S14SeqSeriesNOTES

July 9, 2014 2:56 PM

SeqSeriesNOTES ↴ see below

11U Date:____________ Name: ______________________

Sequences and Series Unit 7

NAME		Learn with	Teach to
HW	/60		
presentation peer teacher	 /5 /10		
TOTAL	/75		

Big idea/Learning Goals

This unit involves discrete functions. Discrete is the opposite of continuous. When you have a domain of real numbers the function you get is **continuous**, where as if the domain is integers/whole/natural numbers, which are separated with spaces, the function is **discrete** or unconnected. Sequences of numbers are different from a series of numbers. **Sequences** are lists of numbers, while **series** are sums of numbers. You will study how to model a pattern sequence with an equation and how to use the equation to solve problems. Equations can be defined explicitly as well as recursively. A **recursion** formula defines terms by using previous terms of the sequence. Finally you will discover how sequences and series are used in real life applications. **Simple interest** means that the interest grows by a *constant* amount each year. **Compound interest** means that the interest grows by a an *increasing* amount each year, because the interest is calculated on the amount deposited as well as on the interest already earned so far.

Corrections for the textbook answers:
Sec 7.1 #11 62 months
Sec 7.3 #1 incorrect solution. $t_n = t_{n-1} - t_{n-2}$ $t_1 = 1, t_2 = 5$ and $t_{10} = -1$
Sec 7.3 #4a) $t_n = 1 - n$, if n odd, $t_n = n$, if n even, -12344 #8 4950
Sec 7.5 #9 $6945
Sec 7.7 #4e) fifth term $270x^2$

Success Criteria

finish sinus. +

skip Handout 3

Date	pages	Topics	# of quest. done? You may be asked to show them	Questions I had difficulty with ask teacher before test!
Day 12 just start	2-4	Introduction to Sequences – 2days Section 7.3 #2, 5,7abcdef,10 + Journal #1 EXTRA two Handouts Handout 1, all even #'s	/9 Handout 2	all even #
		Learn your topic to present		
Day 13	5-6	Arithmetic Sequences Section 7.1 #5def, 6cd, 9cd, 10ab, 11, 13ef, 15 + Journal # 2 EXTRA Handout online	/13	
	7-8	Geometric Sequences Section 7.2 #5cdef, 8cd, 9cd, 11, 12, 13, 14ab + Journal # 3 EXTRA Handout online	/13	
	9-10	Arithmetic Series Section 7.5 #5bcd, 6abd, 7ef, 8ab, 9, 11 + Journal # 4 EXTRA Handout online	/12	
Day 14	11-12	Geometric Series Section 7.6 #3cd, 5abf, 6ef, 7, 11, 12abc 3 + Journal # 5 EXTRA Handout online	/12	
	13-16	Pascal's Triangle & Binomial Expansion & Fibonacci Section 7.7 #5abcdef OR Handout #15-20 Handout #15-24 AND 7.4 #3abc + Journal # 6	/9	
		EXTRA Decimals to Fractions REVIEW		

Reflect – previous TEST mark ___________, Overall mark now___________.

Introduction to Sequences

bottom of page.

1. The sequences you will study will always have a pattern to enable you to find an equation explicitly or recursively for it. The pattern may be arithmetic or geometric or neither.

+ or − **Arithmetic** pattern comes from consecutive terms having a constant difference between them. ↔ Linear

• or ÷ **Geometric** pattern comes from consecutive terms having a constant ratio between them. ↔ exponential

2. To define sequences you must use specific notation. Summarize what these mean

ⓧ input → *n* – position in the list (# of terms)

t_1 or *a* – 1st term value.

d – constant 1st difference

$d = \text{next} - \text{prev.}$

$\boxed{d = t_n - t_{n-1}}$

r – constant 1st ratio

$r = \frac{\text{next}}{\text{prev}}$

$\boxed{r = \frac{t_n}{t_{n-1}}}$

t_6 – term value in position 6

S_7 – sum of 1st 7 terms

S_n – sum of n terms (Formula with n as input.)

t_n – term value at position n (Formula)

t_{n-1} – term value in position n−1 (previous position before n)

2, 4, 6, 8, 10, 12, 14, …

$t_6 = 12$ $t(6) = 12$

a) Which fornulas are explicit,and which recursive ? b) Find 1st four terms, decide if arithmetic/geometric/neither.

3.

$$t_n = \frac{3^n}{2^{n+1}}$$

ⓐ explicit - easy to find t_{100} right away!

ⓑ $t_1 = \frac{3^1}{2^{1+1}} = \frac{3}{2^2} = \frac{3}{4}$

$t_2 = \frac{3^2}{2^{2+1}} = \frac{9}{8}$

$t_3 = \; = \frac{27}{16}$

$t_4 = \; = \frac{81}{32}$

$\times\frac{3}{2}$ $\times\frac{3}{2}$ $\times\frac{3}{2}$

∴ geo

4.

$$t_n = t_{n-1} + 2n - 1, \quad t_1 = 1$$

ⓐ recursive since to get t_{100} you need t_{99}, t_{98}...

ⓑ $t_1 = 1$ provided

$t_2 = t_{2-1} + 2(2) - 1 = t_1 + 3 = 1 + 3 = 4$

$t_3 = t_{3-1} + 2(3) - 1 = t_2 + 5 = 4 + 5 = 9$

$t_4 = \; = t_3 + 7 = 9 + 7 = 16$

neither arithmetic (linear) nor geo (exponential)

$\boxed{t_n = n^2}$ explicit.

5.

$$t_n = \frac{(-1)^n + 1}{n^3}$$

ⓐ explicit.

ⓑ $t_1 = \frac{(-1)^1 + 1}{1^3} = \frac{0}{1} = 0$

$t_2 = \frac{(-1)^2 + 1}{2^3} = \frac{2}{8} = \frac{1}{4}$

$t_3 = \frac{(-1)^3 + 1}{3^3} = \frac{0}{27} = 0$

$t_4 = \frac{(-1)^4 + 1}{4^3} = \frac{2}{64} = \frac{1}{32}$

neither arith. nor geo.

