p1NOTES

February-06-13
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AppVectors NotesNEW

Inserted from: <file://C:\Users\MrsK\Desktop\LacieOct9\2 Math\Math 12\MCB 4U Calc Vect\2013\2 App of Vectors\AppVectorsNotesNEW.doc>

to do:

☐ more space on page 7.

☐ p. 4 #1 more space

☐ p. 5 #3 " "

p. 9 more space if.

☐ p. 9 need to prove equal sides? (did in night school)

p. 16 #3 more sp.

see below

Applications of Vectors Unit - Notes

Tentative TEST date Mon. Mar.4

Big idea/Learning Goals

Some of the topics in this unit overlap with physics. If you never took physics, refer to this page for key ideas you need know:

- speed is rate of change of distance
- acceleration is rate of change of velocity/speed
- the gravitational acceleration due to gravity on earth is $9.8\ m/s^2$
- Newton's first law of motion states that if an object has no net force, then its acceleration is zero. (it is at rest or moving in a straight line with constant speed)
- Newton's second law of motion states a formula: $F = ma$

Corrections for the textbook answers:

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

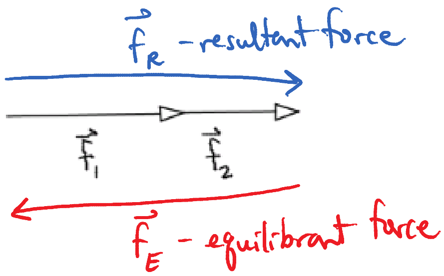
Date	pg	Topics	# of quest. done? You may be asked to show them
	2-3	Forces 7.1	
	4-5	Velocity 7.2	
	6-7	Dot Product (Geometric) 7.3	
	8-9	Dot Product (Algebraic) 7.4	
M 25	10-11	Scalar and Vector Projections 7.5	
T 26	12-14	Cross Product 7.6	
W 27	15-16	Applications of Dot and Cross Products 7.7	
Th 28		Review	

Reflect – previous TEST mark __________, Overall mark now__________.

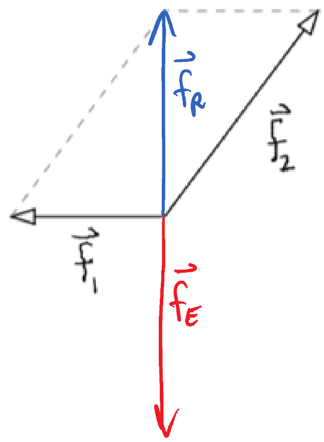
Force

1. When objects are at rest or move at a constant velocity in a straight line, then the forces that act upon the object cancel each other out. In other words, there is no NET force. The counteracting force is called the **equilibrant force**. Draw the resultant force and the corresponding equilibrant force for the following forces:

Collinear Forces at equilibrium

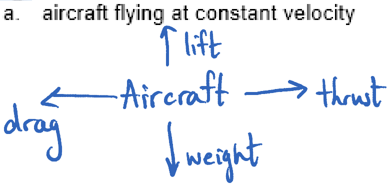


Coplanar Forces at equilibrium

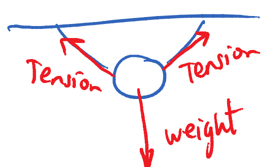


2. Describe the forces that act on the object to keep it in a state of equilibrium.

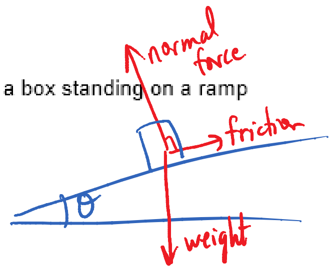
a. aircraft flying at constant velocity



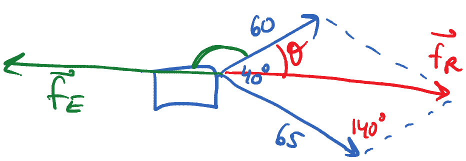
b. an object hanging on two wires



c. a box standing on a ramp



3. Jake and Maria are towing their friends on a toboggan. Jake is exerting a force of 65N and Maria a force of 60 N. Since they are walking side by side, the ropes pull to either side of the toboggan at 40° to each other.
   a. Find the resultant force pulling the toboggan forward from a stop. mag + dir!
   b. Soon the toboggan is travelling at a constant speed. Find the equilibrant force and explain what it represents.



ⓐ $|\vec{f}_R|^2 = 65^2 + 60^2 - 2(65)(60)\cos 140°$

$|\vec{f}_R| \doteq 117.5\text{ N}$

$\frac{\sin\theta}{65} = \frac{\sin 140°}{117.5}$

$\theta = 21°$

$\vec{f}_R = 117.5\text{ N}$ [21° off of 60N force]

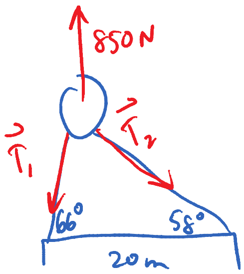
180 − 21°

ⓑ $\vec{f}_E = 117.5\text{ N}$ [159° off of 60N force]

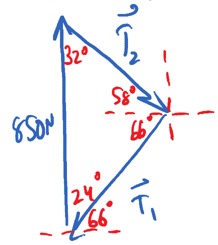
Newtons = $kg \cdot m/s^2$

4. A large balloon is tethered to the top of a building by two wires attached at points 20m apart. If the buoyant force on the balloon is 850N, and the two wires make angles or 58° and 66° with the horizontal, find the tension in each of the wires.

Real Life Pic



Free Body Diagram



$\frac{|\vec{T}_2|}{\sin 24°} = \frac{850}{\sin 124°}$

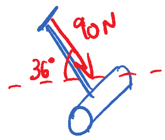
$|\vec{T}_2| \doteq 417\text{ N}$

$\frac{|\vec{T}_1|}{\sin 32} = \frac{850}{\sin 124°}$

$|\vec{T}_1| = 543\text{ N}$

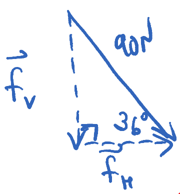
∴ the magnitudes of tension in the wires are 417N and 543 N.

