}$

$y'=1-\frac{1}{2}x^{-1/2}$

$y'=1-\frac{1}{2\sqrt{x}}$

LCD

$y'=\frac{(2\sqrt{x}-1)}{2\sqrt{x}}$

crit pt. $2\sqrt{x}-1=0$

$x=(\frac{1}{2})^2=\frac{1}{4}$

$2\sqrt{x}\neq0$

$x\neq0$

	$-\infty$ … 0	0 … $1/4$	$1/4$ … ∞
$2\sqrt{x}-1$	undef	−	+
$2\sqrt{x}$	undef	+	+
$f'(x)$	undef ∞	−	+
f	undef ●	↘	↗

at $x=0$ where the graph starts

$f(0)=0$

$f'(0)=\text{undef}$

at $x=\frac{1}{4}$ local MIN

c. $y = \sqrt[5]{2-x}$

$y' = \frac{1}{5}(2-x)^{-4/5}(-1)$

$y' = \frac{-1}{5(2-x)^{4/5}}$

to find crit. pts solve $\boxed{\text{numer} = 0}$ $\boxed{\text{denom} \neq 0}$

$-1 = 0$
N/A

$5(2-x)^{4/5} \neq 0$
$2 \neq x$

	−∞		2	3	∞
−1		−		−	
5		+		+	
$(2-x)^{4/5}$		+		+ $\sqrt[5]{(2-3)^4}$	
$f'(x)$		−	∞	−	
f		↘		↘	

→ power of 4 makes the output pos.

$\left(\sqrt[5]{(2-3)}\right)^4$

∴ at $x=2$
thee is a vertical tangent.

d. $y = (3-2x)^3$

$y' = 3(3-2x)^2(-2)$
$y' = -6(3-2x)^2$

$3-2x=0$
$x = \frac{3}{2}$ crit. pt.

	−∞		3/2		∞
−6		−		−	
$(3-2x)^2$		+		+	
f'		−	●	−	
f		↘		↘	

∴ saddle pt.
at $x = 3/2$

eg.

e. $y = |x^3 - 2|$

$$y = \begin{cases} x^3 - 2 & \text{, if } x^3 - 2 \geq 0 \rightarrow x \geq \sqrt[3]{2} \\ -(x^3 - 2) & \text{, if } x^3 - 2 < 0 \rightarrow x < \sqrt[3]{2} \end{cases}$$

$$y' = \begin{cases} 3x^2 & \text{if } x > \sqrt[3]{2} \\ -3x^2 & \text{if } x < \sqrt[3]{2} \end{cases}$$

* not include here since sharp pt creates a hole for deriv.

critical pt

$x = 0$ and $x = \sqrt[3]{2}$

$f'(0) = 0$ undefined (hole)

	$-\infty$		0		$\sqrt[3]{2}$		∞
$3x^2$		NA		NA		+	
$-3x^2$		−		−		NA	
f'		−	●	−	∅	+	

at $x = 0$ saddle pt.

at $x = \sqrt[3]{2}$ cusp (MIN)

f. $y = \sqrt{\dfrac{1-2x}{x^2-1}}$

won't factor even with quad. form.

$$y' = \frac{x^2 - x + 1}{\sqrt{(1-2x)(x^2-1)^3}}$$

$(x+1)^3(x-1)^3$

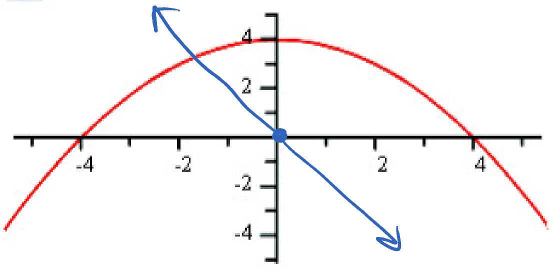
	$-\infty$		-1		$\frac{1}{2}$		1		∞
x^2-x+1 under the root.		+		+		+		+	
$1-2x$		+		+		−		−	
$(x+1)^3$		−		+		+		+	
$(x-1)^3$		−		−		−		+	
overall under root		$\sqrt{+}$		$\sqrt{-}$		$\sqrt{+}$		$\sqrt{-}$	
f'		+		undef		+		undef	
f		/		undef		/		undef.	

∴ at $x = -1, \frac{1}{2}, 1$ the graph splits and has gaps between them can't classify if one side is missing on the crit. pt.

vertical tangent vs VA

2. Draw derivatives of these functions.

a.