6.

$$t_n = t_{n-1} + 2t_{n-2}, \quad t_1 = 1, \quad t_2 = 3$$

ⓐ recursive

ⓑ $t_1 = 1$ given

$t_2 = 3$ given

$t_3 = t_{3-1} + 2t_{3-2} = t_2 + 2t_1 = 3 + 2(1) = 5$

$t_4 = t_{4-1} + 2t_{4-2} = t_3 + 2t_2 = 5 + 2(3) = 11$

1, 3, 5, 11, … neither.

For each of the following identify if it is:

- a. Arithmetic/geometric/neither
- b. Sequences/series
- c. Finite/infinite

and then find

- d. Next term
- e. General explicit formula for t_n.
- f. Recursive formula, if possible

7. $-4, -7, -10, -13, \_\_\_, \ldots, -52.$

$\underbrace{-4,-7}_{-3}, \underbrace{-7,-10}_{-3}$

ⓐ Arithmetic

ⓑ Sequence (list)

ⓒ Finite (stops)

ⓓ -16

ⓔ
$t_1 = -4 - 3(0)$
$t_2 = -4 - 3(1)$
$t_3 = -4 - 3(2)$
$t_4 = -4 - 3(3)$
$\vdots$
$t_{50} = -4 - 3(49)$

explicit relate to 1st term

$\boxed{t_n = -4 - 3(n-1)}$ $-4 - 3n + 3$ explicit

$\boxed{t_n = -3n - 1}$

ⓕ
$t_1 = -4$
$t_2 = t_1 - 3$
$t_3 = t_2 - 3$
$t_4 = t_3 - 3$

recursive relate to prev. term.

$\boxed{t_n = (t_{n-1}) - 3, \; t_1 = -4}$

recursive.

8. $32 + 16 + 8 + 4 + \_\_\_ + \ldots$

$\div 2 \quad \times\frac{1}{2} \quad \times\frac{1}{2}$

ⓐ Geometric

ⓑ Series (sum)

ⓒ Infinite

ⓓ 2

ⓔ
$t_1 = 32$
$t_2 = 32\left(\frac{1}{2}\right)$
$t_3 = 32\left(\frac{1}{2}\right)^2$
$t_4 = 32\left(\frac{1}{2}\right)^3$

$\boxed{t_n = 32\left(\frac{1}{2}\right)^{n-1}}$

explicit.

ⓕ
$t_1 = 32$
$t_2 = t_1\left(\frac{1}{2}\right)$
$t_3 = t_2\left(\frac{1}{2}\right)$
$t_4 = t_3\left(\frac{1}{2}\right)$
$\vdots$

$\boxed{t_n = (t_{n-1})\left(\frac{1}{2}\right), \; t_1 = 32}$

recursive

Develop general formulas:

9. $a, a+d, a+2d, a+3d, \_\_\_\_\_\_\_, \ldots$

$+d \quad +d \quad +d$

Arithmetic

$t_1 = a$
$t_2 = a + d$
$t_3 = a + 2d$
$t_4 = a + 3d$
$\vdots$

explicit.

$\boxed{t_n = a + (n-1)d}$

term value, 1st term, position, constant difference

$t_1 = a$
$t_2 = t_1 + d$
$t_3 = t_2 + d$

$\boxed{t_n = (t_{n-1}) + d, \; t_1 = a}$

recursive.

10. $a, ar, ar^2, ar^3, \_\_\_\_\_\_, \ldots$

$\cdot r \quad \cdot r \quad \cdot r$

Geometric

$t_1 = a$
$t_2 = ar$
$t_3 = ar^2$

$\boxed{t_n = ar^{n-1}}$

explicit

$\boxed{t_n = (t_{n-1}) \cdot r, \; t_1 = a}$

recursive

For each of the following state

a. Arithmetic/geometric/neither
b. Next term
c. General explicit formula for t_n.
d. Recursive formula, if possible

11. $2+6+10+14+\_\_\_+\ldots+66.$

ⓐ Arithmetic $d=4$

ⓑ 18

ⓒ $t_n=a+(n-1)d$

$t_n=2+(n-1)4$

$t_n=2+4n-4$

$\boxed{t_n=4n-2}$

explicit.

ⓓ $t_n=(t_{n-1})+d,\ t_1=a$

$\boxed{t_n=(t_{n-1})+4,\ t_1=2}$

recursive.

12. $5x, 7x, 9x, \_\_\_, \ldots, 99x.$

$+2x\ +2x$

ⓐ Arith $d=2x$

ⓑ $11x$

ⓒ $t_n=5x+2x(n-1)$

$=5x+2xn-2x$

$\boxed{t_n=3x+2xn}$

ⓓ $\boxed{t_n=(t_{n-1})+2x,\ t_1=5x}$

13. $x^4, x^6, x^8, \_\_\_, \ldots, x^{24}.$

$\cdot x^2\ \cdot x^2$

ⓐ Geometric $r=x^2$

ⓑ x^{10}

ⓒ $t_n=ar^{n-1}$

$t_n=(x^4)(x^2)^{n-1}$

$(a^n)^m=a^{nm}$

$t_n=x^4\cdot x^{2n-2}$

$a^n\cdot a^m=a^{n+m}$

$\boxed{t_n=x^{2n+2}}$

explicit.

ⓓ $t_n=(t_{n-1})\cdot r,\ t_1=a$

$\boxed{t_n=(t_{n-1})\cdot x^2,\ t_1=x^4}$

14. $500-200+80-32+\_\_\_\ldots$

$\times -0.4\ \times -0.4$

ⓐ Geo. $r=-0.4$ or $r=-\frac{2}{5}$

ⓑ 12.8 or $\frac{64}{5}$

ⓒ $\boxed{t_n=500\left(-\frac{2}{5}\right)^{n-1}}$

ⓓ $\boxed{t_n=(t_{n-1})\left(\frac{-2}{5}\right),\ t_1=500}$

What if neither?