5. A lawn mower is pushed with a force of 90N directed along the handle, which makes an angle of 36° with the ground.
   a. Determine the horizontal and vertical components of the force on the mower.
   b. Describe the physical meaning of each component.



ⓐ $|\vec{f}_V| = 90 \sin 36° = 53\text{N}$

$|\vec{f}_H| = 90 \cos 36° = 73\text{N}$

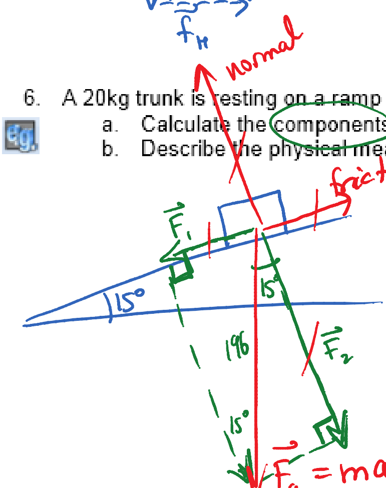


ⓑ $\vec{f}_H$ accelerates the lawn mower horizontally (may be counteracted by friction if go at constant speed)

$\vec{f}_V$ wasted force, counteracted by the normal force of the ground.

6. A 20kg trunk is resting on a ramp inclined at an angle of 15°.
   a. Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp.
   b. Describe the physical meaning of each component.

magnitudes



$\vec{F}_g = ma$

$= (20kg)(9.8 m/s^2)$

$= 196\text{ N}$

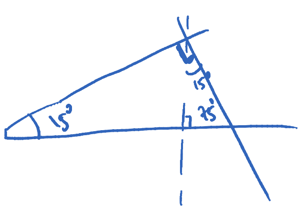
ⓐ $|\vec{F}_1| = 196 \sin 15° \doteq 51\text{ N}$

SOH CAH TOA

$\sin 15° = \frac{|\vec{F}_1|}{196}$

$\cos 15° = \frac{|\vec{F}_2|}{196}$

$|\vec{F}_2| = 196 \cos 15° \doteq 189\text{ N}$

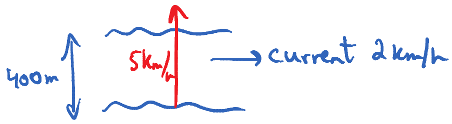


ⓑ $\vec{F}_1$ force is counteracted by frictional force

$\vec{F}_2$ is counteracted by the normal force (ramp pushes back on the box)

Velocity

1. Josh can paddle at a speed of 5 km/h in still water. He wishes to cross a river 400m wide that has a current of 2km/h.
   a. If he steers the canoe in a direction perpendicular to the current, determine the resultant velocity. Find the point on the opposite bank where the canoe touches.
   b. If he wishes to travel straight across the river, determine the direction he must head and the time it will take him to cross the river.



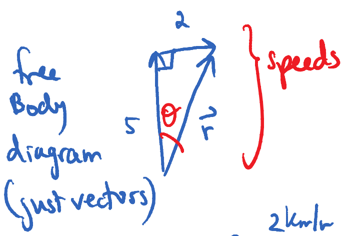
ⓐ $|\vec{r}| = \sqrt{5^2+2^2}$

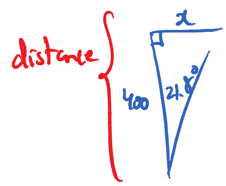
$= \sqrt{29} \sim 5.4\ km/h$

$\theta = \tan^{-1}\left(\frac{2}{5}\right)$

$\theta \doteq 21.8°$

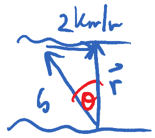
$\therefore \vec{r} = \sqrt{29}\ km/h$ [21.8° off of his original direction]





$\tan 21.8° = \frac{x}{400}$

$160m = x$ ∴ Josh touches at 160m downstream from his starting pt. on the opposite side

ⓑ


$|\vec{r}| = \sqrt{5^2-2^2}$

$= \sqrt{21} \sim 4.6\ km/h$

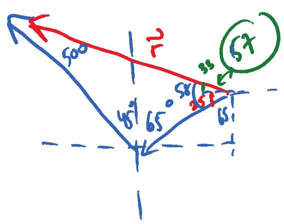
$\sin\theta = \frac{2}{5}$

$\theta = 23.6°$ ∴ he must head 23.6° up the stream off of the perpendicular to the current.

$T = \frac{D}{V} = \frac{0.4\ km}{\sqrt{21}} \sim 0.087\ hr$ or 5.2 min

2. An airplane heading northwest at 500km/h encounters a wind of 120km/h from 65° east of north. Determine the resultant ground velocity of the plane.

N65°E



$|\vec{r}|^2 = 500^2 + 120^2 - 2(500)(120)\cos 110°$

$\boxed{|\vec{r}| = 553}$

$\frac{\sin 110}{553} = \frac{\sin\theta}{500}$

$\boxed{\theta = 58.2°}$

$\vec{r} = 553\ km/h$ [N 56.8 W]

[W 33.2° N]

$\vec{v}_{rel} = 95\,[W] - 110\,[E]$
$= 95\,[W] - -110\,[W] = 205\,[W]$

3. A car travelling east at 110km/h passes a truck going in the opposite direction at 95km/h.
   a. What is the velocity of the truck relative to the car?
   b. The truck turns onto a side road and heads northwest at the same speed. Now what is the velocity of the truck relative to the car?

