Applications of Vectors Unit - Notes

Tentative TEST date_______________

Big idea/Learning Goals

Some of the topics in this unit overlap with physics. If you never took physics, refer to this page for key ideas you need know:

- speed is rate of change of ________________
- acceleration is rate of change of _______________
- the gravitational acceleration due to gravity on earth is ____________
- Newton's first law of motion states that if an object has _______________________________, then its acceleration is __________. (it is at rest or moving in a straight line with constant speed)
- Newton's second law of motion states a formula: ____________________

Corrections for the textbook answers:

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

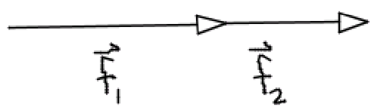
Date	pg	Topics	# of quest. done? You may be asked to show them
	2-3	Forces 7.1	
	4-5	Velocity 7.2	
	6-7	Dot Product (Geometric) 7.3	
	8-9	Dot Product (Algebraic) 7.4	
	10-11	Scalar and Vector Projections 7.5	
	12-14	Cross Product 7.6	
	15-16	Applications of Dot and Cross Products 7.7	
		Review	

Reflect – previous TEST mark ___________, Overall mark now___________.

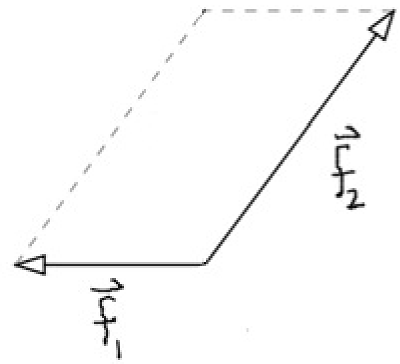
Force

1. When objects are at rest or move at a constant velocity in a straight line, then the forces that act upon the object cancel each other out. In other words, there is no NET force. The counteracting force is called the **equilibrant force**. Draw the resultant force and the corresponding equilibrant force for the following forces:

Collinear Forces at equilibrium



Coplanar Forces at equilibrium



2. Describe the forces that act on the object to keep it in a state of equilibrium.

a. aircraft flying at constant velocity b. an object hanging on two wires c. a box standing on a ramp

3. Jake and Maria are towing their friends on a toboggan. Jake is exerting a force of 65N and Maria a force of 60 N. Since they are walking side by side, the ropes pull to either side of the toboggan at 40^{o} to each other.
   a. Find the resultant force pulling the toboggan forward from a stop.
   b. Soon the toboggan is travelling at a constant speed. Find the equilibrant force and explain what it represents.

4. A large balloon is tethered to the top of a building by two wires attached at points 20m apart. If the buoyant force on the balloon is 850N, and the two wires make angles or 58^{o} and 66^{o} with the horizontal, find the tension in each of the wires.

5. A lawn mower is pushed with a force of 90N directed along the handle, which makes an angle of 36^{o} with the ground.
   a. Determine the horizontal and vertical components of the force on the mower.
   b. Describe the physical meaning of each component.

6. A 20kg trunk is resting on a ramp inclined at an angle of 15^{o}.
   a. Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp.
   b. Describe the physical meaning of each component.

Velocity

1. Josh can paddle at a speed of 5 km/h in still water. He wishes to cross a river 400m wide that has a current of 2km/h.
   a. If he steers the canoe in a direction perpendicular to the current, determine the resultant velocity. Find the point on the opposite bank where the canoe touches.
   b. If he wishes to travel straight across the river, determine the direction he must head and the time it will take him to cross the river.

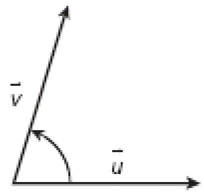
2. An airplane heading northwest at 500km/h encounters a wind of 120km/h from 65^{o} east of north. Determine the resultant ground velocity of the plane.

3. A car travelling east at 110km/h passes a truck going in the opposite direction at 95km/h.
   a. What is the velocity of the truck relative to the car?
   b. The truck turns onto a side road and heads northwest at the same speed. Now what is the velocity of the truck relative to the car?

Relative Velocity

4. A destroyer detects a submarine 8 nautical miles due east travelling northeast at 20knots. If the destroyer has a top speed of 30knots, at what heading should it travel to intercept the submarine?