For each of the following

a. Show that the seq is neither arith or geo

b. state General explicit formula for t_n.

c. state Recursive formula,

15. $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \_\_\_, \ldots$

ⓐ $\boxed{d = \text{next} - \text{prev}}$

$d = \sqrt{6} - \sqrt{3}$ not like $\doteq 0.717\ldots$

$d = \sqrt{9} - \sqrt{6}$ $\doteq 0.550\ldots$

∴ not arith.

$\boxed{r = \frac{\text{next}}{\text{prev}}}$

$r = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$

$r = \frac{\sqrt{9}}{\sqrt{6}} = \sqrt{1.5}$

∴ not geo.

under root it's arith $t_n = a + d(n-1)$

ⓑ $t_n = \sqrt{3 + 3(n-1)} = \sqrt{3n}$

ⓒ $t_n = \sqrt{(t_{n-1})^2 + 3}$, $t_1 = \sqrt{3}$

16. $\frac{1}{2}, \frac{1^{\times 2}}{2^{\times 2}}, \frac{3}{8}, \frac{1^{\times 4}}{④^{\times 4}}, \frac{5}{32}, \ldots$

ⓐ $d = \frac{1}{2} - \frac{1}{2} = 0$

$d = \frac{3}{8} - \frac{1}{2} = -\frac{1}{8}$

not arith.

$r = \frac{1}{2} \div \frac{1}{2} = 1$

$r = \frac{3}{8} \div \frac{1}{2} = \frac{3}{4}$

not geo.

ⓑ "Unreduce" the fractions to see pattern

$\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \ldots$ ← arithmetic ← geo

$a + d(n-1)$

$1 + 1(n-1) = n$

ar^{n-1}

$2(2)^{n-1} = 2^n$

ⓒ $t_n = \frac{n}{2^n}$

ⓓ $t_n = \frac{N_n}{D_n}$, $N_n = (N_{n-1}) + 1$, $N_1 = 1$

$D_n = (D_{n-1}) \cdot 2$, $D_1 = 2$

17. $\frac{1}{8}, \frac{4}{12}, \frac{9}{16}, \frac{16}{20}, \_\_\_, \ldots$

ⓐ $d = \frac{4}{12} - \frac{1}{8} = \frac{5}{24}$

$d = \frac{9}{16} - \frac{4}{12} = \frac{11}{48}$

not arithmetic

$r = \frac{4}{12} \div \frac{1}{8} = 8/3$

$r = \frac{9}{16} \div \frac{4}{12} = \frac{27}{16}$

not geometric

ⓑ $\frac{1}{8}, \frac{4}{12}, \frac{9}{16}, \frac{16}{20}$ ← n^2 ← arith

$a + d(n-1)$

$8 + 4(n-1)$

$t_n = \frac{n^2}{4n + 4}$

ⓒ $t_n = \frac{N_n}{D_n}$, $N_n = \left(\sqrt{(N_{n-1})} + 1\right)^2$, $N_1 = 1$ OR $N_n = (N_{n-1}) + 2n-1$

$D_n = (D_{n-1}) + 4$, $D_1 = 8$

OR $t_n = \frac{\left(\sqrt{(t_{n-1})4n} + 1\right)^2}{4n+4}$

18. $\frac{1}{2}, \frac{8}{5}, \frac{27}{10}, \frac{64}{17}, \_\_\_, \ldots$

ⓐ $d = \frac{8}{5} - \frac{1}{2} = \frac{11}{10}$

$d = \frac{27}{10} - \frac{8}{5} = \frac{11}{10}$

$d = \frac{64}{17} - \frac{27}{10} = \frac{181}{170}$

not arith

$r = \frac{8}{5} \div \frac{1}{2} = 16/5$

$r = \frac{27}{10} \div \frac{8}{5} = \frac{27}{16}$

not geo

ⓑ $\frac{1}{2}, \frac{8}{5}, \frac{27}{10}, \ldots$ ← n^3 ← $n^2 + 1$

$1+1$ $4+1$ $9+1$

$t_n = \frac{n^3}{n^2 + 1}$

ⓒ $t_n = \frac{N_n}{D_n}$, $N_n = \left(\sqrt[3]{N_{n-1}} + 1\right)^3$, $N_1 = 1$

$D_n = D_{n-1} + 2n-1$, $D_1 = 2$

OR

$D_n = \left(\sqrt{(D_{n-1} - 1)} + 1\right)^2 + 1$

Arithmetic Sequences

1. Summarize the general term formula for any arithmetic sequence. Explain what each letter represents.

explicit: $t_n = a + d(n-1)$

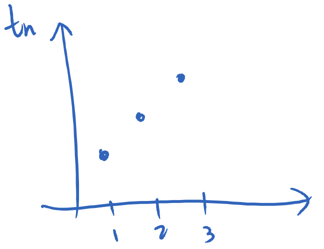
a = 1st term
d = common difference
n = position
t_n = term value.

2. Use function notation to define the arithmetic sequence above. What function type best models it?

$f(x) = a + d(x-1)$

linear (no power on x)

3. Why must the domain be different than usual real numbers? Use graphs to explain..



$n \in \mathbb{N}$

natural #'s
1, 2, 3, ...
no $\emptyset$
no decimals
no negatives

4. Determine a) the explicit formula for a general term, b) the recursive formula, and c) the 30th term of $\overset{+3\ \ +3}{-9,-6,-3},0,\ldots$