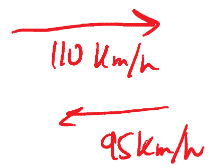
Relative Velocity

relative velocity of B to A $\boxed{\vec{V}_{rel} = \vec{V}_B - \vec{V}_A}$

ⓐ $|\vec{V}_{rel}| = |\vec{V}_{truck}| - |\vec{V}_{car}|$

$= 95 - (-110)$

$= 205$ km/h



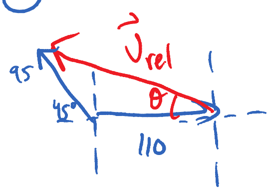
choose a positive direction as the truck

Since the vectors are collinear

$|\vec{V}_{truck} - \vec{V}_{car}| = |\vec{V}_{truck}| - |\vec{V}_{car}|$

p.6 Unit booklet

ⓑ



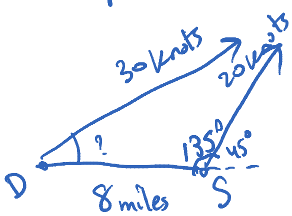
$|\vec{v}_{rel}|^2 = |\vec{V}_{truck} - \vec{V}_{car}|^2 = 95^2 + 110^2 - 2(95)(110)\cos 135°$

$|\vec{v}_{rel}| \doteq 189.5$

$\frac{\sin\theta}{95} = \frac{\sin 135°}{189.5}$ $\quad\theta \doteq 20.8°$ $\quad\therefore \vec{v}_{rel} = 189.5$ km/h $[N69.2°W]$

4. A destroyer detects a submarine 8 nautical miles due east travelling northeast at 20knots. If the destroyer has a top speed of 30knots at what heading should it travel to intercept the submarine?

speed distance



$\frac{\sin\theta}{20} = \frac{\sin 135°}{30}$

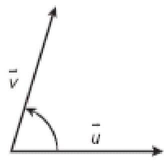
$\theta = 28.1°$

$\therefore$ should travel in direction

N 61.9° E

Two ways to multiply vectors: Dot product $\vec{u}\cdot\vec{v}$ = scalar; Cross product $\vec{u}\times\vec{v}$ = vector

Dot Product (Geometric)



Dot Product with Geometric Vectors

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ ← angle between tails $0° \le \theta \le 180°$

1. Find the dot product of $\vec{u}\bullet\vec{v}$ for each of the following where θ is the angle between vectors.
   a. $|\vec{u}| = 7, |\vec{v}| = 12, \theta = 60°$
   b. $|\vec{a}| = 20, |\vec{b}| = 3, \theta = \frac{5\pi}{6}$
2. For above question a. find $\vec{v}\bullet\vec{u}$. What is property you can conclude from this?
3. Find $\vec{a}\bullet\vec{a}$ and $\vec{b}\bullet\vec{b}$. What can conclude from this?
4. Find $\vec{u}\bullet\vec{0}$ and $\vec{v}\bullet\vec{0}$. What can conclude from this?
5. Prove that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u}\bullet\vec{v} = 0$. must prove both ways.
6. Explain why $(\vec{a}\bullet\vec{b})\bullet\vec{c} \ne \vec{a}\bullet(\vec{c}\bullet\vec{b})$

1a) $\vec{u}\cdot\vec{v} = (7)(12)\cos 60° = 42$

b) try yourself
leave exact answer $-30\sqrt{3}$

2. $\vec{v}\cdot\vec{u} = (12)(7)\cos 60° = 42$
$\therefore \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
ie. dot product is COMMUTATIVE

3. $\vec{a}\cdot\vec{a} = (20)(20)\cos 0°$ ← angle between $\vec{a}$ and itself.
$= 400$
$\vec{b}\cdot\vec{b} = 9$ ie. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

4. $\vec{u}\cdot\vec{0} = (7)(0)\cos\theta = 0$
$\vec{v}\cdot\vec{0} = (12)(0)\cos\theta = 0$ ← always zero scalar #.

5. proof ⇒
assume $\vec{u} \perp \vec{v}$ then $\theta = 90°$
then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos 90°$
$= |\vec{u}||\vec{v}|(0)$
$= 0$

proof ⇐
assume $\vec{u}\cdot\vec{v} = 0$
then $|\vec{u}||\vec{v}|\cos\theta = 0$
$\cos\theta = 0$ ← since $\vec{u}, \vec{v}$ are non zero vectors
$\theta = 90°$
ie. $\vec{u}$ and $\vec{v}$ have an angle of $90°$ between.

Dot Product Properties

1. $\vec{u}\bullet\vec{v} = 0$ iff $\vec{u} \perp \vec{v}$
2. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ commutative
3. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
4. $\vec{u}\cdot\vec{0} = 0$ (scalar)
5. $\vec{u}\cdot(\vec{a}+\vec{b}) = \vec{u}\cdot\vec{a} + \vec{u}\cdot\vec{b}$ distributive
6. $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$

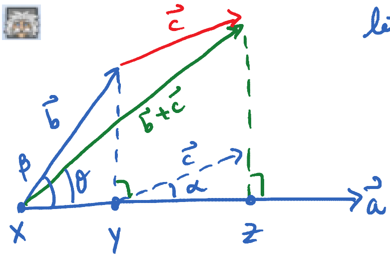
6. $\underbrace{(\vec{a}\cdot\vec{b})}_{\text{scalar \#}}\cdot\vec{c} \ne \vec{a}\cdot(\vec{c}\cdot\vec{b})$ ← ie. NOT ASSOCIATIVE!
can't do scalar # dot product with vector only regular multiplication.

Sorry :(MAY need extra page - if you write big to attach to these notes.