Dot Product (Geometric)



Dot Product with Geometric Vectors

1. Find the dot product of $\vec{u}\bullet\vec{v}$ for each of the following where θ is the angle between vectors.
   a. $\left|\vec{u}\right|=7, \left|\vec{v}\right|=12, \theta=60^{o}$
   b. $\left|\vec{a}\right|=20, \left|\vec{b}\right|=3, \theta=\frac{5\pi}{6}$
2. For above question a. find $\vec{v}\bullet\vec{u}$. What is property you can conclude from this?
3. Find $\vec{a}\bullet\vec{a}$ and $\vec{b}\bullet\vec{b}$. What can conclude from this?
4. Find $\vec{u}\bullet\vec{0}$ and $\vec{v}\bullet\vec{0}$. What can conclude from this?
5. Prove that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular, if and only if $\vec{u}\bullet\vec{v}=0$.
6. Explain why $\left(\vec{a}\bullet\vec{b}\right)\bullet\vec{c}\neq\vec{a}\bullet\left(\vec{c}\bullet\vec{b}\right)$

Dot Product Properties

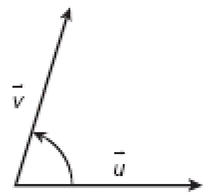
7. Prove the following distributive property: $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

8. Three vectors $\vec{x}, \vec{y}$ and $\vec{z}$ satisfy $\vec{x} + \vec{y} + \vec{z} = \vec{0}$. Calculate the value of $\vec{x} \bullet \vec{y} + \vec{y} \bullet \vec{z} + \vec{z} \bullet \vec{x}$, if $|\vec{x}| = 2, |\vec{y}| = 3$ and $|\vec{z}| = 4$

9. Given $\hat{a}$ and $\hat{b}$ unit vectors, if $|\hat{a} + \hat{b}| = \sqrt{3}$ find $(2\hat{a} - 5\hat{b}) \bullet (\hat{b} + 3\hat{a})$

10. Two vectors $2\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ are perpendicular. Find the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a}| = 2|\vec{b}|$.

Dot Product (Algebraic)



Dot Product with Algebraic Vectors

1. Find $(3\vec{a} + \vec{b}) \bullet (2\vec{b} - 4\vec{a})$, if $\vec{a} = -\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.

2. Find the angle between the following vectors $\vec{u} = (-3, 1, 2)$ and $\vec{v} = (5, -4, -1)$

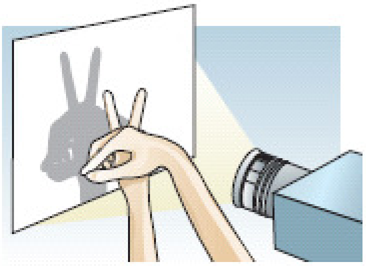
3. Given $\vec{a} = (2, 3, 7)$ and $\vec{b} = (-4, y, -14)$,

   a. for what value of y are the vectors collinear?

   b. for what value of y are the vectors perpendicular?

4. Find any vector $\vec{w}$ that is perpendicular to both $\vec{u} = 3\hat{j} + 4\hat{k}$ and $\vec{v} = 2\hat{i}$.
5. The vectors $\vec{a} = 3\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ are the diagonals of a parallelogram. Show that this parallelogram is a rhombus, and determine the lengths of the sides and the angles between the sides.
6. Find a unit vector that is parallel to the *xy*-plane and perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$.

Scalar & Vector Projections



Scalar Projection

of $\vec{a}$ *on* $\vec{b}$

of $\vec{b}$ *on* $\vec{a}$

Vector Projection of $\vec{a}$ *on* $\vec{b}$

1. a. Find the scalar and vector projections of $\vec{u}$ onto $\vec{v}$, if $\vec{u} = (5, 6, -3)$ and $\vec{v} = (1, 4, 5)$

   b. Find the scalar and vector projections of $\vec{v}$ onto. $\vec{u}$.

2. a. If $\vec{u}$ and $\vec{v}$ are non-zero vectors, but Proj($\vec{u}$ onto $\vec{v}$) $= \vec{0}$, what conclusion can be drawn?

   b. If Proj($\vec{u}$ onto $\vec{v}$) $= \vec{0}$, does it follow that Proj($\vec{v}$ onto $\vec{u}$) $= \vec{0}$? Explain.

3. Find the projection of $\overrightarrow{PQ}$ onto each of the coordinate axes, where P is the point (2, 3, 5) and Q is the point $(-1, 2, 5)$.

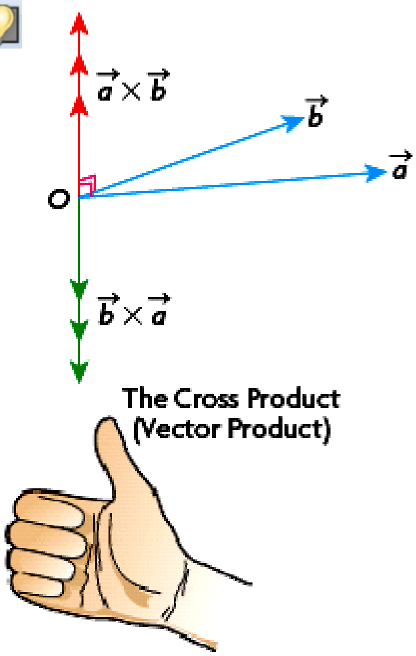
4. Under what circumstances is

   a. $\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \text{Proj}(\vec{v} \text{ onto } \vec{u})$?

   b. $\left|\text{Proj}(\vec{u} \text{ onto } \vec{v})\right| = \left|\text{Proj}(\vec{v} \text{ onto } \vec{u})\right|$?