ⓐ $t_n = a + d(n-1)$
$t_n = -9 + 3(n-1)$
$\boxed{t_n = 3n - 12}$

ⓑ $t_n = (t_{n-1}) + d,\ t_1 = a$
$t_n = (t_{n-1}) + 3,\ t_1 = -9$

ⓒ $t_{30} = 3(30) - 12$
$= 90 - 12$
$t_{30} = 78$

5. Determine the number of terms in the following sequence $\underbrace{3,15,27}_{+12\ +12},\ldots,495$

last term
want the position of this

$t_n = a + d(n-1)$
$t_n = 3 + 12(n-1)$
$\boxed{t_n = 12n - 9}$

$495 = 12n - 9$

$\frac{504}{12} = \frac{12n}{12}$

$42 = n$

∴ there are 42 terms

6. Determine the general term, t_n, for the arithmetic sequence where $t_{10} = 300, t_{15} = 325$.

$t_n = a + d(n-1)$

pt. $(\underset{n}{10}, \underset{t_n}{300})$ pt. $(15, 325)$

$300 = a + d(10-1)$
① $\boxed{300 = a + 9d}$

$325 = a + d(15-1)$
② $\boxed{325 = a + 14d}$

② − ①
$325 = a + 14d$
$(-)\ 300 = a + 9d$
$25 = \cancel{0} + 5d$
$5 = d$

sub in ① $300 = a + 5(9)$
$300 - 45 = a$
$255 = a$

∴ general term
$t_n = 255 + 5(n-1)$
$\boxed{t_n = 5n + 250}$

$3 \overset{+6}{,} 9 \overset{+12}{,} 21, \ldots$

→ Linear

7. Prove whether the following are arithmetic sequences or not.

a. $t_n = (n+3)(n-5) = n^2 + \ldots$

$t_1 = (1+3)(1-5) = -16$ } $d = +1$

$t_2 = (2+3)(2-5) = -15$ } $d = +3$

$t_3 = (3+3)(3-5) = -12$

$\therefore$ not arithmetic

Shortcut

t_n is not linear, it's quad $\therefore$ not arithmetic

b. $\cancel{f(n)}\ t_n = \dfrac{n^2-4}{2+n} = \dfrac{\cancel{(n+2)}(n-2)}{\cancel{(2+n)}}$

$= n-2$

linear

$\therefore$ arithmetic.

c. $t_1 = 3, \quad t_n = 3 + 2t_{n-1}$

$t_2 = 3 + 2t_{2-1} = 3 + 2t_1 = 3 + 2(3) = 9$

$t_3 = 3 + 2t_{3-1} = 3 + 2t_2 = 3 + 2(9) = 21$

$d = 6$

$d = 12$

$\therefore$ not arithmetic

8. Marisa deposits \$12000 in a savings account that pays simple interest at 2%. She makes no other deposits. (**Simple interest** means that the interest grows by a constant amount each year.)

If you have seen finance formulas in a previous course, don't use them here, use sequence formulas only

a. Write the sequence of the year-end balances over four years

b. Determine the general term formula in function notation → $t_n = f(n)$

c. At the end of some year the balance in the account is \$13200. For how long has the original deposit earned interest?

ⓐ

t_1	t_2	t_3	t_4	t_5
yr 0	yr 1	yr 2	yr 3	yr 4
12000	12240	12480	12720	12960

$\xrightarrow{+240}$ $\xrightarrow{+240}$ $\xrightarrow{+240}$ $\xrightarrow{+240}$

offset

ⓑ $t_n = a + d(n-1)$

$t_n = 12000 + 240(n-1)$

$f(n) = 240n + 11760$

ⓒ $13200 = 240n + 11760$

$\dfrac{1440}{240} = \dfrac{240n}{240}$

$6 = n$ $\xrightarrow{\text{offset}}$

$\therefore$ The balance is \$13200 in 5 years

9. If $a = 6, d = -\frac{1}{4}$ and $t_n = -10$ find n

arithmetic

$t_n = a + d(n-1)$

$-10 = 6 - \frac{1}{4}(n-1)$

$-4\left((-16) = \left(-\cancel{\tfrac{1}{4}}(n-1)\right)\right)$

$64 = n - 1$

$\therefore \boxed{65 = n}$

10. If $t_4 = 7x - 3$, and $d = 5 + x$ find a

$t_n = a + d(n-1)$

$7x - 3 = a + (5+x)(4-1)$

$7x - 3 = a + 15 + 3x$

↑ isolate.

$\therefore \boxed{4x - 18 = a}$

Geometric Sequences

1. Summarize the general term formula for any geometric sequence. Explain what each letter represents.

$t_n = a(r)^{n-1}$

a- first term

r - common ratio

n - position

t_n - term value.

2. If the outputs are from the geometric sequence, what function type best models it. Explain.

$f(x) = ar^{x-1}$

exponential since the variable is in the exponent.

3. Determine a) the explicit formula for the general term, b) the recursive formula, and c) the 30th term of $4, 12, 36, 108, \ldots$

$\cdot 3 \quad \cdot 3 \quad \cdot 3$

ⓐ $t_n = ar^{n-1}$

$t_n = 4(3)^{n-1}$

ⓑ $t_n = (t_{n-1}) \cdot r$, $t_1 = a$

$\boxed{t_n = (t_{n-1})(3), \; t_1 = 4}$

ⓒ $t_{30} = 4(3)^{30-1}$

$t_{30} = 4(3)^{29}$ Better

$t_{30} = 2.745\ldots \times 10^{14}$

14 times

E14

Sci. notation.

4. Determine the number of terms in the following sequence $6, -12, 24, -48, \ldots, -768$.

$\cdot(-2) \quad \cdot(-2)$

want position, n.

geo

$t_n = ar^{n-1}$

$t_n = 6(-2)^{n-1}$

$\frac{-768}{6} = \frac{6(-2)^{n-1}}{6}$

$-128 = (-2)^{n-1}$

match bases

$(-2)^{?} = (-2)^{n-1}$

$(-2)^7 = (-2)^{n-1}$

ignore base

$7 = n - 1$

$8 = n$

∴ there are 8 terms.

5. Determine the general term, t_n, for the geometric sequence where $t_5 = 768, t_9 = 196608$.

pt $(5, 768)$ pt. $(9, 196608)$

$t_n = ar^{n-1}$

$768 = ar^{5-1}$ $196608 = ar^{9-1}$

① $\boxed{768 = ar^4}$ ② $\boxed{196608 = ar^8}$

② ÷ ① $\frac{196608}{768} = \frac{\cancel{a}r^8}{\cancel{a}r^4}$

$\frac{a^n}{a^m} = a^{n-m}$

$256 = r^4$

$\sqrt[4]{256} = r$

$\boxed{4 = r}$ sub in ①

$768 = a(4)^4$

$768 = a(256)$

$\boxed{3 = a}$

∴ $\boxed{t_n = 3(4)^{n-1}}$

6. Marisa deposits \$12000 in a savings account that pays compound annual interest at 2%. She makes no other deposits. (**Compound interest** means that the interest grows by a an increasing amount each year, because the interest is calculated on the amount deposited as well as on the interest already earned so far.)
If you have seen finance formulas in a previous course, don't use them here, use sequence formulas only
   a. Write the sequence of the year-end balances over four years
   b. Determine the general term formula in function notation.
   c. At the end of some year the balance in the account is \$13248.97. For how long has the original deposit earned interest?