LS RS

LS RS

7. Prove the following distributive property: $\vec{a}\bullet(\vec{b}+\vec{c})=\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$



let α be angle between $\vec{a}$ and $\vec{c}$
β " " " $\vec{a}$ and $\vec{b}$
θ " " " $\vec{a}$ and $\vec{b}+\vec{c}$

RS: $\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$

$=\underbrace{|\vec{a}||\vec{b}|\cos\beta}_{XY}+|\vec{a}|\underbrace{|\vec{c}|\cos\alpha}_{YZ}$ by CAH rearranged

split vector $\vec{a}$ with points X, Y, Z so that it shows the altitude of the head of vectors $\vec{b}$ and $\vec{b}+\vec{c}$

$=|\vec{a}|XY+|\vec{a}|YZ$

$=|\vec{a}|(XY+YZ)=|\vec{a}|XZ=|\vec{a}||\vec{b}+\vec{c}|\cos\theta=\vec{a}\bullet(\vec{b}+\vec{c})$ $=$ LS DONE :)

again by CAH

8. Three vectors $\vec{x},\vec{y}$ and $\vec{z}$ satisfy $\vec{x}+\vec{y}+\vec{z}=\vec{0}$. Calculate the value of $\vec{x}\bullet\vec{y}+\vec{y}\bullet\vec{z}+\vec{z}\bullet\vec{x}$, if $|\vec{x}|=2, |\vec{y}|=3$ and $|\vec{z}|=4$

sub in three times

let A represent this scalar

then $A=\vec{x}\bullet(-\vec{x}-\vec{z})+\vec{y}\bullet(-\vec{y}-\vec{x})+\vec{z}\bullet(-\vec{y}-\vec{z})$

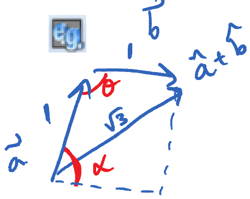
$A=-\vec{x}\bullet\vec{x}-\vec{x}\bullet\vec{z}-\vec{y}\bullet\vec{y}-\vec{x}\bullet\vec{y}-\vec{y}\bullet\vec{z}-\vec{z}\bullet\vec{z}$

$A+\underbrace{\vec{x}\bullet\vec{y}+\vec{x}\bullet\vec{z}+\vec{y}\bullet\vec{z}}_{A}=-(|\vec{x}|^2+|\vec{y}|^2+|\vec{z}|^2)$

$\therefore 2A=-(2^2+3^2+4^2)$

$A=-\frac{29}{2}$

9. Given $\hat{a}$ and $\hat{b}$ unit vectors, if $|\hat{a}+\hat{b}|=\sqrt{3}$ find $(2\hat{a}-5\hat{b})\bullet(\hat{b}+3\hat{a})$



α is tail to tail

$\cos\theta=\frac{1^2+1^2-(\sqrt{3})^2}{2(1)(1)}$

$\theta=120°$

$\therefore \alpha=60°$

FOIL

$=2\hat{a}\bullet\hat{b}+6\hat{a}\bullet\hat{a}-5\hat{b}\bullet\hat{b}-15\hat{b}\bullet\hat{a}$

$=-13\hat{a}\bullet\hat{b}+6|\hat{a}|^2-5|\hat{b}|^2$

$=-13|\hat{a}||\hat{b}|\cos\alpha+6(1)^2-5(1)^2$

$=-13(1)(1)\cos 60°+1$

$=-13\left(\frac{1}{2}\right)+1=-\frac{11}{2}$

10. Two vectors $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ are perpendicular. Find the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a}|=2|\vec{b}|$.

try yourself. Hint $(2\vec{a}+\vec{b})\bullet(\vec{a}-3\vec{b})=0$ since perpendicular

$2\vec{a}\bullet\vec{a}-6\vec{a}\bullet\vec{b}+\vec{b}\bullet\vec{a}-3\vec{b}\bullet\vec{b}=0$

$2|\vec{a}|^2-5\vec{a}\bullet\vec{b}-3|\vec{b}|^2=0$

$2|\vec{a}|^2-5|\vec{a}||\vec{b}|\cos\theta-3|\vec{b}|^2=0$

between $\vec{a}$ and $\vec{b}$

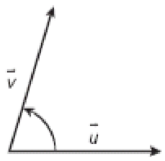
$2(2|\vec{b}|)^2-5(2|\vec{b}|)|\vec{b}|\cos\theta-3|\vec{b}|^2=0$

$-10|\vec{b}|^2\cos\theta=-5|\vec{b}|^2$

$\cos\theta=\frac{1}{2}$

ANS: $\theta=60°$

Dot Product (Algebraic)



Dot Product with Algebraic Vectors

$\vec{u} = (a_1, b_1, c_1)$ $\vec{v} = (a_2, b_2, c_2)$

$\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2 + c_1c_2$

1. Find $(3\vec{a} + \vec{b}) \cdot (2\vec{b} - 4\vec{a})$, if $\vec{a} = -\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.
2. Find the angle between the following vectors $\vec{u} = (-3, 1, 2)$ and $\vec{v} = (5, -4, -1)$
3. Given $\vec{a} = (2, 3, 7)$ and $\vec{b} = (-4, y, -14)$,
   a. for what value of y are the vectors collinear?
   b. for what value of y are the vectors perpendicular?

1. $(3\vec{a} + \vec{b}) \cdot (2\vec{b} - 4\vec{a})$

$\vec{a} = (-1, -3, 1) \rightarrow 3\vec{a} = (-3, -9, 3)$

$\vec{b} = (2, 4, -5) \rightarrow 2\vec{b} = (4, 8, -10)$

$4\vec{a} = (-4, -12, 4)$

$= (-1, -5, -2) \cdot (8, 20, -14)$

$= (-1)(8) + -5(20) + -2(-14)$

$= -8 - 100 + 28$

$= -80$

2. $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$(-3, 1, 2) \cdot (5, -4, -1) = \sqrt{3^2 + 1^2 + 2^2}\sqrt{5^2 + 4^2 + 1^2}\cos\theta$

$-3(5) + 1(-4) + 2(-1) = \sqrt{14}\sqrt{42}\cos\theta$

$-21 = \sqrt{588}\cos\theta$

$\frac{-21}{\sqrt{588}} = \cos\theta$

$150° = \theta$

3. a) $y = -6$

$\vec{a}$ collinear to $\vec{b}$ if $\vec{a} = k\vec{b}$

$(2, 3, 7) = k(-4, y, -14)$

$2 = -4k$ $3 = ky$

$-\frac{1}{2} = k$ $3 = -\frac{1}{2}y$

$-6 = y$

3(b) $\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$

$(2, 3, 7) \cdot (-4, y, -14) = 0$

$-8 + 3y - 98 = 0$

$3y = 106$

$y = \frac{106}{3}$

4. Find any vector $\vec{w}$ that is perpendicular to both $\vec{u} = 3\hat{j} + 4\hat{k}$ and $\vec{v} = 2\hat{i}$.

5. The vectors $\vec{a} = 3\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ are the diagonals of a parallelogram. Show that this parallelogram is a rhombus, and determine the lengths of the sides and the angles between the sides.