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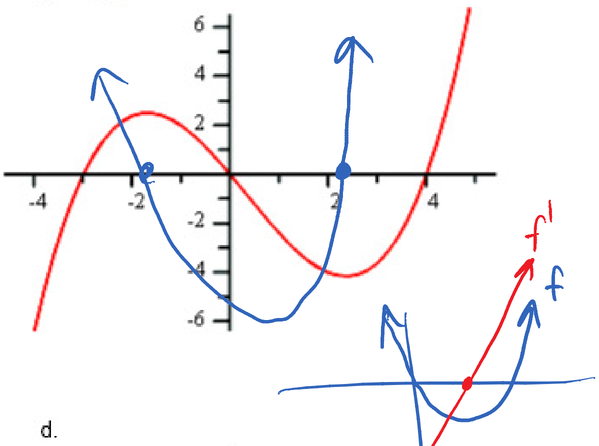
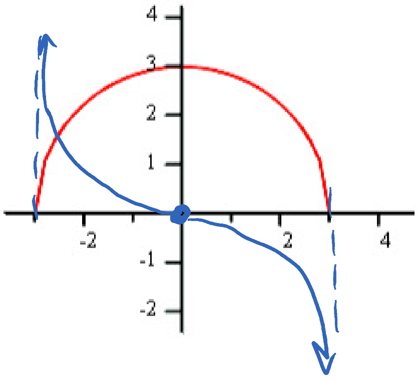
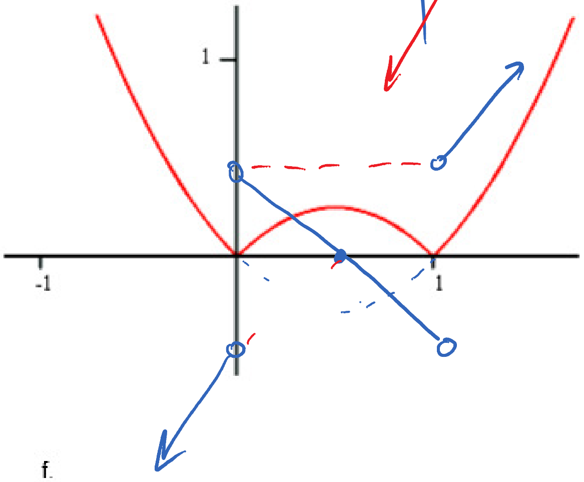


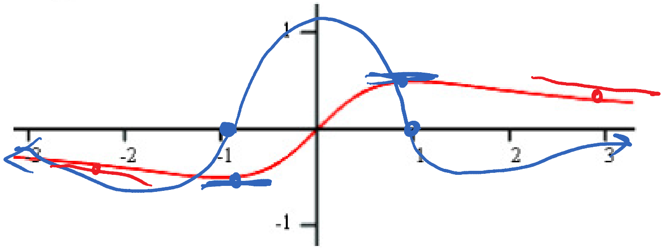
c.

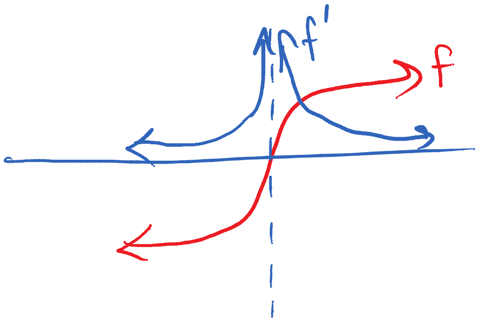


d.

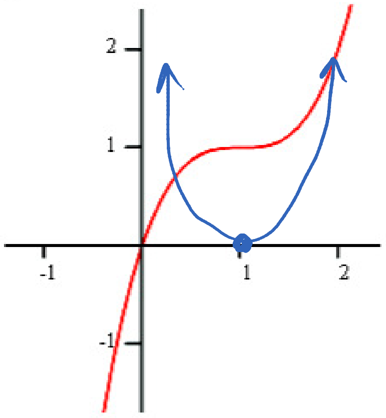


e.





f.