$1+\% = 1+0.02 = 1.02$

Notice the answers to simple interest (done in the last lesson) and compound interest calculations are different. This is what you will study in the next unit.

ⓐ

$$\begin{tikzcd}
t_1 & t_2 & t_3 & t_4 & t_5 \\
0\text{yr} & 1\text{yr} & 2\text{yr} & 3\text{yr} & 4\text{yr} \\
12000 \arrow[r, bend left, "\times 1.02"] \arrow[r, bend right, "+240"'] & 12240 \arrow[r, bend left, "\times 1.02"] \arrow[r, bend right, "+244.8"'] & 12484.80 \arrow[r, bend left, "\times 1.02"] & 12734.50 & 12989.19
\end{tikzcd}$$

ⓑ

$t_n = ar^{n-1}$

$t_n = 12000(1.02)^{n-1}$

$f(n) = 12000(1.02)^{n-1}$

$f(x) = 12000(1.02)^{x-1}$

ⓒ

$\frac{13248.97}{12000} = \frac{12000}{12000}(1.02)^{n-1}$

$1.104... = (1.02)^{n-1}$

trial + error

$5 \doteq n-1$

$6 = n$

$\therefore$ after 5 years.

7. If $a = 2, r = 3$ and $t_n = 1458$ find n

$\overset{\uparrow}{\text{geo}}$

$t_n = ar^{n-1}$

$\frac{1458}{2} = \frac{2}{2}(3)^{n-1}$

$729 = 3^{n-1}$

match bases

$3^{?} = 3^{n-1}$

$3^{6} = 3^{n-1}$

ignore base

$6 = n-1$

$\boxed{7 = n}$

8. If $t_5 = 350y^x$, and $r = 5y^{3-2x}$ find a

$\overset{\uparrow}{\text{geo}}$

$t_n = ar^{n-1}$

$350y^x = a\left(5^1 y^{3-2x}\right)^{(5-1)}$ 4

$\overset{\uparrow}{\text{isolate.}}$

$350y^x = a\left(5^4 y^{12-8x}\right)$

$\frac{350y^x}{625y^{12-8x}} = a$

$\frac{14}{25}y^{9x-12} = a$

$\frac{a^n}{a^m} = \frac{a^{n-m}}{1}$ ↑ in numerator

Arithmetic Series

1. Summarize the TWO sum formulas below. Explain what each letter represents.

arithmetic: $S_n = \frac{n}{2}\left[2a + d(n-1)\right]$

$S_n = \frac{n}{2}\left[\underset{\text{1st term}}{t_1} + \underset{\text{Last term}}{t_n}\right]$

S_n = sum of 1st n terms
n = position in list OR total # of terms
a = 1st term
d = common difference

2. What type of function is the sum formula?

$$S(n) = \frac{n}{2}\left[2a + d(n-1)\right]$$

$$S(x) = \frac{x}{2}\left[2a + d(x-1)\right]$$

$$\frac{x}{2}(2a + dx - d) = ax + \frac{dx^2}{2} - \frac{dx}{2}$$

quadratic

3. The altitude of a plane is 200 km at 12 noon when the pilot begins her descent. If the plane's altitude is 196 km one minute later, 192 km one minute after that and so on. Determine when the plane will reach an altitude of 68 km.

t_n not S_n ← look for "total" "sum"

200,	196,	192,	....
t_1	t_2	t_3	
12:00	12:01	12:02	

$t_n = a + d(n-1)$

$t_n = 200 - 4(n-1)$

$\boxed{t_n = -4n + 204}$

$68 = -4n + 204$

$-136 = -4n$

$34 = n$

∴ At time 12:33 pm the plane is at 68 km height.

4. Colin has begun building a pyramid with dominoes. In total, he has 6519 dominoes to work with. On the bottom row he has used 175 dominoes, and each row of his pyramid has a fixed number of dominoes less than the row below it. Unfortunately, Colin runs out of dominoes before he can finish his pyramid, but he is able to finish the last row he was working on before running out of dominoes. The last row contained 71 dominoes. How many rows of the pyramid was Colin able to complete?

don't know "d"

$S_n = 6519$ n = ?

$t_1 = a = 175$

$t_n = 71$ n = ?

$S_n = \frac{n}{2}\left[t_1 + t_n\right]$

$6519 = \frac{n}{2}\left[175 + 71\right]$

$6519 = \frac{n}{\cancel{2}}(\overset{123}{\cancel{246}})$

$6519 = 123n$

$53 = n$

can find "d"

$S_n = \frac{n}{2}\left[2a + d(n-1)\right]$

$6519 = \frac{53}{2}\left[2(175) + d(53-1)\right]$

⋮

∴ He completed 53 rows.

$S_n=\frac{n}{2}[2a+d(n-1)]$ $\quad$ $S_n=\frac{n}{2}[t_1+t_n]$

5. There are $\overset{n}{20}$ terms in an arithmetic series, the 5th term is -8 and the 9th terms is 4. Find the sum of the series

$S_{20}=?$

$t_5=-8 \rightarrow \text{pt.}(5,-8)$

$t_9=4 \rightarrow \text{pt.}(9,4)$

$t_n=a+d(n-1)$

① $-8=a+d(4)$

② $4=a+d(8)$ subtract.