5. The direction angles of a vector are all equal. Find the direction angles to the nearest degree.

Cross Product



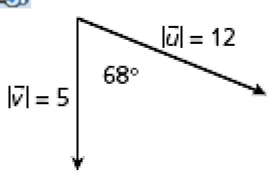
Cross Product

algebraic

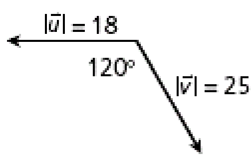
geometric

1. Find $|\vec{u} \times \vec{v}|$ for each of the following pairs of vectors. State whether $\vec{u} \times \vec{v}$ is directed into or out of the page.

a.



b.



2. State whether the following expressions are vectors, scalars, or meaningless

a. $\vec{a} \cdot (\vec{b} \times \vec{c})$
b. $(\vec{a} \cdot \vec{b}) \times (\vec{b} \cdot \vec{c})$
c. $(\vec{a} + \vec{b}) \cdot \vec{c}$
d. $\vec{a} \times (\vec{b} \cdot \vec{c})$
e. $(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c})$
f. $(\vec{a} + \vec{b}) \times \vec{c}$
g. $\vec{a} \cdot (\vec{b} \cdot \vec{c})$
h. $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c})$
i. $(\vec{a} \times \vec{b}) - \vec{c}$
j. $\vec{a} \times (\vec{b} \times \vec{c})$
k. $(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{c})$
l. $(\vec{a} \cdot \vec{b}) - \vec{c}$

3\. A. Given two vectors $\vec{a} = (2, 4, 6)$ and $\vec{b} = (-1, 2, -5)$ calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

B. Using the two vectors given in part A and a third vector $\vec{c} = (4, 3, -1)$ calculate:

i. $\vec{a} \times (\vec{b} + \vec{c})$ ii. $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

C. Compare your results in part B. What property does this demonstrate?

D. Using the three vectors given calculate:

i. $(\vec{a} \times \vec{b}) \times \vec{c}$ ii. $\vec{a} \times (\vec{b} \times \vec{c})$

E. What property does this demonstrate does NOT hold for the cross product? Explain.

4. Prove that the triple scalar product of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has the property that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$. Carry out the proof by expressing both sides of the equation in terms of components of the vectors.

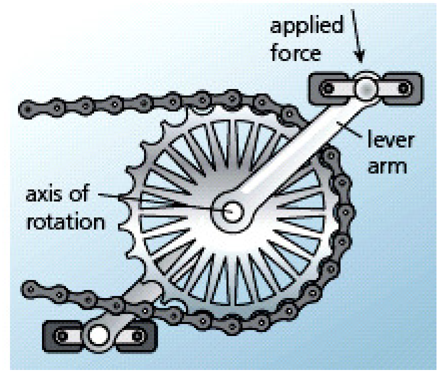
Cross Product Properties

5. Prove that $|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$.

6. a. If $\vec{a} = (1, 3, -1)$, $\vec{b} = (2, 1, 5)$, $\vec{v} = (-3, y, z)$, and $\vec{a} \times \vec{v} = \vec{b}$, find y and z.

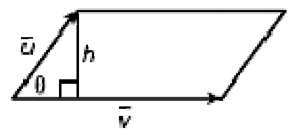
b. Explain why there are infinitely many vectors $\vec{c}$ for which $\vec{a} \times \vec{c} = \vec{b}$.

Applications of Dot & Cross Products

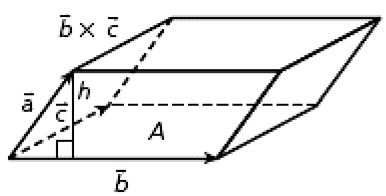


Work	*Area of ||gm*	*Torque*

1. Find the area of the triangle with vertices $P(7, 2, -5)$, $Q(9, -1, -6)$, and $R(7, 3, -3)$.



2. A **parallelepiped** is a box-like solid, the opposite faces of which are parallel and congruent parallelograms. Its edges are three non-coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.
Find the volume formula for the parallelepiped.



3. a. A 25-kg box is located 8 m up a ramp inclined at an angle of 18° to the horizontal. Determine the work done by the force of gravity as the box slides to the bottom of the ramp.

   b. Determine the minimum force, acting at an angle of 40° to the horizontal, required to slide the box back up the ramp. (Ignore friction.)

4. Find the torque produced by a cyclist exerting a force of 115 N on a pedal in the position shown in the diagram, if the shaft of the pedal is 16 cm long.