Asymptotes

Note to me

use $\frac{x}{x^2+1}$ as ex why you need +/- chart

x-int

sub $y=0$, solve

$\boxed{\text{numer}=0}$ if there is one fraction

y-int

sub $x=0$

pos/neg intervals

use +/− chart with zeros and VA's of original f(x)

above/below x-axis

VA $x=$

solve denom=0

	VA		VA	
f	+	+	−	

OA $y=mx+b$

*happens only if degree of numer is bigger by ONE than denom

- OA = quotient in long division

HA $y=$

- ensure the degrees on top and bottom match (insert zeros if needed)
- Look at leading coefficient

textbook does this differently

★text book uses limits instead

Find the asymptotes and sketch what you can of each function

1. $\overset{0}{y}=\frac{2x+4}{3x-6}$

x-int = −2

$2x+4=0$
$2x=-4$
$x=-2$

y-int = −2/3

$\frac{2(0)+4}{3(0)-6}=-2/3$

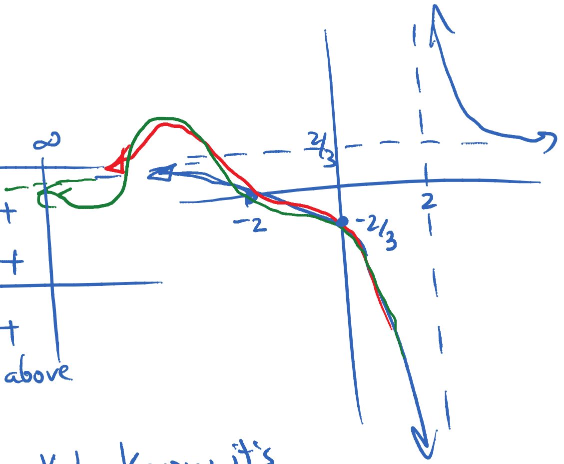
VA x = 2

* use equation format, not just a number

HA $y=2/3$

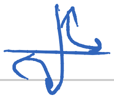
	−∞	−2	2	∞
2x+4	−	+	+	
3x−6	−	−	+	
f	+	−	+	
	above	below	above	

"x-axis"



* to know it's not the "red" or "green" graph you'd need derivatives

OR $f(x)=\frac{2x+4}{3x-6}=2/3+\frac{8}{3x-6}$ → transformed parent rational



2. $y=\frac{1+x^2}{2-x}$

x-int: none

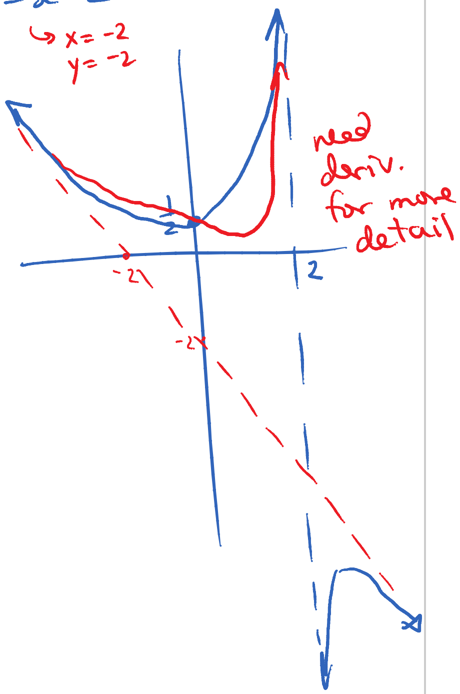
y-int $=\frac{1}{2}$

VA $x=2$

OA $y=-x-2$

↳ $x=-2$
$y=-2$

$$\begin{array}{r} -x-2 \\ -x+2\overline{\big)\,x^2+0x+1} \\ \underline{x^2-2x} \\ 2x \end{array}$$