6. Find a unit vector that is parallel to the xy-plane and perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$.

4. $\vec{w} = (a, b, c)$ $\vec{u} = (0, 3, 4)$ $\vec{v} = (2, 0, 0)$

$\vec{w} \perp \vec{u}$ then $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0 \leftarrow \vec{w} \perp \vec{v}$

$(a, b, c) \cdot (0, 3, 4) = 0$ $(a, b, c) \cdot (2, 0, 0) = 0$

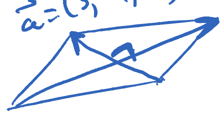
$\cancel{0a} + 3b + 4c = 0$ $2a + \cancel{0b} + \cancel{0c} = 0$

let $b = 2$ anything you wish. $\boxed{a = 0}$

then $3(2) + 4c = 0$

$c = \frac{-3}{2}$ $\therefore \vec{w} = (0, 2, -3/2)$ many other answers depends if you choose another value for b (or c)

5. $\vec{a} = (3, -4, -1)$ $\vec{b} = (2, 3, -6)$



rhombus - sides are equal
- diagonals will meet at 90°

if $\vec{a} \cdot \vec{b} = 0$ then it will be a rhombus

$(3, -4, -1) \cdot (2, 3, -6) \overset{?}{=} 0$

$6 - 12 + 6$

0 ✓

6. let $\hat{v} = (a, b, c)$ be a unit vector

$|\hat{v}| = 1$

$\sqrt{a^2 + b^2 + \cancel{c^2}} = 1$

$c = 0$ since vector is parallel to xy plane.

$\hat{v} \cdot (4, -3, 1) = 0$ for them to be $\perp$

square. $(a, b, 0) \cdot (4, -3, 1) = 0$

$4a - 3b = 0$ $a = \frac{3}{4}b$

$a^2 + b^2 = 1$

$\left(\frac{3}{4}b\right)^2 + b^2 = 1$

$\frac{9}{16}b^2 + \frac{1}{1}b^2 = 1$

$\frac{25}{16}b^2 = 1$

$b^2 = \frac{16}{25}$

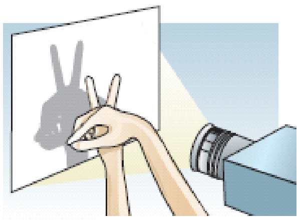
$b = \pm\frac{4}{5}$ $a = \pm\frac{3}{5}$

$\therefore \hat{v} = \left(+\frac{3}{5}, \frac{4}{5}, 0\right)$

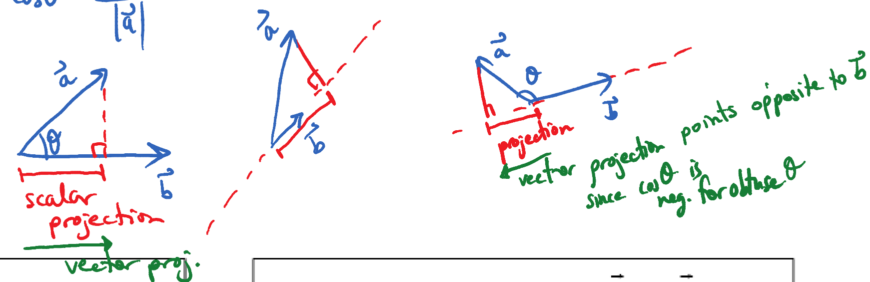
or

$\hat{v} = \left(-\frac{3}{5}, -\frac{4}{5}, 0\right)$

Scalar & Vector Projections



$\cos\theta = \frac{\text{shadow}}{|\vec{a}|}$



Scalar Projection

of $\vec{a}$ on $\vec{b}$ = $|proj(\vec{a} \text{ on } \vec{b})| = |\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

of $\vec{b}$ on $\vec{a}$ = $|proj(\vec{b} \text{ on } \vec{a})| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{a}$ on $\vec{b}$

in the direction of $\vec{b}$

$proj(\vec{a} \text{ on } \vec{b}) = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}_{\text{mag.}} \underbrace{\hat{b}}_{\text{dir}}$

$= \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$ OR $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$

1. a. Find the scalar and vector projections of $\vec{u}$ onto $\vec{v}$, if $\vec{u} = (5, 6, -3)$ and $\vec{v} = (1, 4, 5)$
   b. Find the scalar and vector projections of $\vec{v}$ onto. $\vec{u}$.

2. List in Journal
   a. If $\vec{u}$ and $\vec{v}$ are non-zero vectors, but Proj($\vec{u}$ onto $\vec{v}$) = $\vec{0}$, what conclusion can be drawn?
   b. If Proj($\vec{u}$ onto $\vec{v}$) = $\vec{0}$, does it follow that Proj($\vec{v}$ onto $\vec{u}$) = $\vec{0}$? Explain.

if it tells you to rationalize

(a) $|proj(\vec{u} \text{ on } \vec{v})| = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{5(1)+6(4)+-3(5)}{\sqrt{1^2+4^2+5^2}} = \frac{14}{\sqrt{42}} \times \frac{\sqrt{42}}{\sqrt{42}} = \frac{14\sqrt{42}}{42} = \frac{1}{3}\sqrt{42}$

$proj(\vec{u} \text{ on } \vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{14}{42}(1,4,5) = \frac{1}{3}(1,4,5) = \left(\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right)$

(b) $|proj(\vec{v} \text{ on } \vec{u})| = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \frac{1(5)+4(6)+(-3)(5)}{\sqrt{5^2+6^2+3^2}} = \frac{14}{\sqrt{70}}$