$-12=\cancel{0}-4d$

$\boxed{3=d}$ sub ①

$-8=a+(3)(4)$

$\boxed{-20=a}$

$S_n=\frac{n}{2}[2a+d(n-1)]$

rename

$S_{20}=\frac{20}{2}[2(-20)+3(20-1)]$

$=10[-40+3(19)]$

$=10(-40+57)$

$S_{20}=170$

∴ The sum of 20 terms is 170

6. The 3rd term of an arithmetic series is 9, and the sum of the first 10 terms is -60. Find the partial sum of first 20 terms.

$S_{20}=?$

sub in

$t_3=9 \rightarrow t_n$ formula

$S_{10}=-60 \rightarrow S_n$ formula

(1.) $\boxed{9=a+d(2)}$

(2.) $-60=\frac{10}{2}[2a+d(9)]$

$\frac{-60}{5}=\frac{5}{5}[2a+9d]$

(2.) $\boxed{-12=2a+9d}$

2× ① $18=2a+4d$ subtract

$-30=\cancel{0}+5d$

$\boxed{-6=d}$ sub in ①

$9=a+(-6)(2)$

$\boxed{21=a}$

$S_{20}=\frac{20}{2}[2(21)+-6(19)]$

$=10[42-114]$

$=10(-72)$

$S_{20}=-720$

∴ Sum of 20 terms is -720.

Explanation for why formula works: adding all the terms twice helps to see how formula was developed, mathematician Karl Gauss did this when he was 10 years old in year 1787.

$S_{100}=1+2+3+\ldots \qquad \ldots+97+98+99+100$

$S_{100}=100+99+98+\ldots \qquad \ldots+4+3+2+1$ Add

$2S_{100}=\underbrace{101+101+101+\ldots\ldots 101+101+101+101}_{\text{there are 100 of these}}$

$2S_{100}=101\times100$

$S_{100}=\frac{101\times100}{2}=\frac{\overset{n}{100}}{2}[\underset{t_1}{1}+\underset{t_n}{100}]$

sub in $t_n=a+d(n-1)$

$S_n=\frac{n}{2}[\overset{a}{t_1}+t_n]=\frac{n}{2}[\underbrace{a+a}_{2a}+d(n-1)]$

Geometric Series

1. Summarize the TWO sum formulas below. Explain what each letter represents.

geometric : $S_n = \frac{a(r^n - 1)}{r-1}$ ← not in exponent

$S_n = \frac{(t_{n+1}) - (t_1)}{r-1}$

S_n = sum of n terms
n = position or total # of terms
$t_1 = a$ = 1st term
r = common ratio
t_n = last term
t_{n+1} = term after the last.

2. What type of function is the sum formula if the input is n? If the input is r?

$S(n) = \frac{a(r^n-1)}{r-1}$ $S(r) = \frac{a(r^n-1)}{r-1}$

exponential → $S(x) = \frac{a(r^x-1)}{r-1}$ $S(x) = \frac{a(x^n-1)}{x-1}$ ← Rational : $\frac{1}{x}$

3. Find how many terms in this sequence using t_n

$5 + 20 + 80 + ... + 20480$

$\times 4$ $\times 4$ geo.

$t_n = ar^{n-1}$

$t_n = (5)(4)^{n-1} = \cancel{20^{n-1}}$

$\frac{20480}{5} = \frac{(5)}{5}(4)^{n-1}$

$4096 = 4^{n-1}$

$4^6 = 4^{n-1}$ ∴ $6 = n-1$

$\boxed{7 = n}$

∴ there are 7 terms

4. Find the sum of the geometric series using S_n

$5 + 20 + 80 + ... + 20480 + 81920$

t_7 t_8

$S_n = \frac{a(r^n-1)}{r-1}$

$S_7 = \frac{5(4^7-1)}{4-1}$

$= \frac{5(16384-1)}{3}$

$S_7 = 27305$

OR

$S_n = \frac{t_{n+1} - t_1}{r-1}$

$S_7 = \frac{t_8 - 5}{4-1}$

$S_7 = \frac{81920 - 5}{3}$

$= 27305$

5. A doctor prescibes 200mg of medication on the first day of treatment. The dosage is halved on each successive day for one week. To the nearest milligram, what is the total amount of medication administered?

$200 \xrightarrow{\times 0.5} 100 \xrightarrow{\times 0.5} 50, ...$

1st day 2nd day

t_1 t_2

no offset.

$r = \frac{100}{200} = \frac{1}{2} = 0.5$

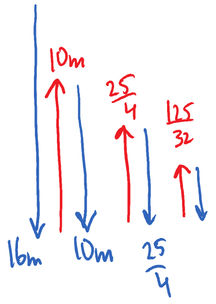
$S_n = \frac{a(r^n-1)}{r-1}$

$S_7 = \frac{200\left(\left(\frac{1}{2}\right)^7 - 1\right)}{\frac{1}{2}-1} = 200\left[\frac{\frac{1}{128} - 1\frac{\cdot 128}{\cdot 128}}{-\frac{1}{2}}\right]$

$= 200\left(\frac{-127}{128}\right) \div \left(\frac{-1}{2}\right)$

$= 200\left(\frac{-127}{128}\right) \times \left(\frac{-2}{1}\right) = \frac{3175}{8} \doteq 396.9$ mg

6. Amy drops a ball from a height of 16m. Each time the ball touches the ground, it bounces up to 5/8 of the maximum height of the previous bounce. Determine the total vertical distance the ball has travelled when it touches the ground on the seventh bounce.



down distances : $a=16$

$r=\frac{5}{8}$

$n=7$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_7 = \frac{16\left(\left(\frac{5}{8}\right)^7-1\right)}{\left(\frac{5}{8}-1\right)} \doteq 41.08 \text{ m}$$

up distances:

$a=10$

$r=\frac{5}{8}$

$n=6$

$$S_6 = \frac{10\left(\left(\frac{5}{8}\right)^6-1\right)}{\left(\frac{5}{8}-1\right)} \doteq 25.08 \text{ m}$$

∴ The total distance

41.08 m + 25.08 m

66.16 m

Explanation for why formula works:

For the series $S_9 = 3+15+75+....+1\ 171\ 875$ find the sum without the formula

Hint subtract $rS_9 - S_9$ and notice what happens. Note that $t_1 = 3, t_9 = 1171875, t_{10} = 5859375$, how does that relate to the formula given on the previous page?

$$\times r \quad S_9 = 3 + 15 + 75 + \cdots + \overset{t_n}{1171875} \quad ①$$

×5 ×5 ×5 ×5 ×5

$$5S_9 = 15 + 75 + 375 + 1171875 + \overset{t_{n+1}}{5859375} \quad ②$$

subtract ② − ①

$5S_9 - S_9$ = everything cancels except $5859375 - 3$

$$(5-1)S_9 = 5859375 - 3$$

$$S_9 = \frac{\overset{t_{n+1}}{5859375} - \overset{t_1}{3}}{\underset{\uparrow r}{5-1}}$$

we know: $t_n = ar^{n-1}$ multiply by r for next term.