need deriv. for more detail

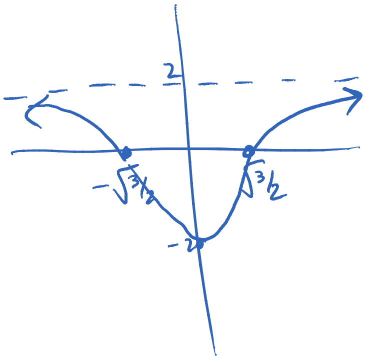
3. $y=\frac{2x^2-3}{x^2+1}$

X-int = $\pm\sqrt{\frac{3}{2}}$

y-int = 2

VA none

HA y=2



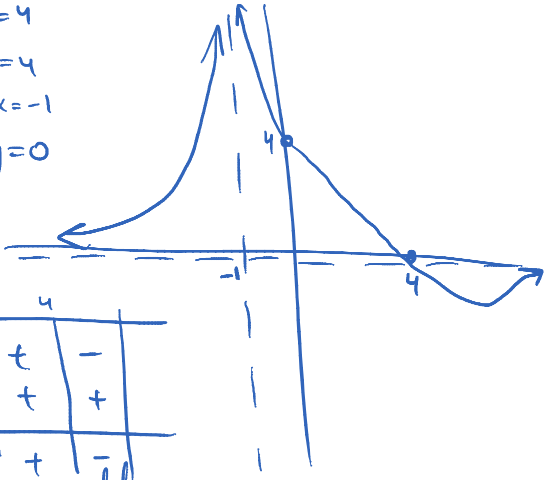
4. $y=\frac{4-x}{x^2+2x+1}=\frac{4-x}{(x+1)^2}$

x-int = 4

y-int = 4

VA x=-1

HA y=0



	-∞	-1	4
4-x	+	+	-
$(x+1)^2$	+	+	+
f	+	+	- below

5. $y=\frac{0x^2+3-2x}{1x^2+6x-16}=\frac{(3-2x)}{(x+8)(x-2)}$

* watch out for Holes if factors cancel.

x-int = 3/2

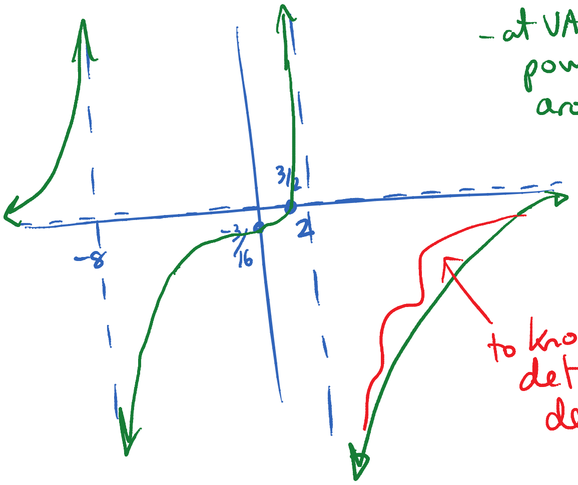
y-int = $-\frac{3}{16}$

VA x=-8 and x=2

HA y=0

instead of using +/- chart on f(x) to fill in details, you can

- choose a big # x=100
- f(100) = neg, draw below
- at VA x=2, factor has power of one ∴ arms around VA go in opposite direction



to know more detail need derivatives

6. $y=\frac{4-x^2}{x+3}=\frac{(2-x)(2+x)}{x+3}$

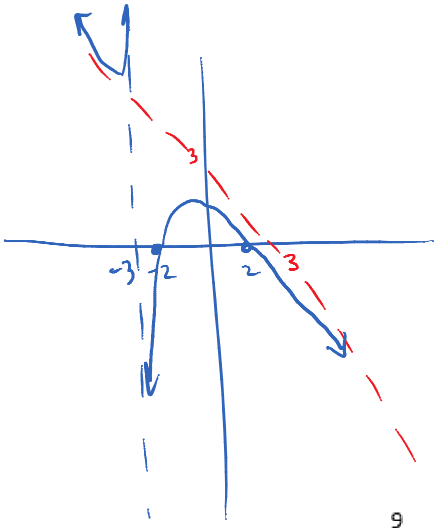
x-int -2 and 2

y-int 4/3

VA x=-3

OA y=-x+3

$$\begin{array}{r} -x+3 \\ x+3\,\overline{)\,-x^2+0x+4} \\ \underline{-x^2-3x} \\ 3x \end{array}$$



Concavity and Points of Inflection

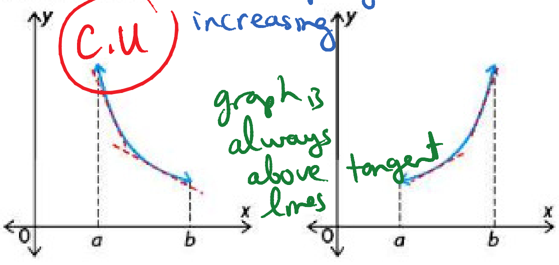
f'(x)

CONCAVE UP – where slopes of f(x) are increasing

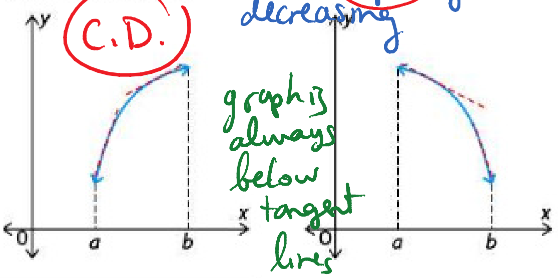
C.U

CONCAVE DOWN – where slopes of f(x) are decreasing

C.D.