$proj(\vec{v} \text{ on } \vec{u}) = \frac{(\vec{v}\cdot\vec{u})}{(\vec{u}\cdot\vec{u})}\vec{u} = \frac{14}{70}(5,6,-3) = \frac{1}{5}(5,6,-3)$

2. (a) if $proj(\vec{u} \text{ on } \vec{v}) = \vec{0}$

$(|\vec{u}|\cos\theta)\hat{v} = \vec{0}$

$|\vec{u}| \neq 0$ $|\vec{v}| \neq 0$ $\therefore \cos\theta = 0$

$\theta = 90°$

(b) Yes. since if $\vec{u} \perp \vec{v}$ then $\vec{v} \perp \vec{u}$

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3. Find the projection of $\overrightarrow{PQ}$ onto each of the coordinate axes, where P is the point $(2, 3, 5)$ and Q is the point $(-1, 2, 5)$.

4. Under what circumstances is

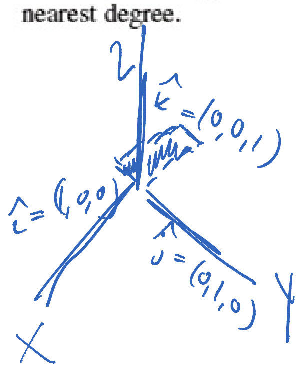
List in Journal

a. $\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \text{Proj}(\vec{v} \text{ onto } \vec{u})$?

b. $|\text{Proj}(\vec{u} \text{ onto } \vec{v})| = |\text{Proj}(\vec{v} \text{ onto } \vec{u})|$?

5. The direction angles of a vector are all equal. Find the direction angles to the nearest degree.

3.



$\overrightarrow{PQ} = -1-2,\ 2-3,\ 5-5$

$\overrightarrow{PQ} = (-3, -1, 0)$

$\text{proj}(\overrightarrow{PQ} \text{ on } \hat{i}) = \dfrac{\overrightarrow{PQ} \cdot \hat{i}}{\hat{i} \cdot \hat{i}}\hat{i} = \dfrac{(-3)(1)+0+0}{1}(1,0,0) = (-3,0,0)$

$\text{proj}(\overrightarrow{PQ} \text{ on } \hat{j}) = (0,-1,0)$

$\text{proj}(\overrightarrow{PQ} \text{ on } \hat{k}) = (0,0,0)$

4 a) $\text{Proj}(\vec{u} \text{ on } \vec{v}) = \text{Proj}(\vec{v} \text{ on } \vec{u})$

$\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$

$\dfrac{1}{|\vec{v}|^2}\vec{v} = \dfrac{1}{|\vec{u}|^2}\vec{u}$

$\vec{v} = \dfrac{|\vec{v}|^2}{|\vec{u}|^2}\vec{u}$

$\vec{v} = k\vec{u}$

$\therefore \vec{v}, \vec{u}$ are collinear.

also

shadows won't be equal in size unless magnitudes are also equal

b) $|\text{Proj}(\vec{u} \text{ on } \vec{v})| = |\text{Proj}(\vec{v} \text{ on } \vec{u})|$

$|\vec{u}|\cos\theta = |\vec{v}|\cos\theta$

$\therefore |\vec{u}| = |\vec{v}|$

$|\vec{u}|\cos\theta\,\hat{v} = |\vec{v}|\cos\theta\,\hat{u}$

$|\vec{u}|\,\hat{v} = |\vec{v}|\,\hat{u}$

remove dir. since collinear

$\therefore |\vec{u}| = |\vec{v}|$ and collinear

$\therefore \vec{u} = \vec{v}$ same vectors!

5. $\overrightarrow{OP} = (a,b,c)$

$\alpha = \beta = \gamma$

$\cos\alpha = \cos\beta = \cos\gamma$

$\dfrac{a}{|\overrightarrow{OP}|} = \dfrac{b}{|\overrightarrow{OP}|} = \dfrac{c}{|\overrightarrow{OP}|} \quad \therefore a=b=c$

unit vector $= (\cos\alpha, \cos\beta, \cos\gamma)$

$|\hat{u}| = 1$

$\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$

$\sqrt{3\cos^2\alpha} = 1$

$3\cos^2\alpha = 1$

$\cos^2\alpha = 1/3$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\alpha = 55°$

~~or 125°~~ must be in 1st quadrant

$(a,b,c) = (a,a,a) = a(1,1,1)$

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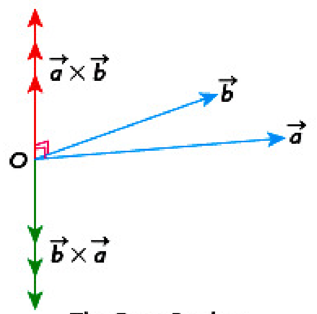
Cross Product

↘ minus ↗



The Cross Product (Vector Product)

Right Hand Rule
connect head of 1st to tail of 2nd vector
then follow the curvature with your right hand – thumb will point where the normal is.

Cross Product

algebraic

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_1 & b_2 & b_3 \end{matrix}$$

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

geometric

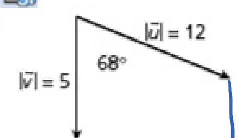
$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$$

θ: between two tails.

$\hat{n}$: normal (perpendicular) to both $\vec{a}$ and $\vec{b}$

1. Find $|\vec{u}\times\vec{v}|$ for each of the following pairs of vectors. State whether $\vec{u}\times\vec{v}$ is directed into or out of the page.

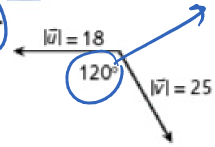
a.



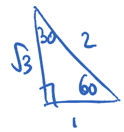
$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$
$$= (12)(5)\sin 68°$$
$$\doteq 55.6$$

$\therefore \vec{u}\times\vec{v} = 55.6$ units [into the page]

b.



Special $\therefore$ use exact values.