$t_{n+1} = \underbrace{ar^{n-1}(r)}_{\text{simplify}}$

$t_{n+1} = ar^n$ sub

$$S_n = \frac{t_{n+1} - t_1}{r-1}$$

$$S_n = \frac{ar^n - a}{r-1}$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$\frac{3(5^9-1)}{4}$$

$S_9 = 1464843$

Pascal's Triangle & Binomial Expansion

1. Expand and simplify each binomial. Look for patterns.

a. $(a+b)^0 = 1$

b. $(a+b)^1$

$= 1a + 1b$

c. $(a+b)^1(a+b)$ Foil

$= a^2 + ab + ab + b^2$

$= 1a^2 + 2ab + 1b^2$

d. $(a+b)^3$

$= (a+b)^2(a+b)$

$= (a^2 + 2ab + b^2)(a+b) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$

e. $(a+b)^4$

$= (a+b)^3(a+b)$

$= (a^3 + 3a^2b + 3ab^2 + b^3)(a+b) = a^4 + 3a^3b^1 + 3a^2b^2 + ab^3 + a^3b^1 + 3a^2b^2 + 3ab^3 + b^4$

$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

f. $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1b^5$

add 5 add 5 add 5

- will have 6 terms
- Row that has "5"

2. Complete the pattern in the Pascal's triangle
   - The value of the first cell and the last cell in every row is ONE
   - The value of any other cell in the triangle is equal to the SUM of the two cells immediately above it.

1 → 11^0

1 1 ~> 11^1

1 2 1 ~> 11^2

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1 ~> ~~11^5~~ pattern breaks down

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

3. Use the found pattern of the Pascal's triangle to expand the following

a. $(\underset{a}{2x}+\underset{b}{1})^{5} = 1(2x)^5 + 5(2x)^4(1)^1 + 10(2x)^3(1)^2 + 10(2x)^2(1)^3 + 5(2x)^1(1)^4 + 1(1)^5$

$5(2^4) \qquad 10(2^3)$

$= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

6 terms

b. $(\underset{a}{\sqrt{m}} - \underset{b=-3}{3})^6 = 1(\sqrt{m})^6 + 6(\sqrt{m})^5(-3)^1 + 15(\sqrt{m})^4(-3)^2 + 20(\sqrt{m})^3(-3)^3 + 15(\sqrt{m})^2(-3)^4 + 6(\sqrt{m})^1(-3)^5 + 1(-3)^6$

not pos!!

$15(m^{1/2})^4(9) \qquad 20(m^{1/2})^3(-27)$

$= m^3 - 18m^{5/2} + 135m^2 - 540m^{3/2} + 1215m - 1458m^{1/2} + 729$

c. $(3x^3 - \frac{2}{x})^4 = 1(3x^3)^4 + 4(3x^3)^3\left(\frac{-2}{x}\right)^1 + 6(3x^3)^2\left(\frac{-2}{x}\right)^2 + 4(3x^3)^1\left(\frac{-2}{x}\right)^3 + 1\left(\frac{-2}{x}\right)^4$

$6(3^2x^6)\left(\frac{4}{x^2}\right)$

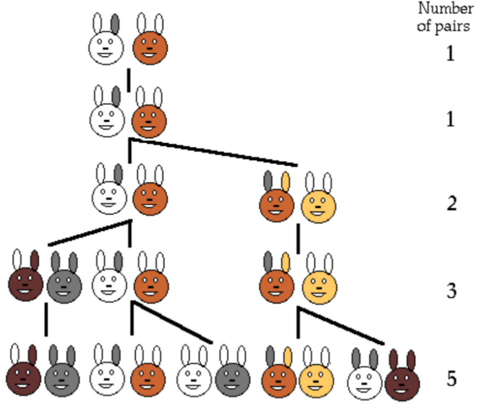
$= 81x^{12} - 216x^8 + 216x^4 - 96 + \frac{16}{x^4}$

Fibonacci Sequence

The information below is taken from:
www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html

The Fibonacci sequence actually first appeared in Indian mathematics in 200 BC, then in the west, Leonardo of Pisa, known as Fibonacci, posed an idealized (biologically unrealistic) rabbit puzzle in 1202 AD, that also relates to Fibonacci numbers. Here it is:

> "A newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. How many pairs will there be in one year?"



1. What is the **Fibonacci sequence**? ✓ $t_9 = t_7 + t_8$

1, 1, 2, 3, 5, 8, 13, 21, 34,

recursive: $t_n = (t_{n-2}) + (t_{n-1})$, $t_1 = 1$, $t_2 = 1$

can't do explicit.

2. Give some reasons why the rabbit puzzle is not realistic?

- rabbits do die
- a litter may be larger than just 1 male 1 female.

Even though the problem is not realistic, Fibonacci does what mathematicians often do at first, simplify the problem and see what happens - and the series bearing his name *does* have *lots* of other interesting and practical applications. For example the honeybee ancestry: parent, grand parents ...

First, some unusual facts about honeybees such as: not all of them have two parents!

- In a colony of honeybees there is one special female called the queen.
- There are many worker bees who are female too but unlike the queen bee, they produce no eggs.
- There are some drone bees who are male and do no work.
- Males are produced by the queen's unfertilised eggs, so male bees only have a mother but no father!
- All the females are produced when the queen has mated with a male and so have two parents.
- Females usually end up as worker bees but some are fed with a special substance called royal jelly which makes them grow into queens ready to go off to start a new colony when the bees form a swarm and leave their home (a hive) in search of a place to build a new nest.