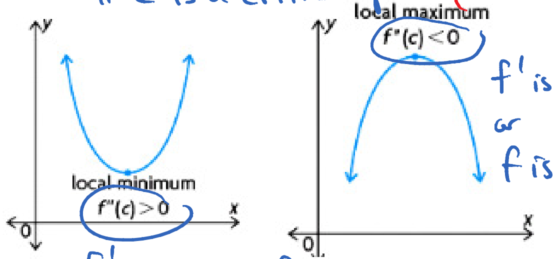
graph is always above tangent lines



graph is always below tangent lines

Second Derivative TEST for Max/Min

if c is a critical point (f'(c)=0 or undefined)



f' is dec or f is CD

f' is inc or f is C.U.

Possible and actual POINT OF INFLECTION:

solving $f''(x)=0$ or undefined gives possible P.O.Inf.

to see if it's an actual P.O.Inf need to check if concavity changes at that pt.
↳ use +/- chart on f''

Three ways of checking if a point is local max/min:

#1. using f(x) will learn in next unit (3.2 lesson)
#2. using f'(x) 1st derivative test
#3 using f''(x) 2nd derivative test.

1. Analyze any $\underbrace{\text{critical points}}_{f'(x)=0 \text{ or undefined}}$ and $\underbrace{\text{points of inflection}}_{f''(x)=0 \text{ undefined}}$ of $y=\frac{1}{3}x^3-\frac{5}{2}x^2+6x-1$, then sketch the function.

$y' = x^2-5x+6=0$

$(x-3)(x-2)=0$

∴ x=3 and x=2 are critical pts

y=3.5   y≐3.6

local MAX
local MIN
vertical tangent
cusp
saddle pt.

−∞ 2 3 ∞

x−3	−	−	+
x−2	−	+	+
f'	+ ↗	− ↘	+

∴ x=2 local MAX

at x=3 local MIN

$y''=2x-5=0$

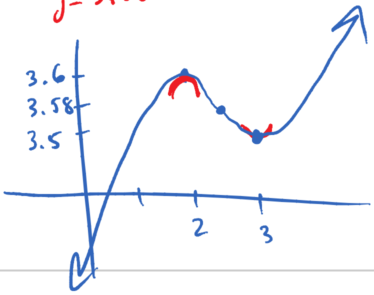
∴ $x=5/2$ possible inf. pt. → is / → not P.O.Inf

y≐3.58

−∞ 5/2 ∞

2x−5	−	+
f''	−	+
f	CD	CU

∴ $x=5/2$ is a P.O.Inf.



HA at $y=0$

2. Analyze and sketch $y=\dfrac{2}{x^2+4}=2(x^2+4)^{-1}$

$y'=-2(x^2+4)^{-2}(2x)$

$y'=\dfrac{-4x}{(x^2+4)^2}$

$\boxed{\therefore \text{ crit. pt. } x=0 \quad y=\tfrac{1}{2}}$

$y''=-4x(-2)(x^2+4)^{-3}(2x)+-4(x^2+4)^{-2}$

$=(x^2+4)^{-3}\left[\overset{12x^2-16}{16x^2-4x^2-16}\right]$

$=\dfrac{4(3x^2-4)}{(x^2+4)^3}=\dfrac{4(\sqrt{3}x+2)(\sqrt{3}x-2)}{(x^2+4)^3}$

$\boxed{\therefore \text{ possible POInf. } x=\frac{-2}{\sqrt{3}},\ x=\frac{2}{\sqrt{3}} \quad y=0.375 \quad y=0.375}$

x=0 is here

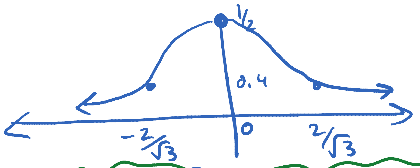
$-\infty \qquad -2/\sqrt{3} \qquad 2/\sqrt{3} \qquad \infty$

4	+	+	+
$(x^2+4)^3$	+	+	+
$\sqrt{3}x+2$	−	+	+
$\sqrt{3}x-2$	−	−	+
f''	+	−	+
f	CU	CD	CU

$\therefore$ since at $x=0$ CD it's a MAX

since at $x=\pm\frac{2}{\sqrt{3}}$ it changes concavity

both are actual pts. of inf.