$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$
$$= (18)(25)\sin 120°$$
$$= 450\left(\frac{\sqrt{3}}{2}\right)$$

$\therefore \vec{u}\times\vec{v} = 225\sqrt{3}$ units [out of page]

2. State whether the following expressions are vectors, scalars, or meaningless

a. $\vec{a}\bullet(\vec{b}\times\vec{c})$ scalar
b. $(\vec{a}\bullet\vec{b})\times(\vec{b}\bullet\vec{c})$ meaningless
c. $(\vec{a}+\vec{b})\bullet\vec{c}$ scalar
d. $\vec{a}\times(\vec{b}\bullet\vec{c})$ meaningless
e. $(\vec{a}\times\vec{b})\bullet(\vec{b}\times\vec{c})$ scalar
f. $(\vec{a}+\vec{b})\times\vec{c}$ vector
g. $\vec{a}\bullet(\vec{b}\bullet\vec{c})$ meaningless
h. $(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})$ vector
i. $(\vec{a}\times\vec{b})-\vec{c}$ vector
j. $\vec{a}\times(\vec{b}\times\vec{c})$ vector
k. $(\vec{a}\bullet\vec{b})+(\vec{b}\bullet\vec{c})$ scalar
l. $(\vec{a}\bullet\vec{b})-\vec{c}$ meaningless

3. A. Given two vectors $\vec{a} = (2, 4, 6)$ and $\vec{b} = (-1, 2, -5)$ calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

B. Using the two vectors given in part A and a third vector $\vec{c} = (4, 3, -1)$ calculate: $(3, 5, -6)$

i. $\vec{a} \times (\vec{b} + \vec{c})$ ii. $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

C. Compare your results in part B. What property does this demonstrate?

D. Using the three vectors given calculate:

i. $(\vec{a} \times \vec{b}) \times \vec{c}$ $(-32, 4, 8)$ ii. $\vec{a} \times (\vec{b} \times \vec{c})$

E. What property does this demonstrate does NOT hold for the cross product? Explain.

do at home + compare with online

3A.)

$\vec{a} \times \vec{b} = (-20-12, -6+10, 4+4)$
$= (-32, 4, 8)$

$\vec{b} \times \vec{a} = (12+20, -10+6, -4-4)$
$= (32, -4, -8)$

since $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ NOT commutative

$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$

B.)

i) $\vec{a} \times (\vec{b} + \vec{c}) = (-24-30, 18+12, 10-12)$
$= (-54, 30, -2)$

ii) $\vec{a} \times \vec{c} = (-4-18, 24+2, 6-16)$
$= (-22, 26, -10)$

$\therefore \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
$= (-32, 4, 8) + (-22, 26, -10)$
$= (-54, 30, -2)$

same

C.) $\therefore$ distributive

D.)

$(\vec{a} \times \vec{b}) \times \vec{c} = (-4-24, 32-32, -96-16)$
$= (-28, 0, -112)$

$\vec{b} \times \vec{c} = (-2+15, -20-1, -3-8)$
$= (13, -21, -11)$

E.) not associative

$\vec{a} \times (\vec{b} \times \vec{c}) = (-44+126, 78+22, -42-52)$
$= (82, 100, -94)$

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$$\begin{matrix} v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 & w_1 & w_2 & w_3 \end{matrix} \qquad \begin{matrix} u_1 & u_2 & u_3 & u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \end{matrix}$$

4. Prove that the triple scalar product of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has the property that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$. Carry out the proof by expressing both sides of the equation in terms of components of the vectors.

LS $(u_1, u_2, u_3) \cdot (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$

$u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1$

RS $(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) \cdot (w_1, w_2, w_3)$

$u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3$

LS = RS

Cross Product Properties

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ not commutative
2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ distributive
   watch the order!
3. $k(\vec{a} \times \vec{b}) = \vec{a} \times k\vec{b} = k\vec{a} \times \vec{b}$
4. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$ not associative
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
   TRIPLE scalar product
   used to prove coplanar if equals zero
6. $\vec{a} \times \vec{a} = \vec{0}$

5. Prove that $|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$.

6. a. If $\vec{a} = (1, 3, -1)$, $\vec{b} = (2, 1, 5)$, $\vec{v} = (-3, y, z)$, and $\vec{a} \times \vec{v} = \vec{b}$, find y and z.
b. Explain why there are infinitely many vectors $\vec{c}$ for which $\vec{a} \times \vec{c} = \vec{b}$.

5. sq. both sides

$|\vec{a} \times \vec{b}|^2 = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$

RS.

$RS = |\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$

$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$

common

$= |\vec{a}|^2 |\vec{b}|^2 [1 - \cos^2\theta]$

Pythag

$= |\vec{a}|^2 |\vec{b}|^2 (\sin^2\theta)$

$= |\vec{a} \times \vec{b}|^2 = LS$

$\therefore LS = RS$

6. a)

$$\begin{matrix} 1 & 3 & -1 & 1 & 3 & -1 \\ -3 & y & z & -3 & y & z \end{matrix}$$

$\vec{a} \times \vec{v} = (3z + y,\ 3 - z,\ y + 9) = (2, 1, 5) = \vec{b}$

$3z + y = 2$

$3 - z = 1$

$y + 9 = 5$

$z = 2$

$y = -4$

check

$3(2) + (-4) \stackrel{?}{=} 2$ ✓

b)

$$\begin{matrix} 1 & -3 & -1 & 1 & -3 & -1 \\ x & y & z & x & y & z \end{matrix}$$

$\vec{a} \times \vec{c} = (-3z + y,\ -x - z,\ y + 3x) = (2, 1, 5) = \vec{b}$

① $y - 3z = 2$

② $-x - z = 1$

③ $3x + y = 5$

① − ③ $-3z - 3x = 3$ ÷3

$-x - z = 1$

same as ②

ie. one equation but 2 unknowns

$\therefore \infty$ many sol.