3. Fill in the table below, are the numbers from the Fibonacci sequence, yes or no?

Number of	parents	grand-parents	great-grand-parents	great, great grand parents	gt, gt, gt, grand parents
for male bee	1 only mom	2 (mom + dad)	3 (2 moms, 1 dad)	5	8
for female bee	2 (mom + dad)	3	5	8	13

Looking at your answers to the previous question, your friend Dee says to you:

- You have 2 parents. They each have two parents, so that's 4 grand-parents you've got.
- They also had two parents each making 8 great-grand-parents in total ... and 16 great-great-grand-parents ... and so on.
- So the farther back you go in your Family Tree the more people there are.
- It is the same for the Family Tree of *everyone* alive in the world today. It shows that the farther back in time we go, the more people there must have been. So it is a logical deduction that the population of the world *must* be getting smaller and smaller as time goes on!

4. Is there an error in Dee's argument? If so, what is it?

- people have more than one child

∴ a sister and a brother share same parents

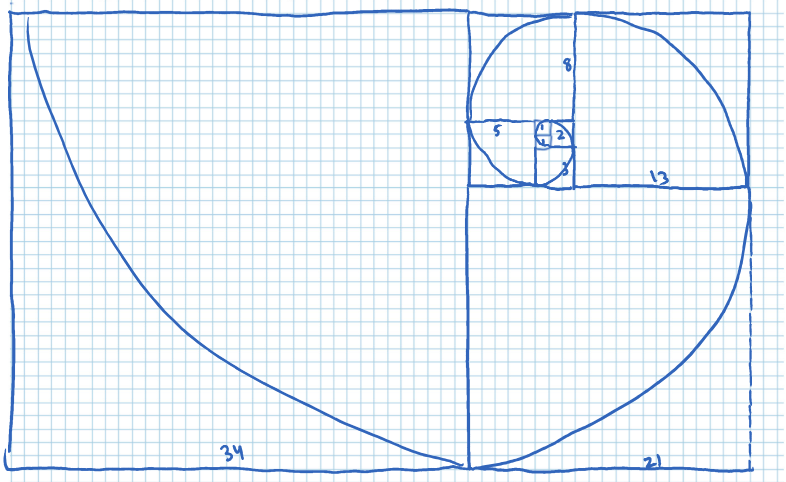
5. Show that the ratios $\frac{t_n}{t_{n-1}}$ of the successive numbers in the Fibonacci sequence get closer and closer to a number called the **Golden Number**, ϕ , called phi, which is approximately 1.618034…

1, 1, 2, 3, 5, 8, 13, 21, 34

$\frac{1}{1}=1$, $\frac{2}{1}=2$, $\frac{3}{2}=1.5$, $\frac{5}{3}=1.\overline{6}$, $\frac{8}{5}=1.6$, $\frac{13}{8}=1.625$, $\frac{21}{13}=1.6153...$ etc $\xrightarrow[\text{approach}]{}$ $\phi=1.618034...$

$\boxed{\phi=\frac{1+\sqrt{5}}{2}}$

6. Draw the Fibonacci Rectangles that create a spiral that is seen in nature



"All life is biology. All biology is physiology. All physiology is chemistry. All chemistry is physics. All physics is math."

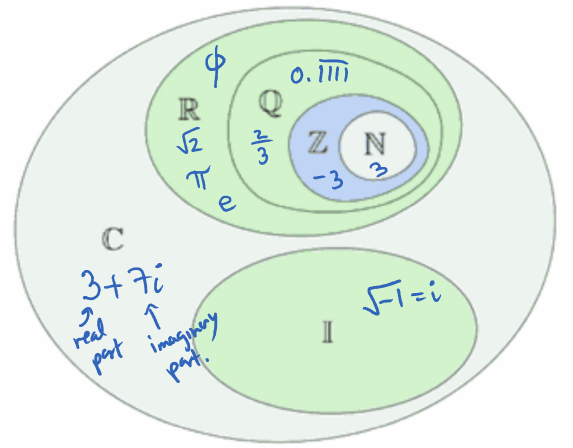
Dr. Stephen Marquardt

The Golden Ratio has a pervasive appearance in real life and the universe.

Please visit
http://www.goldennumber.net/life.htm
(note to teacher: screen shots of this website is in oneNote)

EXTRA

Number Sets



Give an example of a number in each set, record in the corresponding circle.

What type of number is $\sqrt{-4}$? imaginary #.

$\sqrt{4}\sqrt{-1}$

2i

What type of number is 7?

N, Z, R, C

7+0i

What type of number is -3.91?

Q, R, C

What type of number is $\frac{\sqrt{3}}{2}$? ← non repeating ← non terminating.

$\overline{Q}$, R, C

What type of number is 5.67777777…?

Q

Explain:

N – natural #'s – counting #'s {1,2,3,4,…}

W = {0,1,2,3,…}

Z – integers = {…-3,-2,-1,0,1,2,3,…}

Q – rational #'s = $\{\frac{a}{b}, a,b \in Z, b \neq 0\}$ ⇒ decimals that terminate ⇒ repeating decimals.

$\overline{Q}$ – irrational #'s. → non repeating/non terminating

R – real #'s {Q and $\overline{Q}$}

I – imaginary #'s.

C – complex #'s.

Decimals to Fractions

How to convert terminating decimals to fractions

N=34.765 $\times \frac{1000}{1000} = \frac{34765}{1000} = \frac{6953}{200}$

N=2.0168 →

$\frac{2521}{1250}$

How to convert repeating decimals to fractions

① N=45.$\boxed{23}$2323….

② 100N=4523.2323…

② - ① 99N = 4478

$N = \frac{4478}{99}$

N=5.12222222…

① 10N=51.2222…

② 100 N=512.2222…

right before the repeat.

② - ① 90N = 461.0000

$N = \frac{461}{90}$