3. Analyze and sketch $y=\sqrt[3]{4x-x^3}$

$y'=\frac{1}{3}(4x-x^3)^{-2/3}(4-3x^2)$ ← use this!

$y'=\dfrac{(2+\sqrt{3}x)(2-\sqrt{3}x)}{3\left(x(4-x^2)\right)^{2/3}}$

$y'=\dfrac{(2+\sqrt{3}x)(2-\sqrt{3}x)}{3\left[x\ (2-x)\ (2+x)\right]^{2/3}}$

crit. pt. $x=2,-2,0,-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}$

$y= 0,\ 0,\ 0,\ -1.45,\ 1.45$

$y''=-\frac{2}{9}(4x-x^3)^{-5/3}(4-3x^2)(4-3x^2)+\frac{1}{3}(4x-x^3)^{-2/3}(-6x)$

$y''=(4x-x^3)^{-5/3}\left[\underbrace{-\frac{2}{9}(16-24x^2+9x^4)-2x(4x-x^3)}_{-\frac{8}{9}\left[3x^2+4\right]}\right]$

$=\dfrac{-8(3x^2+4)}{9\left[x(2-x)(2+x)\right]^{5/3}}$

$\therefore$ possible inf. pts

$x=0,-2,2$

$\star$ do +/- chart on f'' only then see where critical pts fall in.

$-\infty \qquad -2 \quad \downarrow -\frac{2}{\sqrt{3}} \quad 0 \quad \downarrow 2/\sqrt{3} \quad 2 \qquad \infty$

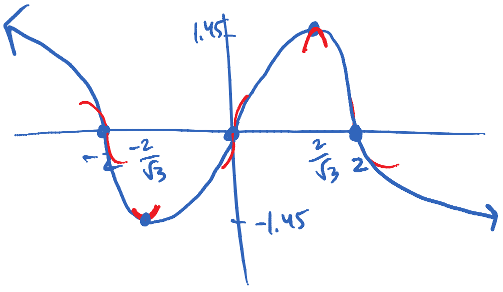
$-8/9$	−	−	−	−
$3x^2+4$	+	+	+	+
under root				
x	−	−	+	+
$2-x$	+	+	+	−
$2+x$	−	+	+	+
f''	−	+	−	+
f	CD	CU	CD	CU

odd root

$\therefore x=0,-2,2$ are actual inf. pts.

since it's CU at $x=-\frac{2}{\sqrt{3}}$ it's a MIN

since it's CD at $x=\frac{2}{\sqrt{3}}$ it's a MAX



as $x\to\infty$ $y\to-\infty$

$x\to-\infty$ $y\to+\infty$

V3

All Together

from $f(x)$: → sub y=0 → sub x=0
- x & y intercepts
- √neg ← Domain & VA → denom ≠ 0
- OA/HA long division / lead.coeff.
- ~~pos/neg intervals~~ can skip

from $f'(x)$:
- Critical points $f'=0$ or $f'=$ undefined
- ~~inc/dec intervals~~ can skip
- Max/Min values / cusp / vertical tang / saddle pt.

from $f''(x)$:
- possible Inf.Pt. $f''=0$ or undef.
- CU/CD intervals
- actual Inf.Pt does it change from CU to CD.

At the end, find all y-values of ALL the 'special points' from above in order to plot them accurately.

1. Analyze and sketch $y=\dfrac{4-x+0x^2}{1x^2+2x+1}=\dfrac{4-x}{(x+1)^2}$

eg.