ex. pick $x = 0$

$z = -1$

$y = 5$

$\therefore \vec{c} = (0, -1, 5)$ is one solution of many

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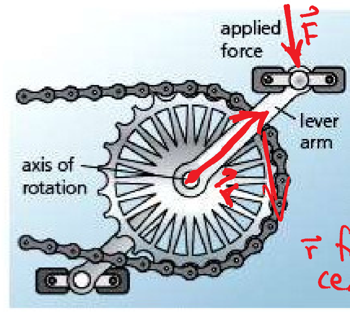
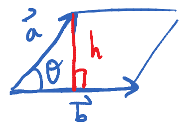
Applications of Dot & Cross Products

Work (in physics) is done whenever a force causes linear displacement

Torque is the turning effect done by a force that causes angular displacement



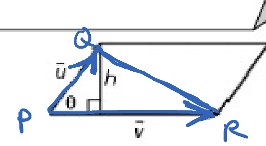


SOH

$\sin\theta = \frac{h}{|\vec{a}|}$

Work	Area of //gm	Torque
$W = \vec{F} \cdot \vec{d}$ $W = \|\vec{F}\|\|\vec{d}\|\cos\theta$ ← tail to tail N·m or Joules	Area $= lw$ $= \|\vec{b}\|\|\vec{a}\|\sin\theta$ $= \|\vec{a}\times\vec{b}\|$	$\vec{T} = \vec{r}\times\vec{F}$ $\vec{T} = \|\vec{r}\|\|\vec{F}\|\sin\theta\,\hat{n}$ Nm or J

1. Find the area of the triangle with vertices $P(7, 2, -5)$, $Q(9, -1, -6)$, and $R(7, 3, -3)$.



use area of //gm formula but cut it in half

$\vec{PQ} = (9-7, -1-2, -6--5) = (2, -3, -1)$

$\vec{PR} = (7-7, 3-2, -3--5) = (0, 1, 2)$

$$\begin{matrix} 2 & -3 & -1 & 2 & -3 & -1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{matrix}$$

$\vec{PQ}\times\vec{PR} = (-6+1, 0-4, 2-0)$

$= (-5, -4, 2)$

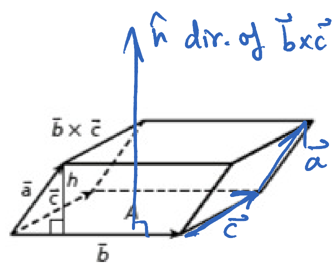
$\therefore$ Area of $\Delta = \frac{|\vec{PQ}\times\vec{PR}|}{2}$

$= \frac{\sqrt{5^2+4^2+2^2}}{2} = \frac{\sqrt{45}}{2}$

$= \frac{3\sqrt{5}}{2}$

2. A **parallelepiped** is a box-like solid, the opposite faces of which are parallel and congruent parallelograms. Its edges are three non-coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Find the volume formula for the parallelepiped.



$V = (\text{Area of base})(\text{height})$

$= |\vec{b}\times\vec{c}|\,|\text{projection of } \vec{a} \text{ onto } \hat{n}|$

$= |\vec{b}\times\vec{c}|\,\frac{\vec{a}\cdot\hat{n}}{|\hat{n}|}$

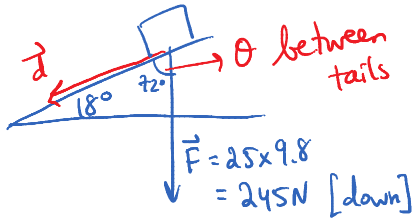
$\hat{n}$ can be replaced with $\vec{b}\times\vec{c}$ since dir. is the only thing that matters

$= \cancel{|\vec{b}\times\vec{c}|}\,\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\cancel{|\vec{b}\times\vec{c}|}} = |\vec{a}\cdot(\vec{b}\times\vec{c})|$ insert abs. values to ensure no negatives.

3. a. A 25-kg box is located 8 m up a ramp inclined at an angle of 18° to the horizontal. Determine the work done by the force of gravity as the box slides to the bottom of the ramp.

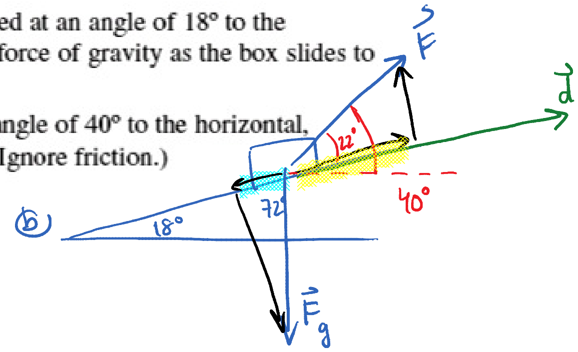
b. Determine the minimum force, acting at an angle of 40° to the horizontal, required to slide the box back up the ramp. (Ignore friction.)

(a)



$W = \vec{F} \cdot \vec{d}$
$= |\vec{F}||\vec{d}|\cos\theta$
$= (245)(8)\cos 72°$
$\doteq 606$ Nm or Joules

(b)



Break both $\vec{F}_g$ and $\vec{F}$ forces into components ensuring they are parallel to the ramp

Now for the box to move, the up component needs to be slightly bigger than down component

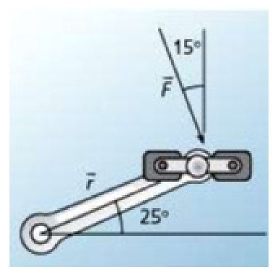
use SOH CAH TOA

$|\vec{F}|\cos 22° > |\vec{F}_g|\cos 72°$

$|\vec{F}| > \frac{245\cos 72°}{\cos 22°}$

$|\vec{F}| > 81.7N$

4. Find the torque produced by a cyclist exerting a force of 115 N on a pedal in the position shown in the diagram, if the shaft of the pedal is 16 cm long.



$\vec{T} = \vec{r} \times \vec{F}$
$= |\vec{r}||\vec{F}|\sin\theta\,\hat{n}$
$= (0.16)(115)\sin 100°\,\hat{n}$
$\doteq 18.1$ Joules [into the page]