x-int = 4

y-int = 4

VA x = −1

HA y = 0

$$y'=\frac{(x^2+2x+1)(-1)-(4-x)(2x+2)}{(x^2+2x+1)^2}$$

$$y'=\frac{(x+1)^2(-1)-2(4-x)(x+1)}{(x^2+2x+1)^2}$$

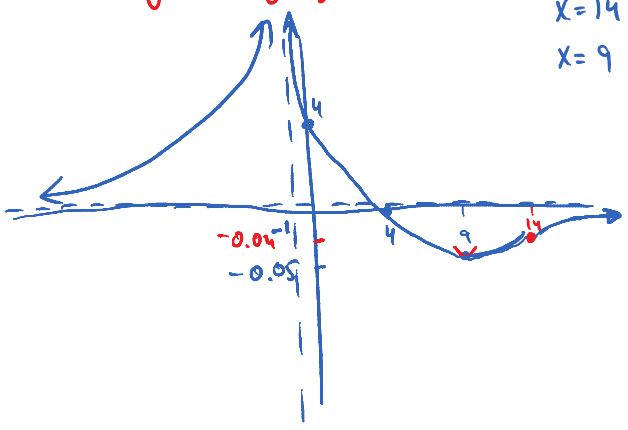
$$y'=\frac{\cancel{(x+1)}\left[(x+1)(-1)-2(4-x)\right]}{\left[(x+1)^{\cancel{2}}\right]^{2\,3}}$$

$$y'=\frac{-x-1-8+2x}{(x+1)^3}$$

$$y'=\frac{x-9}{(x+1)^3}$$

∴ crit. pts. x = −1 x = 9

y = VA $y=\frac{-1}{20}$



$$y''=\frac{(x+1)^3(1)-(x-9)(3)(x+1)^2(1)}{(x+1)^6}$$

$$=\frac{\cancel{(x+1)^2}\left[(x+1)(1)-3(x-9)\right]}{(x+1)^{\cancel{6}\,4}}$$

$$=\frac{x+1-3x+27}{(x+1)^4}$$

$$=\frac{-2x+28}{(x+1)^4}=\frac{-2(x-14)}{(x+1)^4}$$

∴ possible P.O.Inf. x = 14 and x = −1

$y=\frac{-2}{45}$ y = VA

$-\infty$ -1 14 ∞

−2	−	−	−
x−14	−	−	+
$(x+1)^4$	+	+	+
f''	+	+	−
f	CU	CU	CD

∴ x = −1 there is a VA

x = 14 is actual P.O.Inf.

x = 9 is a local MiN

x = 9 falls here the graph is CU there and f'(9) = 0

$\frac{p}{q}$: $\frac{\pm 18}{1}$, $\frac{\pm 9}{1}$ $\frac{\pm 6}{1}$ $\frac{\pm 3}{1}$ $\frac{\pm 2}{1}$ $\frac{\pm 1}{1}$

2. Analyze and sketch $y = 1x^4 - 5x^3 + x^2 + 21x - 18$ q p

$f(1) = 0$ ∴ $(x-1)$ is a factor
$f(3) = 0$ ∴ $(x-3)$ is a factor

1	1	-5	1	21	-18
	↓ copy	1 add	-4	-3	18
	1 ×1	-4	-3	18	0 ☺

$y = (x-1)(x^3 - 4x^2 - 3x + 18)$

3	1	-4	-3	18
	↓	3	-3	-18
	1	-1	-6	0 ☺

∴ $y = (x-1)(x-3)(x^2 - x - 6)$
$y = (x-1)(x-3)(x-3)(x+2)$
$y = (x-1)(x-3)^2(x+2)$

∴ x-int = 1, 3, -2
cut bounce cut

y-int = -18

no VA's, HA's, OA's

$y' = 4x^3 - 15x^2 + 2x - 21$

$f'(-1) = 0$ ∴ $(x+1)$ is a factor

-1	4	-15	2	-21
	↓	-4	19	21
	4	-19	21	0 ☺

∴ $y' = (x+1)(4x^2 - 19x + 21)$
$= (x+1)(4x-7)(x-3)$

∴ crit. pts. $x = -1$ $x = 1.75$ $x = 3$
$y = -32$ $y \doteq 4.4$ $y = 0$

$y'' = 12x^2 - 30x + 2$
$y'' \doteq 12(x - 0.07)(x - 2.43)$ } using quad formula

∴ possible POInf $x \doteq 0.07$ $x \doteq 2.43$
$y = -16.5$ $y \doteq 2.1$

	−∞ to 0.07	0.07 to 2.43	2.43 to ∞
12	+	+	+
x−0.07	−	+	+
x−2.43	−	−	+
f″	+	−	+
f	CU	CD	CU

x=-1 here ∴ CU means local MIN

x=1.75 here CD means local MAX

x=3 here CU means local MIN

∴ at x = 0.07 and x = 2.43 actual POInf